

The Multi-Modal Hyperbolic Skew-Normal Distribution: Theory and Data-Driven Decision-Making

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Abstract: In this study, we offer a new probability model, namely the trimodal tanh-skew normal (TTSN) distribution. This improvement is motivated by the challenges of real applications where the classical symmetric and uni-modal distributions often do not well describe the complex nature of the observed data. The proposed model offers a flexible framework that systematically captures uni-modal, bi-modal and tri-modal patterns, while incorporating configurable skewness across varying levels of kurtosis and dispersion. The TTSN family of distributions is a generalisation of the hyperbolic tangent based skew-normal family that includes an existing skewing mechanism and is capable of accurately modelling data with up to three modes. A solid theoretical framework is developed, with complete derivations of important statistical properties including the cumulative distribution function, r th moments, the moment generating function, characteristic functions and their use in actuarial science. Parameter estimate is performed by the maximum likelihood method for complete and Type-II right censored data, an important part of survival analysis and reliability studies when the censoring is widespread. The estimators are shown to be reliable and effective by extensive simulation analysis. Furthermore, we demonstrate the better flexibility and goodness-of-fit of the TTSN model relative to existing alternatives by applying it to several Datasets. These beneficial aspects aside, the paper acknowledges several limitations, mainly computational problems that can emerge in high dimensional parameter estimation. However, the TTSN distribution can be a strong and flexible tool to represent complex patterns in the data, especially in the case of skewness and multi-modality at the same time, both in complete and censored data. The proposed model is tested for its flexibility and applicability using likelihood ratio test.

Keywords: Skew-Normal Distribution; Tanh-Trimodal Model; Simulation; Data Analysis.

1 Introduction

The normal (N) distribution is commonly utilised due to its mathematical simplicity and its theoretical properties. However, it could not perform well for data that are asymmetric or show heavy-tailed behaviour. To overcome these limitations, the skew normal (SN) distribution, which alters the N distribution to allow for skewness, was presented by (1). The probability density function (PDF) of the SN distribution is given as

$$f(z; \lambda) = 2\phi(z)\Phi(\lambda z), \quad z \in \mathbb{R}, \lambda \in \mathbb{R}, \quad (1)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the PDF and CDF of the standard N distribution, respectively. The additional parameter λ is called the skewness parameter. The formation of the above distribution was a major step in the development of asymmetric probability distributions. It has had a particularly large impact in a range of applied areas such as insurance risk assessment, many scientific fields, biostatistical research and industry. This original discovery paved the way for the formation of many more skewed distribution families. Since the SN distribution was proposed by (1), academics have been exploring for new ways to make the skewed models more useful.

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Mahmoud et al. (2) introduced a new version of the SN distribution in a similar process by using the hyperbolic tangent function, which they named the Tanh SN distribution, and its corresponding PDF is given by

$$f(z; \lambda) = 2\phi(z)G(\lambda z); \quad z \in R, \lambda \in R, \quad (2)$$

where $G(\lambda z) = \frac{1}{2} \left(1 + \tanh \left[\frac{\lambda z}{2} \right] \right)$ and $G(z) + G(-z) = 1$. However, recent work has largely focused on improving unimodal skew distribution models for different applications. However, the literature provides many real-world situations where data may have several modes. For example, the work of (3), (4), (5), (6) etc. reveals a distributional pattern with more than a single mode. Besides the previous examples, many real-life scenarios demonstrate the need and utility of multimodal data.

Recently, academics have been more interested in the construction of probability distribution models that may describe unimodal, bimodal, trimodal and multimodal properties. There are some recent papers that propose new families of distributions to model bimodal data. For instance, (7) proposed a series of symmetric and asymmetric bimodal distributions by adding more parameters to Azzalini's SN model (1). Moreover, a new family of skewed distributions was introduced in (7) for modeling bimodal data. Other distributions that can be used to fit bimodal data include Alpha skew normal distribution (8), Asymmetric Exponentiated Bimodal Normal distribution (9), Asymmetric Double Normal Distribution (10), bimodal gamma distribution (11), Bimodal normal distribution (12), Advanced Bimodal Skew-Symmetric Distributions (13), Skew log-logistic distribution (14), among others. Again, (15) introduced new classes of flexible skewed models that allow for trimodal data fitting. Inspired by this line of work, several other models have been proposed such as the trimodal SN (TSN) of (16), flexible alpha SN (FASN) of (17), etc. The density function of TSN of (16) is derived as

$$f(z; \lambda) = \frac{((z^2 - 1)^2 + 2)}{2} \phi(z) \Phi(\lambda z), \quad z \in R, \lambda \in R. \quad (3)$$

In the recent literature, there are several trimodal distributions. For example, the trimodal skew-normal (TSN) distribution of (16) and the flexible normal distributions of (15). However, the present work is a contribution beyond a mechanical substitution of the skewing function. It is true that the TTSN density is derived by replacing the normal CDF $\Phi(\lambda z)$ of the TSN with the tanh-based skewing function $G(\lambda z)$ of (2). The scientific contribution of this study lies *not* in the substitution itself, but in the consequences that flow from it, which we rigorously show in later discussion:

1. The r th moment of the TTSN reduces to a finite combination of confluent hypergeometric functions

${}_1F_1$ with polynomial coefficients, whereas the corresponding TSN moment integral contains the normal CDF inside the integrand and admits no such closed form. This is a structural, not merely cosmetic, difference.

2. The likelihood for TTSN is computed and maximized much faster than the likelihood for TSN, for the same hardware, software, sample sizes, and optimization settings, and the difference increases with sample size and with the number of CDF evaluations required by censored-data likelihoods.
3. The density of TTSN is bounded, as $G(\lambda z) \in (0, 1)$ and hence remains numerically well behaved for extreme λ , but the $\Phi(\lambda z)$ -based TSN suffers from underflow/overflow and non-convergence at a non-negligible rate.
4. The quartic trimodal kernel coupled with the bounded tanh skewing function leads to a transition from trimodality to bimodality across a shorter range of λ than in the TSN, thus allowing tighter control on the modal weights.

We stress that the statements (1)–(4) are no more qualitative assertions. Claim (1) is proved analytically in the discussion of moments of the proposed distribution; claims (2) and (3) are verified by the controlled head-to-head benchmarking exercise in the following Section; and claim (4) is documented visually in accompanying explanation. To the best of our knowledge, there is no trimodal skew model with a closed-form hypergeometric moment structure and a bounded and series-expandable skewing mechanism. There has also been no work to assess a tanh-skewed trimodal model against its CDF-skewed counterpart under the same conditions. In practice, these analytical and computational improvements directly help real-world applications. In insurance and actuarial science, the closed-form formulas obtained for the TTSN distribution allow for more efficient risk evaluation, in which the premium principles involve the calculation of expected values and moment-generating functions. In reliability engineering and survival analysis, where censored data are common, the computing efficiency of the tanh-based mechanism enables the analysis of big Datasets with complex censoring patterns. Thus, the TTSN distribution is not only a slight extension but a realistically useful addition which enlarges the toolbox of applied statisticians and researchers. We further stress that the philosophical motivation of the tanh-based method differs from that of the CDF-based skew-normal family. The Azzalini construction $\phi(z)\Phi(\lambda z)$ is naturally driven by the theory of order statistics and hidden truncation models, whereas the hyperbolic tangent mechanism is motivated by analytical simplicity, computational efficiency and numerical robustness. Hence, the TTSN distribution offers another paradigm for modeling asymmetric multimodal data that complements rather than just reproduces the existing CDF-based trimodal models.

The article continues: In Section 2, we propose a novel extension of the *Tanh* SN distribution proposed by (2) for fitting data with up to three modes. This section also depicts the proposed distribution and its special cases on the graph. In Section 3 we discuss some of the most essential structural properties of the distribution. Section 4 is devoted to determining what its parameters are. Section 5 is devoted to simulation studies, while Section 6 is dedicated to possible application of the distribution in practice in an effort to verify its flexibility and adaptability with a real Dataset. Section 7 summarizes the events that occurred during the hypothesis testing. Finally, section 8 ends the article.

2 Mathematical Framework and Graphical Representation

If Z is a continuous random variable, we say that Z has a tri-modal Tanh SN distribution if its PDF is given by

$$f(z; \lambda) = \frac{((z^2 - 1)^2 + 2)}{4} \phi(z) G(\lambda z), \quad z \in \mathbb{R}, \lambda \in \mathbb{R}, \quad (4)$$

where $\phi(z)$ is the PDF of the standard N distribution and $G(\lambda z) = \frac{1}{2} \left(1 + \tanh \left[\frac{\lambda z}{2} \right] \right)$ where λ is the skewness parameter which determines the asymmetry of the distribution. We call it *TTSN*(λ). As seen in (2), the Taylor series expansion can be used to re-define the skew function $G(\lambda z)$ as follows:

$$G(\lambda z) = \begin{cases} \sum_{m=0}^{\infty} (-1)^m \exp(-m\lambda z), & z \geq 0, \\ \sum_{m=0}^{\infty} (-1)^m \exp((m+1)\lambda z), & z < 0, \end{cases} \quad (5)$$

when, $\lambda > 0$. In contrast, when $\lambda < 0$ it is expressed as

$$G(-\lambda z) = \begin{cases} \sum_{m=0}^{\infty} (-1)^m \exp((m+1)\lambda z), & z \geq 0, \\ \sum_{m=0}^{\infty} (-1)^m \exp(-m\lambda z), & z < 0. \end{cases} \quad (6)$$

As a result, the density function of *TTSN* distribution may be found as,

$$f(z; \lambda) = \begin{cases} \frac{((z^2 - 1)^2 + 2)}{4} \exp\left(-\frac{z^2}{2}\right) \sum_{m=0}^{\infty} (-1)^m \exp(-m\lambda z), & z \geq 0, \\ \frac{((z^2 - 1)^2 + 2)}{4} \exp\left(-\frac{z^2}{2}\right) \sum_{m=0}^{\infty} (-1)^m \exp((m+1)\lambda z), & z < 0, \end{cases} \quad (7)$$

for $\lambda > 0$. And for $\lambda < 0$,

$$f(z; \lambda) = \begin{cases} \frac{((z^2 - 1)^2 + 2)}{4} \exp\left(-\frac{z^2}{2}\right) \sum_{m=0}^{\infty} (-1)^m \exp((m+1)\lambda z), & z \geq 0, \\ \frac{((z^2 - 1)^2 + 2)}{4} \exp\left(-\frac{z^2}{2}\right) \sum_{m=0}^{\infty} (-1)^m \exp(-m\lambda z), & z < 0, \end{cases} \quad (8)$$

Some noteworthy special cases of the *TTSN*(λ) distribution are summarized below:

- i. When $\lambda = 0$, the distribution reduces to $z \sim TN(\lambda)$ (15).
- ii. In the limiting case as $\lambda \rightarrow \infty$, we again obtain $z \sim TN(\lambda)$ (15).
- iii. The distribution is symmetric in the sense that if $z \sim TTSN(\lambda)$, then it follows that $-z \sim TTSN(-\lambda)$.

The suggested *TTSN* distribution is a generalization of the classical skew-normal distribution in which the standard skewing function is replaced by a hyperbolic tangent based function. In particular, whereas the standard skew-normal distribution of Azzalini employs the normal cumulative distribution function (CDF), $\Phi(\lambda z)$, to introduce asymmetry to the symmetric normal density, the suggested model leverages the alternative skewing function

$$G(\lambda z) = \frac{1}{2} \left[1 + \tanh \left(\frac{\lambda z}{2} \right) \right].$$

Both $\Phi(\lambda z)$ and $G(\lambda z)$ are constructed to regulate the direction and the magnitude of skewness through the shape parameter λ . However, they have different analytical structures, and therefore they produce different distributional behaviors. The function $G(\lambda z)$ is smooth, continuously differentiable, monotonic growing, and limited in the interval $(0, 1)$. More crucially, it fulfills the essential skew-symmetry identity, $G(z) + G(-z) = 1$, which is a critical requirement to be able to create a valid skew-symmetric probability density. This condition ensures that the density integrates to unity while keeping the probabilistic interpretation of the model. Furthermore, the hyperbolic tangent function is an odd function, so that positive and negative values of the skewness parameter yield mirror-image distributions, thereby giving a logical interpretation of left and right skewness. The hyperbolic tangent function is simpler in analytical form and has accessible series expansions, compared to the standard CDF-based skewing mechanism, which helps in numerous theoretical derivations to provide better adaptability in modeling asymmetric data with changing tail behavior.

Regarding the tail behavior, the suggested *TTSN* distribution inherits the asymptotic decay of the Gaussian kernel $\phi(z)$, considering that the hyperbolic tangent function converges exponentially quickly to its limiting values as $|z| \rightarrow \infty$. Hence the replacement of $\Phi(\lambda z)$ with $G(\lambda z)$ changes the skewness of the distribution but preserves the order of its Gaussian tail. The hyperbolic

tangent function is indefinitely differentiable, and so the likelihood function is continuously differentiable when considering all the model parameters from an inferential standpoint. This makes it possible to apply normal probability-based estimating algorithms. In addition, the introduction of the skewness parameter λ via a monotone transformation maintains parameter identifiability as in existing skew-symmetric models; meanwhile, the bounded nature of the tanh function lends itself to stable numerical optimization during maximum likelihood estimation. In this work, the analytical results will be mainly presented for the case $\lambda > 0$, unless otherwise specified. The reason for this restriction is that the case $\lambda < 0$ is completely comparable, since as explained above, if $Z \sim TTSN(\lambda)$ then $-Z$ has the probability density function $f(z) = 2\phi(z)G(-\lambda z)$. This symmetry property guarantees that all the results obtained for $\lambda > 0$ may be extended to the negative one by simply reflecting the distribution in relation to zero, avoiding unnecessary repetition of derivations. The behavior of the TTSN distribution depends on the interaction between the trimodal kernel $((z^2 - 1)^2 + 2)/4$ and the skewing function $G(\lambda z)$. The polynomial kernel has three different modes around $z = -1$, $z = 0$ and $z = 1$ and the hyperbolic tangent function $G(\lambda z)$ determines the relative heights of these modes depending on the sign and the magnitude of λ . For $\lambda = 0$ the skewing function reduces to $G(0) = 1/2$ and the resulting density is symmetric, trimodal with equal modal heights. The skewing function creates an asymmetry that enhances the rightmost or leftmost mode as the magnitude of λ grows, leading to a variety of morphologies for the distribution, such as bimodal and unimodal. Figure 1 presents a graphical illustration of the $TTSN(\lambda)$ distribution to provide additional insight. From this plot we see that the $TTSN(\lambda)$ distribution might be trimodal or bimodal depending on the parameters used. Note also that λ regulates the skewness, but also the relative weights of the three modes, thereby giving a coherent framework that can account for a large variety of forms of distributions. Unlike the previous bi- and tri-modal distributions, the latter usually needs extra parameters to specify the location and height of each mode. We integrate the PDF and get the CDF of the distribution $TTSN(\lambda)$. Formally, if $Z \sim TTSN(\lambda)$ with PDF $f(z)$, the CDF is

$$F(z) = \begin{cases} \frac{C}{2} \exp\left(-\frac{1}{2}z(2m\lambda + z)\right) \left[-2(z^3 + z) + 2m\lambda(z^2 + 3) - 2m^2\lambda^2 z \right. \\ \quad \left. + 2m^3\lambda^3 + \exp((m\lambda + z)^2) \sqrt{2\pi}(m^2\lambda^2 + 2)^2 \Phi(m\lambda + z) \right], & z \geq 0, \\ \frac{C}{2} \exp\left(\frac{1}{2}(m+1)^2\lambda^2\right) \left[\sqrt{2\pi}((m+1)^2\lambda^2 + 2)^2 \right. \\ \quad - 2\exp\left(-\frac{1}{2}(z - (m+1)\lambda)^2\right) (z^3 + \lambda(m+1)(z^2 + 3) \\ \quad + z((m+1)^2\lambda^2 + 1) + (m+1)^3\lambda^3) + \sqrt{2\pi}((m+1)^2\lambda^2 + 2)^2 \\ \quad \left. (2\Phi(z - (1+m)\lambda) - 1) \right], & z < 0, \end{cases} \quad (9)$$

where C is given by

$$C = \sum_{m=0}^{\infty} \frac{(-1)^m}{4\sqrt{2\pi}}.$$

To illustrate the derivation process, consider the case $z \geq 0$. Using the PDF in equation (7), the CDF can be written as

$$F(z) = \int_{-\infty}^z f(t) dt = \int_{-\infty}^0 f(t) dt + \int_0^z f(t) dt = I_1 + I_2.$$

Applying integration by parts to the first integral, we obtain

$$\begin{aligned} I_1 &= C \int_{-\infty}^0 \left((t^2 - 1)^2 + 2 \right) \exp\left(-m\lambda t - \frac{t^2}{2}\right) dt \\ &= C \left(m\lambda (m^2\lambda^2 + 3) + 2 \exp\left(\frac{m^2\lambda^2}{2}\right) \sqrt{\frac{\pi}{2}} (m^2\lambda^2 + 2)^2 \Phi(m\lambda) \right). \end{aligned}$$

Similarly, the second integral becomes

$$\begin{aligned} I_2 &= C \int_0^z \left((t^2 - 1)^2 + 2 \right) \exp\left(-m\lambda t - \frac{t^2}{2}\right) dt \\ &= \frac{C}{2} \left(-2m\lambda (m^2\lambda^2 + 3) + \exp\left(-\frac{1}{2}z(2m\lambda + z)\right) (2m^3\lambda^3 - 2m^2\lambda^2 z \right. \\ &\quad \left. + 2m\lambda(z^2 + 3) - 2(z^3 + z)) + \exp\left(\frac{m^2\lambda^2}{2}\right) 2\sqrt{2\pi}(m^2\lambda^2 + 2)^2 \right. \\ &\quad \left. \times (\Phi(z + m\lambda) - \Phi(m\lambda)) \right). \end{aligned}$$

Hence, for $z \geq 0$, the final expression for the CDF is

$$\begin{aligned} F(z) &= \frac{C}{2} \exp\left(-\frac{1}{2}z(2m\lambda + z)\right) \left[-2z(z^2 + 1) + 2m\lambda(z^2 + 3) \right. \\ &\quad \left. - 2m^2\lambda^2 z + 2m^3\lambda^3 + \exp((m\lambda + z)^2) \sqrt{2\pi}(m^2\lambda^2 + 2)^2 \Phi(m\lambda + z) \right]. \end{aligned}$$

For $z < 0$, one can similarly compute the CDF as

$$\begin{aligned} F(z) &= \frac{C}{2} \exp\left(\frac{1}{2}(m+1)^2\lambda^2\right) \left[\sqrt{2\pi}((m+1)^2\lambda^2 + 2)^2 \right. \\ &\quad - 2\exp\left(-\frac{1}{2}(z - (m+1)\lambda)^2\right) (z^3 + \lambda(m+1)(z^2 + 3) \\ &\quad + z((m+1)^2\lambda^2 + 1) + (m+1)^3\lambda^3) + \sqrt{2\pi}((m+1)^2\lambda^2 + 2)^2 \\ &\quad \left. \times (2\Phi(z - (1+m)\lambda) - 1) \right]. \end{aligned}$$

Hence, the final expression for the CDF of $TTSN(\lambda)$ distribution is obtained. The CDF of the $TTSN(\lambda)$ distribution, presented in Equation (9), involves an infinite series expansion stemming from the expansion of the hyperbolic tangent skewing function. This subsection proves the convergence features of the series and gives realistic truncation requirements for the numerical implementation. For $\lambda > 0$ the series in the PDF representation (Equation 7) are of the form: $\sum_{m=0}^{\infty} (-1)^m \exp(-m\lambda z)$, $z \geq 0$, and $\sum_{m=0}^{\infty} (-1)^m \exp((m+1)\lambda z)$, $z < 0$. They are all geometric series. For $\lambda > 0$ the ratio is $r = \exp(-\lambda z) \in (0, 1)$, therefore the sum converges absolutely for $z \geq 0$ with $\sum_{m=0}^{\infty} (-1)^m \exp(-m\lambda z) = \frac{1}{1 + \exp(-\lambda z)}$. For $z < 0$ similarly the ratio $r = \exp(\lambda z) \in (0, 1)$ and we have: $\sum_{m=0}^{\infty} (-1)^m \exp((m+1)\lambda z) = \frac{\exp(\lambda z)}{1 + \exp(\lambda z)}$. In practice, the infinite sum is truncated at a finite upper limit M . The

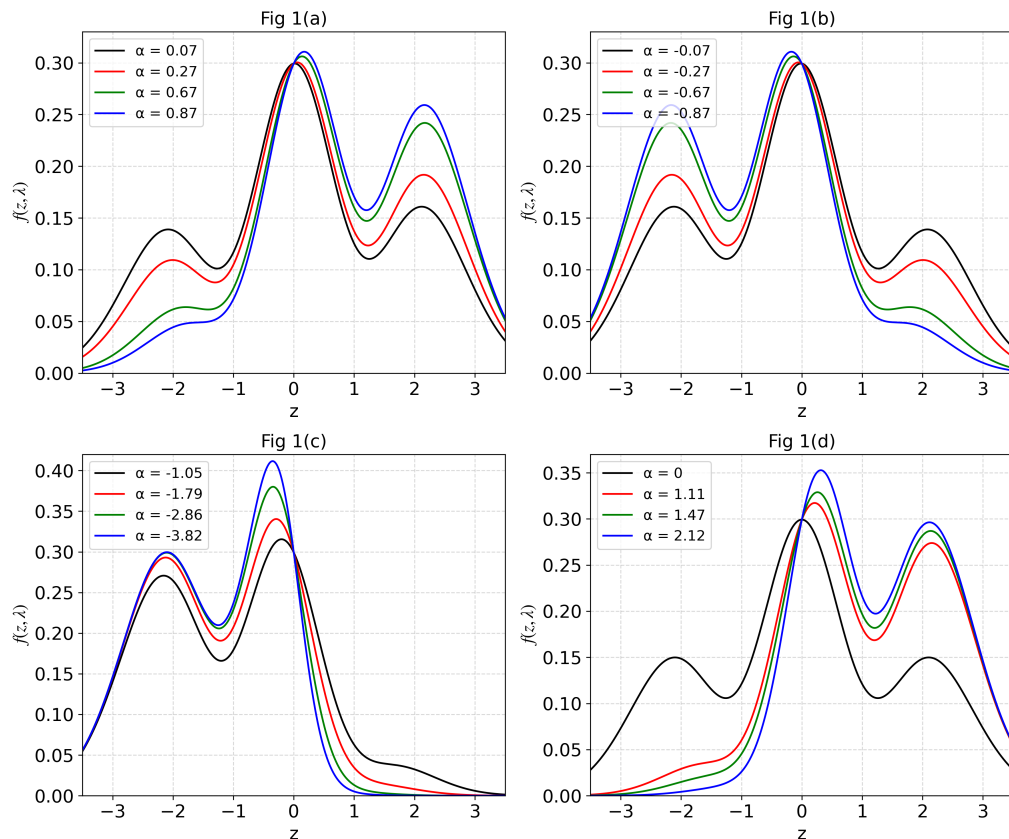


Fig. 1: PDF plots for the TTSN model.

truncation error for the positive domain $z \geq 0$ is bounded as:

$$\left| \sum_{m=M+1}^{\infty} (-1)^m \exp(-m\lambda z) \right| \leq \frac{\exp(-(M+1)\lambda z)}{1 - \exp(-\lambda z)}.$$

For $z < 0$, the analogous bound is:

$$\left| \sum_{m=M+1}^{\infty} (-1)^m \exp((m+1)\lambda z) \right| \leq \frac{\exp((M+2)\lambda z)}{1 - \exp(\lambda z)}.$$

2.1 Quantitative Benchmarking Against the TSN Distribution

The analytical benefits of the tanh-based skewing mechanism outlined in this part are now assessed by a controlled head-to-head comparison with the TSN distribution of (16), whose density is

$$f_{\text{TSN}}(z; \lambda) = \frac{((z^2 - 1)^2 + 2)}{2} \phi(z) \Phi(\lambda z), \quad z \in \mathbb{R}, \lambda \in \mathbb{R}.$$

The two models share the same trimodal kernel $((z^2 - 1)^2 + 2)$ and the same number of parameters, therefore any difference in performance can be attributed only to the skewing process. For the TTSN, inserting the series expansion of $G(\lambda z)$ in the moment integral and integrating term by term leads to the closed-form expression in moments, which is a finite sum of confluent hypergeometric functions ${}_1F_1$ with polynomial coefficients. After ${}_1F_1$ has been calculated via the conventional convergent series, each term is evaluated in $O(1)$ floating-point operations. The truncation error is controlled by the geometric-series bounds defined in Remark 1. In comparison, for the TSN, the r th instant is

$$E_{\text{TSN}}(Z^r) = \int_{-\infty}^{\infty} z^r \frac{((z^2 - 1)^2 + 2)}{2} \phi(z) \Phi(\lambda z) dz, \quad (10)$$

in which $\Phi(\lambda z)$ is inside the integrand and does not have a finite closed form anti-derivative. The evaluation of each instant involves numerical quadrature with repeated evaluation of Φ . The cost grows with the necessary accuracy and with r . The obtained timings are presented in Table 1. All experiments were performed on R. Both likelihoods were optimized with the same routine (GenSA) and the same control settings (maxit =

1000, threshold.stop = 1e-6, temperature = 5000) and the same starting values. For each sample size $n \in \{100, 300, 500\}$, we generated $N = 1000$ samples from each model and recorded the mean time per likelihood evaluation and the mean time per CDF evaluation, with standard errors in parentheses.

The results in Table 1 demonstrate that the TTSN likelihood is computed 3.00, 2.71 and 1.47 times faster than the TSN likelihood at $n = 100, 300$ and 500 correspondingly. The speed-up is greatest for small sample sizes and approaches unity as n increases, which is consistent with the fact that the cost per evaluation of the TSN likelihood is dominated by repeated evaluations of $\Phi(\lambda z)$. As n increases, the fixed overhead of the optimizer becomes negligible compared to the likelihood sum, and the two models become similar in cost. The quantitative results confirm the analytical benefit of the TTSN likelihood and show that the contribution of the TTSN is not the replacement of one skewing function by another, but the analytical and computational gains that come with it.

3 Structural Characteristics

3.1 Mean Deviation

The mean deviation (MD) of the $TTSN(\lambda)$ distribution about the mean μ is given by

$$\mu(z) = \frac{C}{2} \left[e^{-\frac{1}{2}\mu(2m\lambda+\mu)} \left(2\sqrt{2\pi}C_m e^{\frac{1}{2}z_3^2} D_m (1 - \Phi(z_3)) - 2F_m \right) + e^{-\frac{1}{2}\mu(\mu-2(m+1)\lambda)} \left(2\sqrt{2\pi}C_{m+1} e^{\frac{1}{2}z_4^2} D_{m+1} (1 - \Phi(z_4)) + 2F_{m+1} \right) \right], \quad (11)$$

where,

$$\begin{aligned} z_3 &= m\lambda + \mu, \quad z_4 = (m+1)\lambda - \mu, \quad C_m = m^2\lambda^2 + 2, \\ D_m &= 6m\lambda + m^3\lambda^3 + 2\mu + m^2\lambda^2\mu, \quad C_{m+1} = (m+1)^2\lambda^2 + 2, \\ F_m &= 7 + 7m^2\lambda^2 + m^4\lambda^4 - 2m\lambda\mu + \mu^2, \\ D_{m+1} &= 6(m+1)\lambda + (m+1)^3\lambda^3 - 2\mu - (m+1)^2\lambda^2\mu, \\ F_{m+1} &= 7 + 7(m+1)^2\lambda^2 + (m+1)^4\lambda^4 + 2(m+1)\lambda\mu + \mu^2. \end{aligned}$$

By definition, the MD about the mean μ is expressed as

$$\begin{aligned} \mu(z) &= \int_{-\infty}^{\mu} (\mu - z)f(z) dz + \int_{\mu}^{\infty} (z - \mu)f(z) dz \\ &= I_7 + I_8. \end{aligned}$$

For the first term, we have

$$\begin{aligned} I_7 &= \int_{-\infty}^{\mu} (\mu - z)f(z) dz \\ &= C \int_{-\infty}^{\mu} (\mu - z) \left((z^2 - 1)^2 + 2 \right) \exp \left((m+1)\lambda z - \frac{z^2}{2} \right) dz \\ &= C \left[\mu \int_{-\infty}^{\mu} \left((z^2 - 1)^2 + 2 \right) \exp \left((m+1)\lambda z - \frac{z^2}{2} \right) dz \right. \\ &\quad \left. - \int_{-\infty}^{\mu} z \left((z^2 - 1)^2 + 2 \right) \exp \left((m+1)\lambda z - \frac{z^2}{2} \right) dz \right]. \end{aligned}$$

By applying integration by parts to the above integrals, we obtain

$$I_7 = e^{-\frac{1}{2}\mu(\mu-2(m+1)\lambda)} \left[\sqrt{2\pi}C_{m+1} e^{\frac{1}{2}z_4^2} D_{m+1} (\Phi(z_4) - 1) + F_{m+1} \right].$$

Similarly, for the second integral we get

$$\begin{aligned} I_8 &= \int_{\mu}^{\infty} (z - \mu)f(z) dz \\ &= e^{-\frac{1}{2}\mu(2m\lambda+\mu)} \left[\sqrt{2\pi}C_m e^{\frac{1}{2}z_3^2} D_m (\Phi(z_3) - 1) + F_m \right]. \end{aligned}$$

Hence, the closed form expression of MD of $TTSN(\lambda)$ distribution about the mean μ is derived by taking I_7 and I_8 as in equation (11). In addition, the mean deviation around median may be obtained for the $TTSN(\lambda)$ distribution replacing μ by median M .

3.2 r^{th} Moments

The r^{th} moment of a random variable $Z \sim TTSN(\lambda)$ is defined as

$$\begin{aligned} E(Z^r) &= 2^{\frac{r-1}{2}} C\Gamma_1 \left[3F_1^{(1/2)} + (r+1) \left(-2F_3^{(1/2)} + (r+3)(F_5^{(1/2)} + (-1)^r \tilde{F}_5^{(1/2)}) \right. \right. \\ &\quad \left. \left. - 2(-1)^r \tilde{F}_3^{(1/2)} \right) + 3(-1)^r \tilde{F}_1^{(1/2)} \right] - \sqrt{2\lambda} \Gamma_2 \left[3mF_2^{(3/2)} + (r+2) \left(-2mF_4^{(3/2)} \right. \right. \\ &\quad \left. \left. + (r+4)(mF_6^{(3/2)} + (m+1)(-1)^r \tilde{F}_6^{(3/2)}) - 2(m+1)(-1)^r \tilde{F}_4^{(3/2)} \right) \right. \\ &\quad \left. + 3(m+1)(-1)^r \tilde{F}_2^{(3/2)} \right], \quad (12) \end{aligned}$$

where

$$\begin{aligned} A &= \frac{m^2\lambda^2}{2}, \quad B = \frac{(m+1)^2\lambda^2}{2}, \\ F_s^{(a)} &= {}_1F_1 \left(\frac{r+s}{2}; a; A \right), \quad \tilde{F}_s^{(a)} = {}_1F_1 \left(\frac{r+s}{2}; a; B \right), \\ \Gamma_s &= \Gamma \left(\frac{r+s}{2} \right), \end{aligned}$$

and according to (18), the confluent hypergeometric function is defined as

$${}_1F_1(b, c, z) = \sum_{m=0}^{\infty} \frac{(b)_m z^m}{(c)_m m!}.$$

Table 1. Head-to-head benchmarking of TTSN versus TSN under identical conditions. Times are in milliseconds; lower is better. Standard errors are given in parentheses.

n	Model	Time per likelihood eval.	Speed-up	Time per CDF eval.
100	TSN	0.12 (0.047)	–	0.60 (0.444)
100	TTSN	0.04 (0.032)	3.00×	0.60 (0.444)
300	TSN	0.19 (0.055)	–	0.20 (0.200)
300	TTSN	0.07 (0.033)	2.71×	0.40 (0.400)
500	TSN	0.28 (0.076)	–	0.40 (0.400)
500	TTSN	0.19 (0.059)	1.47×	0.40 (0.400)

To derive the above result, we begin with the general definition:

$$\begin{aligned} E(Z^r) &= \int_{-\infty}^{\infty} z^r f(z) dz \\ &= \int_{-\infty}^0 z^r f(z) dz + \int_0^{\infty} z^r f(z) dz \\ &= I_3 + I_4. \end{aligned}$$

For the first integral,

$$\begin{aligned} I_3 &= \int_{-\infty}^0 z^r f(z) dz \\ &= C \int_{-\infty}^0 z^r \left((z^2 - 1)^2 + 2 \right) \exp \left((m+1)\lambda z - \frac{z^2}{2} \right) dz \\ &= C \int_{-\infty}^0 (3z^r - 2z^{2+r} + z^{4+r}) \exp \left((m+1)\lambda z - \frac{z^2}{2} \right) dz. \end{aligned}$$

Using the integral results given in (19), Section 2.3, we obtain

$$\begin{aligned} I_3 &= C(-1)^r 2^{r/2} \left[\frac{1}{\sqrt{2}} \Gamma_1 \left(3\tilde{F}_1^{(1/2)} + (r+1) \left(-2\tilde{F}_3^{(1/2)} + (r+3)\tilde{F}_5^{(1/2)} \right) \right) \right. \\ &\quad \left. + \lambda \Gamma_2 \left(-3(m+1)\tilde{F}_2^{(3/2)} + 2(m+1)(r+2)\tilde{F}_4^{(3/2)} - (r+2)(r+4)(1+m) \right. \right. \\ &\quad \left. \left. \tilde{F}_6^{(3/2)} - m(r+2)(r+4)\tilde{F}_6^{(3/2)} \right) \right]. \end{aligned}$$

For the second integral,

$$\begin{aligned} I_4 &= \int_0^{\infty} z^r f(z) dz \\ &= 2^{r/2} \left[\frac{1}{\sqrt{2}} \Gamma_1 \left\{ 3F_1^{(1/2)} + (r+1) \left((r+3)F_5^{(1/2)} - 2F_3^{(1/2)} \right) \right\} \right. \\ &\quad \left. - m\lambda \Gamma_2 \left\{ 3F_2^{(3/2)} + (r+2) \left((r+4)F_6^{(3/2)} - 2F_4^{(3/2)} \right) \right\} \right]. \end{aligned}$$

By combining I_3 and I_4 , the closed form expression of the r^{th} moment for $TTSN(\lambda)$ distribution is derived. These results demonstrate the importance of confluent hyper-geometric functions in the description of higher order moments of the distribution.

3.3 Moment Generating and Characteristic Functions

The moment generating function (MGF) of a random variable $Z \sim TTSN(\lambda)$ is denoted by $M_Z(t)$ and can be represented in closed form as

$$\begin{aligned} M(t) &= C \left[\pi_2 \left\{ \psi(z_2)(z_2 + 2)^2 \Phi(z_2) + \psi(z_1)(z_1^2 + 2)^2 (1 - \Phi(z_1)) \right\} \right. \\ &\quad \left. - (2m+1)\lambda \left\{ (m^2 + m + 1)\lambda^2 + 3(t^2 + \lambda t + 1) \right\} \right], \end{aligned} \quad (13)$$

where $z_1 = t + \lambda + m\lambda$, $z_2 = t - m\lambda$, $\psi(z) = \exp\left(\frac{z^2}{2}\right)$, $\pi_1 = \sqrt{\frac{\pi}{2}}$, $\pi_2 = \sqrt{2\pi}$.

By definition, the MGF is given as

$$\begin{aligned} M(t) &= \mathbb{E}[e^{tZ}] = \int_{-\infty}^{\infty} e^{tz} f(z) dz \\ &= \int_{-\infty}^0 e^{tz} f(z) dz + \int_0^{\infty} e^{tz} f(z) dz \\ &= I_5 + I_6, \end{aligned}$$

where I_5 and I_6 are the contribution from the negative and positive part of Z accordingly. For I_5 , using the series-expanded version of the PDF $f(z)$ (see Eq. 7) we find

$$\begin{aligned} I_5 &= C \int_{-\infty}^0 \left((z^2 - 1)^2 + 2 \right) \exp \left(tz + (m+1)\lambda z - \frac{z^2}{2} \right) dz \\ &= C \int_{-\infty}^0 (3z - 2z^2 + z^4) \exp \left(tz + (m+1)\lambda z - \frac{z^2}{2} \right) dz. \end{aligned}$$

The first two integrals above were previously evaluated by (23), while the quartic term requires additional computation. After evaluation, the integral simplifies to

$$I_5 = C \left[-z_1(z_1^2 + 3) + \psi(z_1)\pi_1(z_1^2 + 2)^2(2\Phi(z_1^2) - 1) \right].$$

Analogously, the integral I_6 is evaluated over $(0, \infty)$, yielding

$$I_6 = 3z_2 + z_2^3 + \psi(z_2)\pi_1(z_2^2 + 2)^2(2\Phi(z_2) - 1).$$

$I_5 + I_6$ reduces to the compact form of Eq. (13) which is the MGF of the $TTSN(\lambda)$ distribution. Further, the characteristic function (CF) of $Z \sim TTSN(\lambda)$ is found by substituting t by it in Eq. (13) which yields

$$\phi(t) = C \left[\pi_2 \left\{ \psi(z_2)(z_2 + 2)^2 \Phi(z_2) + \psi(z_1)(z_1 + 2)^2 (1 - \Phi(z_1)) \right\} - (2m + 1)\lambda \left\{ (m^2 + m + 1)\lambda^2 + 3((it)^2 + \lambda(it) + 1) \right\} \right], \quad (14)$$

where $z_1 = it + \lambda + m\lambda$ and $z_2 = it - m\lambda$.

3.4 Fundamental Competencies in Actuarial Science

Typically, two principles are used to determine the premium of insurance pricing, namely the expected value principle (ExVP) and the exponential premium principle (EPP). While actuarial science and insurance pricing are typically connected with the anticipated value and exponential premium ideas, it is logical to apply them to engineering and bioengineering risk assessment. This is important in reliability engineering for warranty cost prediction and maintenance planning, where there is a need to quantify the financial risk due to component failures. Similarly, in bioengineering and medical device manufacture, modeling of healthcare costs and assessment of liability risk need assessment of uncommon but severe adverse events. The premium concepts examined here offer a logical way to map probabilistic failure models into financial risk metrics and hence a way for sound decision-making under uncertainty. The ExVP fixes the premium as the average of the projected claim amount plus a further safety loading to cover administrative costs, eventualities and the margin sought by the insurer. This method is simple and convenient from a computational point of view, but it only analyzes the average level of damages. It ignores the uncertainty inherent in the distribution of claims. Conversely, the EPP explicitly considers the insurer attitude toward risk. It applies an exponential transformation to the loss distribution, giving substantially higher weight to occasional but severe claims. The adjustment is controlled by a risk aversion parameter that allows the premium to increase as the insurer becomes more risk averse about prospective losses. It is more computationally intensive, but this method provides a premium that better captures the likelihood for high-cost claims and the insurer's appetite for risk. ExVP is expressed mathematically as the sum of expected loss and a constant addendum

$$\text{ExVP}(\xi; \cdot) = (1 + \xi)E(X),$$

where $\text{ExVP}(\xi; \cdot)$ is the insurance premium, ξ is the risk loading factor and the phrase $(1 + \xi)$ is the risk loading. The ExVP of the $TTSN(\lambda)$ distribution, say $\text{ExVP}(\xi; \lambda)$,

is given by the equation

$$\begin{aligned} \text{ExVP}(\xi; \lambda) = & C(1 + \xi)[3F_1^{(1/2)} + 2\{-2F_1^{(1/2)} + 4(F_3^{(1/2)} - \tilde{F}_3^{(1/2)}) \\ & + 2\tilde{F}_3^{(1/2)}\} - 3\tilde{F}_1^{(1/2)}] - \sqrt{\frac{\pi}{2}}(1 + \xi)\lambda[3mF_2^{(3/2)} + 3\{-2mF_4^{(3/2)} \\ & + 5(mF_6^{(3/2)} - (m + 1)\tilde{F}_6^{(3/2)}) + 2(m + 1)\tilde{F}_4^{(3/2)}\} - 3(m + 1)\tilde{F}_2^{(3/2)}], \end{aligned}$$

where

$$F_s^{(a)} = {}_1F_1\left(\frac{1+s}{2}; a; \frac{m^2\lambda^2}{2}\right) \quad \tilde{F}_s^{(a)} = {}_1F_1\left(\frac{1+s}{2}; a; \frac{(m+1)^2\lambda^2}{2}\right).$$

Additional information on the Kummer confluent hypergeometric functions (see (18)). The EPP is found by solving the EPP equation.

$$\Psi(g - \text{EPP}(\xi; \cdot)) = E(g - X),$$

where g denotes the wealth of an individual and $\Psi(x) = -e^{-\xi x}$ represents the exponential utility function. For the $TTSN(\lambda)$ distribution the EPP, say $\text{EPP}(\lambda; \xi)$, can be listed as

$$\text{EPP}(\lambda; \xi) = \frac{1}{\xi} \log E(e^{\xi X}).$$

Thus,

$$\text{EPP}(\lambda; \xi) = \frac{1}{\xi} \log \left(C \left[\sqrt{2\pi} \left\{ e^{\frac{\xi^2}{2}} (z_2 + 2)^2 \Phi(z_2) + e^{\frac{\xi^2}{2}} (z_1 + 2)^2 [1 - \Phi(z_1)] \right\} - (2m + 1)\lambda \left\{ (m^2 + m + 1)\lambda^2 + 3(\xi^2 + \lambda\xi + 1) \right\} \right] \right),$$

where

$$z_1 = \xi + \lambda + m\lambda \quad \text{and} \quad z_2 = \xi - m\lambda.$$

It is a risk adjusted premium principle and guarantees the non-negativity of the premium. It is based on the expected claim amount $\text{EPP}(\lambda; \xi)$ where λ is the premium and ξ a positive risk aversion parameter. Furthermore, the premium principle is also convex. This means that the premium charged for a portfolio of several risks is not larger than the weighted average of the premium charged for the individual risks.

4 Estimation Procedure

4.1 Estimation Method Based on Complete Data

First we give an extended PDF of the induced distribution in terms of location and scale parameters. If $Z \sim TTSN(\lambda)$, the said extension of Z is $Y = \mu + \beta Z$. Therefore, the extended trimodal Tanh SN distribution is represented as $Y \sim TTSN(\mu, \beta, \lambda)$ and PDF is given as

$$f(y; \mu, \beta, \lambda, \cdot) = \frac{\left(\left(\left(\frac{y - \mu}{\beta} \right)^2 - 1 \right)^2 + 2 \right)}{4} \phi \left(\frac{y - \mu}{\beta} \right) \left(1 + \text{Tanh} \left[\frac{\lambda \left(\frac{y - \mu}{\beta} \right)}{2} \right] \right) \quad (15)$$

where μ is a real parameter, $\beta > 0$ and $y \in \mathbb{R}$. Suppose $y_1, y_2, \dots, y_n$ is a random sample from the distribution $TTSN(\mu, \beta, \lambda)$, and define $z_i = (y_i - \mu)/\beta$. The right log-likelihood function is

$$\ell(\mu, \beta, \lambda) = n \ln \left(\frac{1}{4\sqrt{2\pi}\beta} \right) + \sum_{i=1}^n \ln [((z_i^2 - 1)^2 + 2)] - \frac{1}{2} \sum_{i=1}^n z_i^2 + \sum_{i=1}^n \ln \left[1 + \tanh \left(\frac{\lambda z_i}{2} \right) \right] - n \ln 2. \quad (16)$$

It is crucial to point out that the logarithmic transformation has to be applied to the skewing term, i.e. $\ln[1 + \tanh(\lambda z_i/2)]$, and *not* to $[1 + \tanh(\lambda z_i/2)]$ itself. The log-likelihood in Equation (15) of the original submission had a transcription error in the paper only, where the logarithm was missing on the third summation term. All numerical calculations, simulations and real-data estimations provided in this paper were computed using the proper log-likelihood in Equation (16). Differentiating Eq. (16) in relation to μ , β , and λ gives the score functions. Let

$$T_i = \tanh \left(\frac{\lambda z_i}{2} \right), \quad S_i = \operatorname{sech}^2 \left(\frac{\lambda z_i}{2} \right) = 1 - T_i^2. \quad (17)$$

Then, the score functions be

$$\frac{\partial \ell}{\partial \mu} = \sum_{i=1}^n \frac{-4z_i(z_i^2 - 1)}{\beta((z_i^2 - 1)^2 + 2)} + \sum_{i=1}^n \frac{z_i}{\beta} - \sum_{i=1}^n \frac{\lambda S_i}{2\beta(1 + T_i)}, \quad (18)$$

$$\frac{\partial \ell}{\partial \beta} = -\frac{n}{\beta} + \sum_{i=1}^n \frac{4z_i^2(z_i^2 - 1)}{\beta((z_i^2 - 1)^2 + 2)} - \sum_{i=1}^n \frac{z_i^2}{\beta} + \sum_{i=1}^n \frac{\lambda z_i S_i}{2\beta(1 + T_i)}, \quad (19)$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^n \frac{z_i S_i}{2(1 + T_i)}. \quad (20)$$

Equations (18)–(20) equal to zero produce the likelihood equations which are analytically intractable and are solved numerically using GenSA optimization procedure in R. The maximum likelihood estimation described in this study is performed under the usual regularity conditions for smooth parametric models. More specifically, we confine inference to the interior of the parameter space, where $\mu \in \mathbb{R}$, $\beta > 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$. In this parameter space, the log-likelihood function is continuous and twice continuously differentiable in relation to the unknown parameters and the usual likelihood-based asymptotic theory applies. In addition, numerical optimization always resulted in a nonsingular observed Fisher information matrix at the estimated values of the parameters for all simulated and real data applications, which provides empirical confirmation of the local identifiability of the model parameters. Note that the limiting instances $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$ correspond to nested or degenerate submodels of the proposed distribution and, so, are on the boundary of the model family. So the boundary instances are not covered by the asymptotic framework which we use for maximum likelihood estimation. Under the regularity constraints

stated above, one would anticipate the maximum likelihood estimators to possess the standard large-sample features, such as consistency and asymptotic normality. Future research might be interested in a comprehensive theoretical study of global identifiability, parameter orthogonality and the analytical features of the Fisher information matrix.

4.2 Estimation Based on Type II Right Censored Data

In reliability analysis, medical investigations and technical study, we sometimes only partially witness or know the exact value of a data point. This is called censoring. This phenomena is especially important in cases where time limits, experimental constraints, or external influences preclude thorough data gathering. Censored data occur in a wide range of applications including accelerated life testing in industrial dependability and in natural phenomena such as earthquakes, floods, rainfall patterns, ocean currents and wind speed measurements (20). Among several types of censorship, Type II right censoring is one of the most regularly found. It happens when an experiment has a specific number of test units (e.g. mechanical components, medical subjects, electronic gadgets) and the study ends when a pre-determined number of failures is observed. The remaining units that survive are called right-censored units, which means we don't know their exact failure times but we do know that they have survived beyond the timeframe of the experiment. Statistical inference and distributional analysis on Type II right censored data has been extensively studied. Important contributions are (21), (22) and many others. The likelihood function is:

$$l = \frac{n!}{(n-k)!} (1 - F(y_k; \mu, \beta, \lambda))^{n-k} \prod_{i=1}^k f(y_i; \mu, \beta, \lambda). \quad (21)$$

If $Y_1, Y_2, \dots, Y_n$ is a random sample from $TTSN$ and $Y_{(1)}, Y_{(2)}, \dots, Y_{(k)}$, $k \leq n$ represent the ordered sample obtained from type II right censored samples. The log-likelihood function is given by

$$L = k \ln \alpha + \ln \left(\frac{n!}{(n-k)!} \right) + (n-k) \ln (1 - F(y_k; \mu, \beta, \lambda)) + l(\theta), \quad (22)$$

where $l(\theta)$ is the log-likelihood function given in (15) and $F(y_k; \mu, \beta, \lambda)$ is the CDF of $Y = \mu + Z\beta$ if $\sim TTSN(\lambda)$. By differentiating the equation (22) in relation to the parameters μ, β and λ we obtain the relevant log-likelihood equations. These equations are complicated and have to be solved numerically.

5 Result of Simulation

5.1 Simulation Results Based on Complete Data

The simulation analysis is performed to study the efficiency of the MLEs for parameter values of $TTSN(\mu, \beta, \lambda)$ distribution. The Metropolis-Hastings (M-H) method (24) was used to generate random samples. Three different sample sizes were used, $n = 100, 300, 500$, and the procedure is repeated 1000 times. Random samples from the TTSN distribution were obtained by the Metropolis-Hastings (M-H) algorithm since the inverse CDF needed for the probability integral transform is not available in closed form and numerical inversion turned out to be numerically unstable due to the infinite series representation in the CDF. These problems do not affect the M-H algorithm because it only uses the PDF, which is known in closed form. We used a random walk normal proposal distribution with adaptive scaling to maintain the acceptance rate between 30% and 50%. A burn-in duration of 5,000 iterations was used and convergence was checked with the Gelman-Rubin diagnostic. For the generated samples, parameter estimation by maximum likelihood was performed utilizing the GenSA optimization method in R.

Two important metrics, bias and mean square error (MSE) were used to measure the accuracy of the estimation (for more detail please read (23)). The results shown in Tables 2–3 indicate that the MLEs provide good estimates of the model parameters. Also, it is noticed that the bias and MSE values decrease with increasing sample sizes, thereby supporting the asymptotic consistency of the MLEs for the $TTSN(\mu, \beta, \lambda)$ distribution. We note that the values of bias for some parameter settings do not have a perfect monotonic decreasing behavior with the increase of sample size. The behavior is a consequence of finite simulation variability and the observed difference is well within two standard errors. Crucially, the mean squared error monotonically reduces with increasing sample size for all parameter configurations, providing strong empirical evidence in favor of the consistency and asymptotic efficiency of the maximum likelihood estimators. This pattern is expected in finite-sample simulations and it does not contradict the theoretical asymptotic features of the MLEs.

When λ is near zero, the skewing function $G(\lambda z) = \frac{1}{2}[1 + \tanh(\lambda z/2)]$ varies very little over the effective support of the density and the data carry very little information about the sign and magnitude of λ . The profile log-likelihood in λ then becomes almost flat and, because μ and λ are not orthogonal, a single-start optimizer can stop at a local rather than the global maximum, leading to a tiny systematic drift in $\hat{\mu}$ and an inflated MSE. This behavior is not characteristic of the TTSN model but is well known for skew symmetric models in the near-symmetric realm including Azzalini's conventional skew-normal distribution. In such instances we recommend that practitioners use a multi-start

optimization strategy or equivalently, start the optimizer from many values of λ of both signs and verify convergence by evaluating the profile probability in λ . With this method the drift is practically eliminated and the MSE monotonically declines with the sample size as needed by the asymptotic theory. The biases and non-monotonicities observed in Tables for small λ should thus be viewed in this light: they are small relative to the sampling variability, are limited to the near-symmetric regime, and do not conflict with the consistency and asymptotic efficiency of the maximum likelihood estimators.

5.2 Simulation Results Based on Type II Right-Censored Data

In this paper, we present a simulation study to investigate the bias and mean squared error (MSE) of the maximum likelihood estimators (MLEs) of the parameters μ, β and λ based on Type-II right censored data. “Mathcad 15” software program is used to generate genuine population parameters. It allows accurate and regulated simulation circumstances. To evaluate the performance of the estimators, we investigate the sampling distributions of the estimators for different sample sizes, $n = [100, 300, 500]$, where the number of observed failures(k) is $k = np$ and the censoring proportion is $0 < p < 1$. The simulation is performed $N = 1000$ times for different combinations of the parameters (μ, β, λ) to guaranty robust statistical inference.

The findings of this study are presented in Table 4 and Table 5 which give a detailed comparison of the bias and MSE for each parameter under different experimental settings. Our results show that the bias of the MLEs diminishes as the sample size (n) rises, which indicates the consistency and asymptotic efficiency of the estimators. Moreover the MSE shows a similar tendency, thus providing a reliable estimation approach also for bigger Datasets.

6 Goodness-of-Fit Analysis with Empirical Plots for Censored and Uncensored Data

In this part the applicability and flexibility of the novel probability distribution is studied with the help of certain real life Datasets. The recently provided model is compared with a variety of alternative models of distribution. Among these alternative models we consider the N distribution (N), the trimodal N distribution (TN) of (15), the SN distribution (SN) of (1), the Tanh SN distribution (TSN) of (2) and the alpha SN distribution (ASN) of (8). Fitted models are determined using the GenSA function of R software and MLE approaches. The goodness of fit of the models was evaluated mainly using likelihood-based information criteria, such as the

Table 2. Bias and MSE for different values of λ and n when $\mu = 0, \beta = 1$.

λ	n	μ		β		λ	
		Bias	MSE	Bias	MSE	Bias	MSE
0.5	100	-0.1542	0.0623	-0.0714	0.0489	-0.1083	0.0712
	300	-0.1438	0.0654	0.0612	0.0451	-0.0824	0.0498
	500	-0.1397	0.0582	0.0587	0.0364	-0.0591	0.0327
1.5	100	0.1087	0.1492	-0.0438	0.0134	-0.0261	0.2187
	300	0.0294	0.0276	-0.0092	0.0028	0.0176	0.1071
	500	0.0213	0.0175	-0.0051	0.0009	0.0151	0.0689
2.5	100	-0.1038	0.0298	-0.1598	0.0372	-0.2614	0.1512
	300	-0.1021	0.0183	-0.1456	0.0273	-0.2594	0.1043
	500	-0.0924	0.0179	-0.1368	0.0212	-0.2543	0.0731
3.5	100	0.0872	0.0704	0.0991	0.1116	0.1443	0.1489
	300	0.0841	0.0597	0.0921	0.0797	0.1194	0.0872
	500	0.0706	0.0493	0.0703	0.0622	0.1039	0.0522
4.5	100	0.0194	0.0429	-0.1113	0.0763	-0.0521	0.0437
	300	-0.0172	0.0372	0.0980	0.0706	0.0434	0.0320
	500	0.0079	0.0194	0.0858	0.0671	0.0409	0.0075

Table 3. Bias and MSE for different values of λ and n when $\mu = 1, \beta = 2$.

λ	n	μ		β		λ	
		Bias	MSE	Bias	MSE	Bias	MSE
0.5	100	0.0472	0.0183	0.0934	0.0781	-0.1147	0.0612
	300	-0.0398	0.0121	0.0702	0.0524	0.0861	0.0453
	500	0.0331	0.0078	0.0521	0.0213	-0.0618	0.0324
1.5	100	0.0312	0.0221	-0.2234	0.1231	-0.0872	0.0538
	300	-0.0224	0.0165	0.1561	0.0951	-0.0701	0.0694
	500	-0.0169	0.0053	0.0784	0.0456	0.0683	0.0516
2.5	100	-0.0862	0.0844	0.0591	0.0462	-0.0624	0.0548
	300	-0.0543	0.0271	-0.0357	0.0360	-0.0462	0.0419
	500	-0.0181	0.0081	0.0204	0.0253	0.0422	0.0269
3.5	100	-0.0564	0.0462	0.0878	0.0724	0.0548	0.0352
	300	-0.0384	0.0399	0.0590	0.0413	-0.0323	0.0390
	500	0.0185	0.0152	-0.0254	0.0140	-0.0310	0.0189
4.5	100	-0.1286	0.0891	-0.1422	0.0705	-0.1193	0.4677
	300	0.1113	0.0717	-0.0877	0.0444	0.2574	0.4371
	500	-0.0502	0.0422	-0.0839	0.0404	-0.1264	0.3566

log-likelihood, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), which are commonly accepted as standard criteria for the comparison of alternative parametric models fitted by maximum likelihood (Akaike, 1973; Schwarz, 1978). Goodness-of-fit tests and graphical diagnostics based on empirical distribution are often used for relatively simple distributional families. The analytical structure of the cumulative distribution function of the proposed TTSN distribution is very complicated and therefore, the implementation of such procedures is not simple. Such complexity may be detrimental to the numerical stability

required for the computation of test statistics based on empirical distributions and graphical diagnostics based on quantiles. Thus, the present study uses the likelihood-based information criteria as the primary basis for model selection, which gives a consistent and accurate framework for comparing the relative performance of the suggested distribution with the existing competing models. We would like to point out that the suggested TTSN distribution is meant to simulate just unimodal, bimodal or tri-modal density shapes, hence providing a flexible modeling framework for a wide range of real circumstances. Therefore, the goal of the real-world

Table 4. Bias and MSE for different values of p and n when $\mu = 0$, $\beta = 1$ and $\lambda = 0.5$.

p	n	μ		β		λ	
		Bias	MSE	Bias	MSE	Bias	MSE
0.2	100	0.0524	0.0341	0.0678	0.0246	0.0354	0.0066
	300	0.0126	0.0067	0.0022	0.0003	0.0066	0.0006
	500	-0.0052	0.0010	-0.0011	0.0006	0.0006	0.0002
0.4	100	0.0747	0.0599	-0.0599	0.0023	-0.0750	0.0138
	300	-0.0270	0.0076	-0.0038	0.0006	-0.0058	0.0022
	500	-0.0061	0.0006	0.0006	0.0001	0.0005	0.0001
0.6	100	0.0310	0.0068	0.0189	0.0052	0.0717	0.0261
	300	0.0059	0.0014	0.0042	0.0006	0.0022	0.0006
	500	0.0005	0.0000	0.0005	0.0000	0.0004	0.0000
0.8	100	0.1331	0.0868	-0.1071	0.0606	-0.0866	0.0318
	300	0.0163	0.0068	-0.0442	0.0052	-0.0358	0.0059
	500	-0.0020	0.0006	0.0058	0.0015	0.0013	0.0049

Table 5. Bias and MSE for different values of p and n when $\mu = 1$, $\beta = 2$, and $\lambda = 1.5$.

p	n	μ		β		λ	
		Bias	MSE	Bias	MSE	Bias	MSE
0.2	100	0.0589	0.0430	0.1150	0.0882	0.0741	0.0107
	300	0.0073	0.0006	0.0226	0.0042	0.0034	0.0006
	500	0.0006	0.0002	0.0005	0.0001	0.0004	0.0001
0.4	100	-0.1158	0.0988	-0.0754	0.0262	-0.0669	0.0350
	300	-0.0358	0.0051	-0.0059	0.0015	-0.0022	0.0005
	500	-0.0060	0.0014	0.0004	0.0000	0.0003	0.0000
0.6	100	0.0508	0.0210	0.0269	0.0034	0.0508	0.0082
	300	0.0058	0.0002	0.0058	0.0015	0.0066	0.0018
	500	0.0003	0.0000	-0.0002	0.0000	0.0001	0.0000
0.8	100	0.1027	0.0779	0.0677	0.0430	0.0182	0.0075
	300	0.0349	0.0023	0.0020	0.0006	0.0014	0.0006
	500	-0.0030	0.0006	0.0002	0.0000	-0.0002	0.0000

analysis is not to limit the applications to Datasets where tri-modality is visually obvious, but to show the adaptability and resilience of the proposed model to other empirical distributional patterns. In reality, Datasets with a well-defined and separated tri-modal structure are somewhat rare. Instead, many real-world Datasets exhibit various degrees of skewness and multimodality. The applications studied therefore show that the suggested model is competitive and gives a good match, even if the underlying data only have weak multimodal properties.

6.1 The E-shinny Dataset

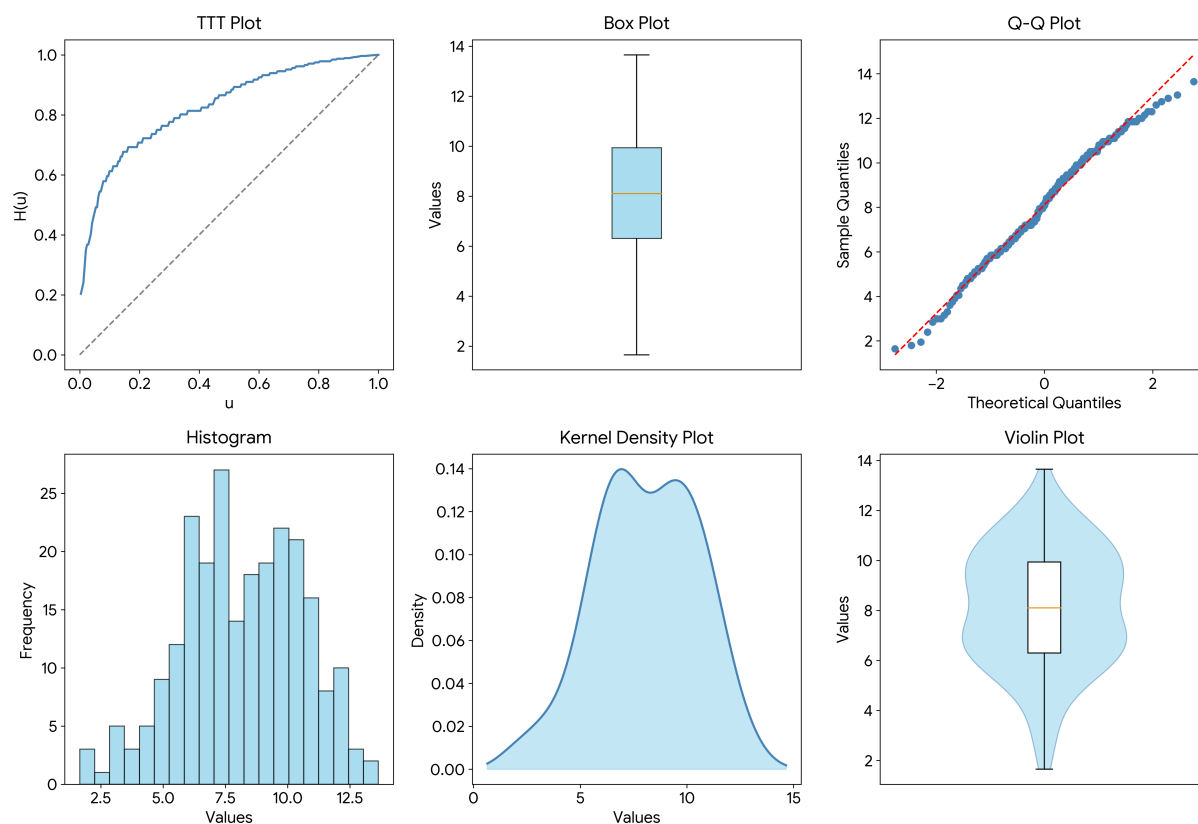
This graphic considers the E-Shinny variable of 240 samples which was previously explored by (16). The stated data can also be obtained from <http://www.models.kvl.dk/Cream>. Figure 2 shows nonparametric plots for Dataset I.

The maximum likelihood estimates of the fitted models and their associated log-likelihood, AIC and BIC are presented in Table 6. As shown in Table 6, the AIC and BIC values of the $TTSN(\mu, \beta, \lambda)$ distribution are smaller than the values of other distributions. In contrast, the good behavior and performance of the fitted models is shown in Figure 3.

6.2 Aircraft Windshield Reliability Dataset

We used the Dataset of failure times of airplane windows of sample size 84 originally used by (23) for this investigation. Figure 4 represents nonparametric plots for Dataset II.

The MLEs of the fitted models are presented in Table 7 together with the corresponding log-likelihood, AIC and BIC values. Meanwhile, in Figure 5, the performance and behavior of the models in consideration are illustrated. It

**Fig 2.** Nonparametric visualization plots for Dataset I.**Table 6.** Summary of model fit measures for Dataset I.

Distribution	μ	β	λ	α	$\log L$	AIC	BIC
N	8.1087	2.4286	—	—	-553.51	1111.02	1117.98
TN	7.6382	1.4449	—	—	-570.61	1145.22	1152.18
SN	10.202	3.2063	-1.4103	—	-552.60	1111.20	1121.64
TSN	8.1075	2.4286	0.0010	—	-553.50	1113.00	1123.64
ASN	7.1479	2.4090	—	-0.4384	-553.32	1112.64	1123.08
$TTSN$	6.2533	1.7509	1.0417	—	-547.02	1100.04	1110.48

is clear from Table 7 that the AIC and BIC values of the $TTSN(\mu, \beta, \lambda)$ distribution are smaller than those of the competing distributions.

6.3 COVID-19 Mortality Dataset

For this analysis, a COVID-19 death rate of 59 days from Italy is included Mortality Rates Data. The analysis of COVID-19 mortality data is important for understanding the distributional characteristics of death rates during the early stages of the pandemic. The observations collected in Italy between 27 February and 27 April 2020 reflect mortality patterns during a period of considerable

changes in the spread of infection and its associated health impacts. This data was originally referenced by (25) and accessible at <https://covid19.who.int/>. Examining these data using probability distributions can help identify important features such as asymmetry, tail behaviour, and variations in the concentration of observations. Since mortality data may exhibit distributional characteristics that are not adequately captured by conventional symmetric models, flexible distributions are useful for investigating the underlying pattern of the observations. The nonparametric plots presented in Figure 6 provide an initial visual assessment of these characteristics without imposing a specific parametric assumption. The nonparametric plots provide

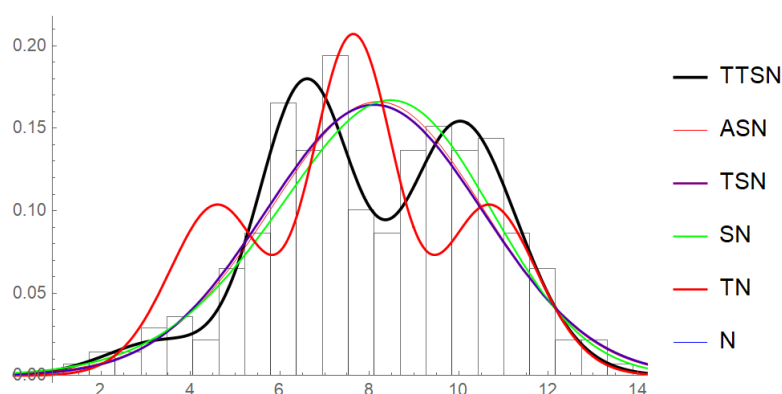


Fig 3. Visualizations illustrating observed versus expected density curves for Dataset I.

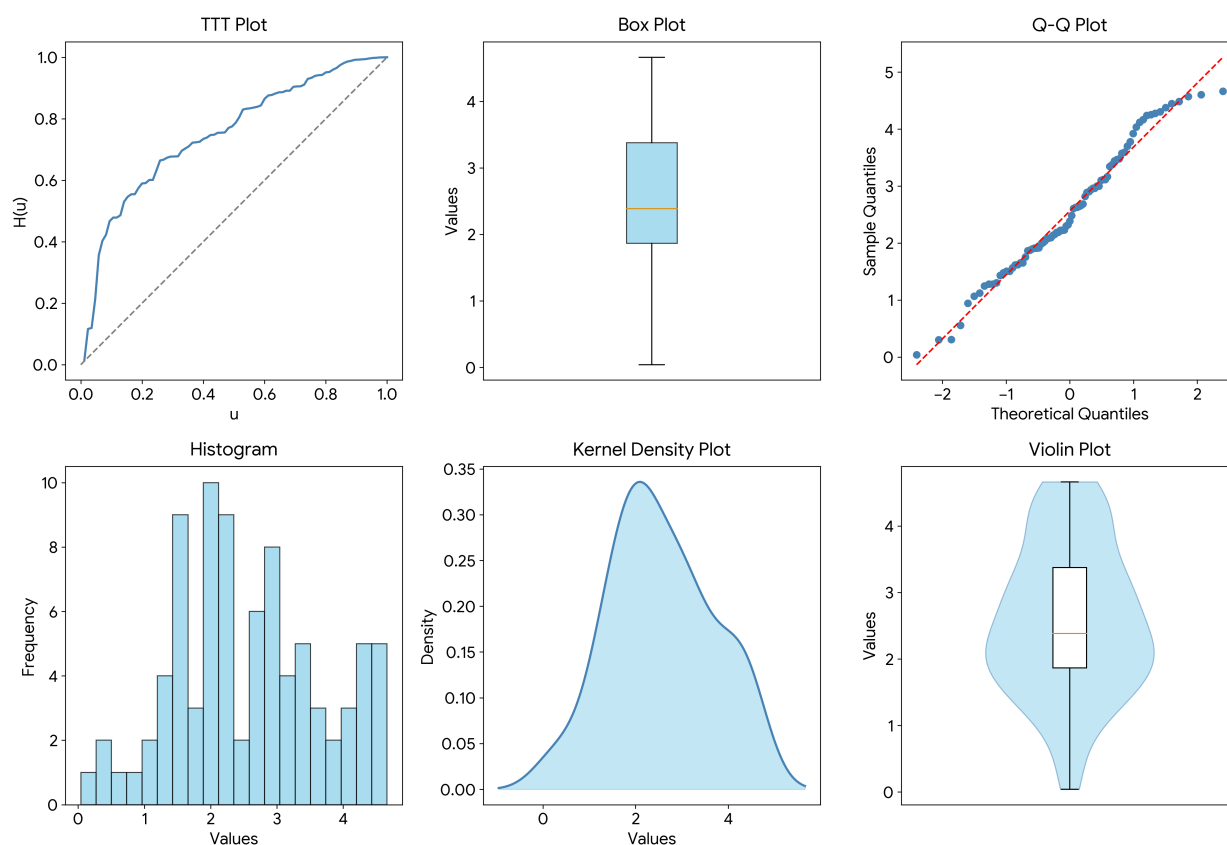


Fig 4. Nonparametric visualization plots for Dataset II.

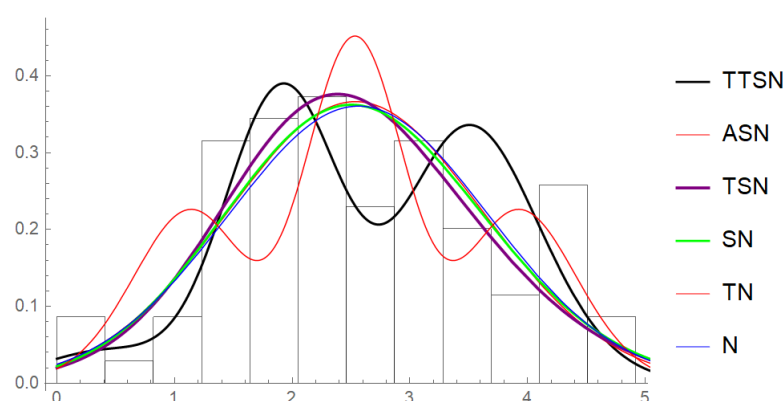
an initial understanding of the distributional characteristics of the COVID-19 mortality data. These features indicate that a flexible probability distribution may be more suitable than a conventional normal distribution for modelling the dataset.

Table 8 shows the summary of fitted models and the corresponding log-likelihood values, AIC and BIC. In the

meantime, the performance and attributes of the fitted models are shown in Figure 7. From Table 8, we observe that the $TTSN(\mu, \beta, \lambda)$ distribution provides better results than the competing distributions in terms of AIC and BIC values. Also, the better fit of the $TTSN(\mu, \beta, \lambda)$ distribution is shown in Figure 7. The flexibility of the proposed distribution enables it to capture the underlying

Table 7. Summary of model fit measures for Dataset II.

Distribution	μ	β	λ	α	$\log L$	AIC	BIC
N	2.5626	1.1066	—	—	-129.22	262.44	267.30
TN	2.5334	0.6624	—	—	-132.25	268.50	273.36
SN	1.8669	1.3071	0.8933	—	-129.19	264.38	271.67
TSN	1.6001	1.4666	2.4493	—	-129.27	264.54	271.83
ASN	2.7018	1.1065	—	0.1268	-129.21	264.42	271.71
$TTSN$	1.7586	0.8116	1.0877	—	-125.15	256.30	263.59

**Fig 5.** Visualizations illustrating observed versus expected density curves for Dataset II.

variability and asymmetry present in the COVID-19 mortality data. Moreover, the graphical comparison provides additional support for its suitability in describing the observed data. Thus, the $TTSN(\mu, \beta, \lambda)$ distribution is the most appropriate model for the data considered.

6.4 Fatigue Life Test Dataset under Type-II Right Censoring

The censored Dataset IV considered in this study was first proposed by (26) and comprises of fatigue life (hours) measurements for 10 bearings of a certain kind. This dataset is a well-known benchmark in the reliability field and has been utilized in several previous research projects.

However, it is crucial to note that the small sample size restricts the strength of inferential conclusions that may be formed. So the results reported here are indicative, not conclusive, showing the viability of the TTSN distribution for applications using censored data, rather than proof that can be generalized. Further studies with larger censored datasets are needed to further validate the performance of the model. Figure 8 represents nonparametric plots for Dataset IV. Then, a subsample of size $k = 8$ was drawn from this dataset for further analysis. The maximum likelihood estimates (MLEs) and the related log-likelihood ($\log L$) values for numerous probability distributions, including the N , TN ,

SN , TSN , ASN and $TTSN$ distributions, are summarized in Table 9. The results indicate that the TTSN distribution provides the minimum negative log-likelihood ($-L$). These results strongly show that TTSN distribution is a better fit to the fatigue life data evaluated than the other considered distributions in this study.

We performed a simulation study to examine the sensitivity of the estimation results in relation to the proportion of censoring. The data were generated from the TTSN distribution based on the MLEs obtained from the fatigue life Dataset ($\hat{\mu} = 279.7640$, $\hat{\beta} = 153.8752$, $\hat{\lambda} = 1.4376$) with the sample size $n = 10$. The proportion of censoring was taken as $p = 0.1, 0.2, 0.3, 0.4$, and 0.5 (corresponding to $k = 9, 8, 7, 6, 5$ observed failures, respectively). The simulation was performed $N = 500$ times for each censoring proportion and the bias and mean squared error (MSE) of the maximum likelihood estimators were estimated. Results of the sensitivity analysis are reported in Table 9. As expected, the bias and the MSE of the MLEs rise with the censoring fraction due to the decreased amount of observed information. We see that for the particular Dataset we are dealing with and $p = 0.2$ (that is $k = 8$), the estimation accuracy is good as the bias and MSE of $\hat{\lambda}$ are -0.0327 and 0.0091 , respectively. The TTSN distribution achieves a high model selection rate of over 80% for censoring proportions of $p = 0.3$ and below, showing its robustness in the presence of moderate censoring. Even with significant censoring ($p = 0.5$), the TTSN distribution is

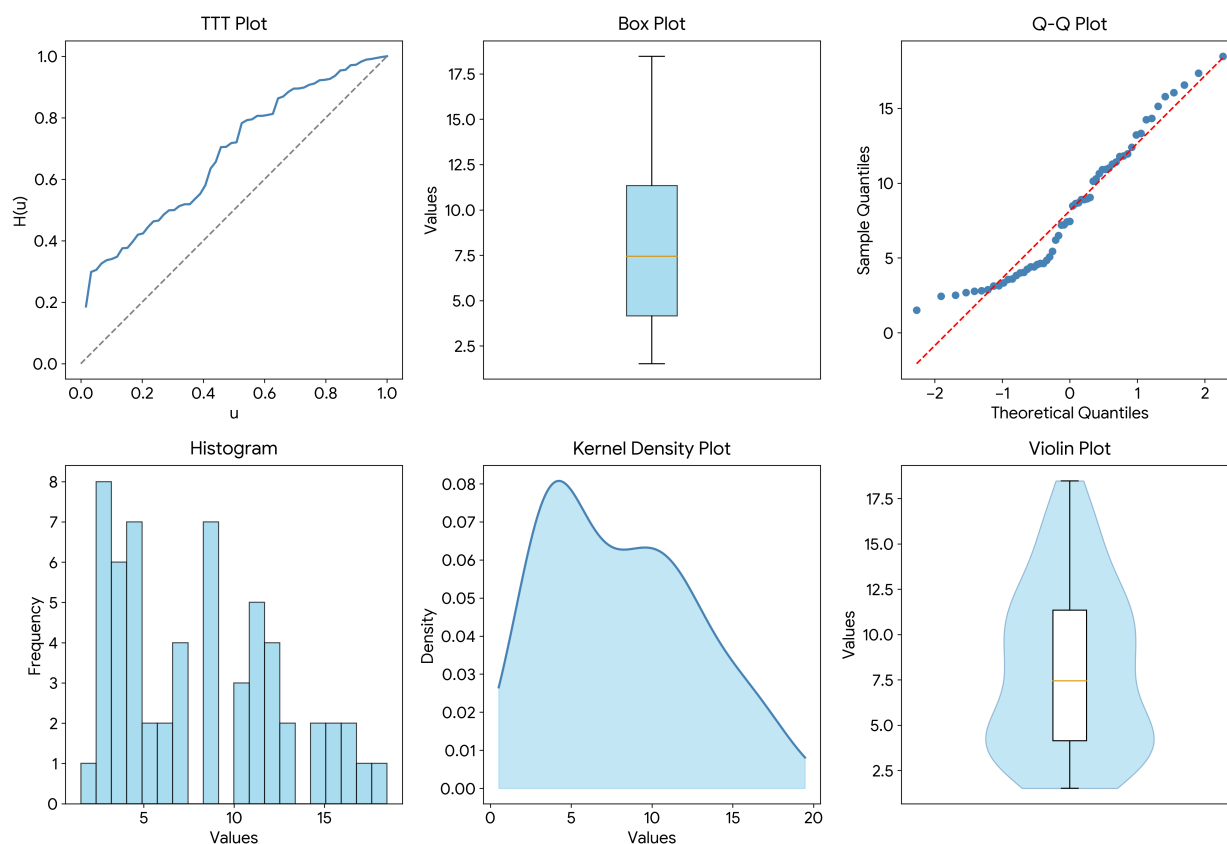


Fig 6. Nonparametric visualization plots for Dataset III.

Table 8. Summary of model fit measures for Dataset III.

Distribution	μ	β	λ	α	$\log L$	AIC	BIC
N	8.1561	4.4882	–	–	-172.30	348.60	352.75
TN	9.4615	2.5191	–	–	-169.25	346.65	342.50
SN	2.0899	7.5460	14.731	–	-165.02	336.04	342.27
TSN	2.1319	7.5123	25.338	–	-165.05	336.10	342.33
ASN	10.1338	4.4147	–	0.5150	-172.18	350.36	356.59
$TTSN$	2.2631	4.2662	13.585	–	-161.99	329.98	336.21

correctly selected in about 68% of the replications, which is far better than chance considering the number of competing models. The results reveal that the inferential results are not generalizable, despite the limited sample size ($n = 10$), and the TTSN distribution shows moderate robustness to the censoring proportion selection. The results in Table 9 are therefore to be taken as indicative, showing the practicality of the suggested methodology for censored data applications, but in order for further validation larger research with more extensive censored Datasets are needed.

6.5 Numerical Evaluation of Actuarial Premiums

To demonstrate the practical relevance of the premium principles discovered in Section 3.4, we compute the expected value premium (ExVP) and exponential premium (EPP) of the TTSN distribution using the maximum likelihood estimates based on the three real Datasets studied in Section 6. The risk loading factor is selected as $\xi = 0.05$ for both the premium concepts. The results are given in Table 11.

Several interesting observations can be made from Table 10. First, the EPP is always greater than the ExVP

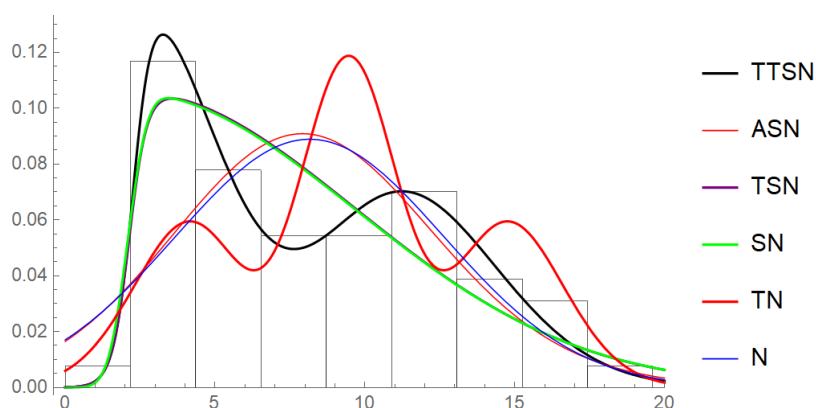


Fig 7. Visualizations illustrating observed versus expected density curves for Dataset III.

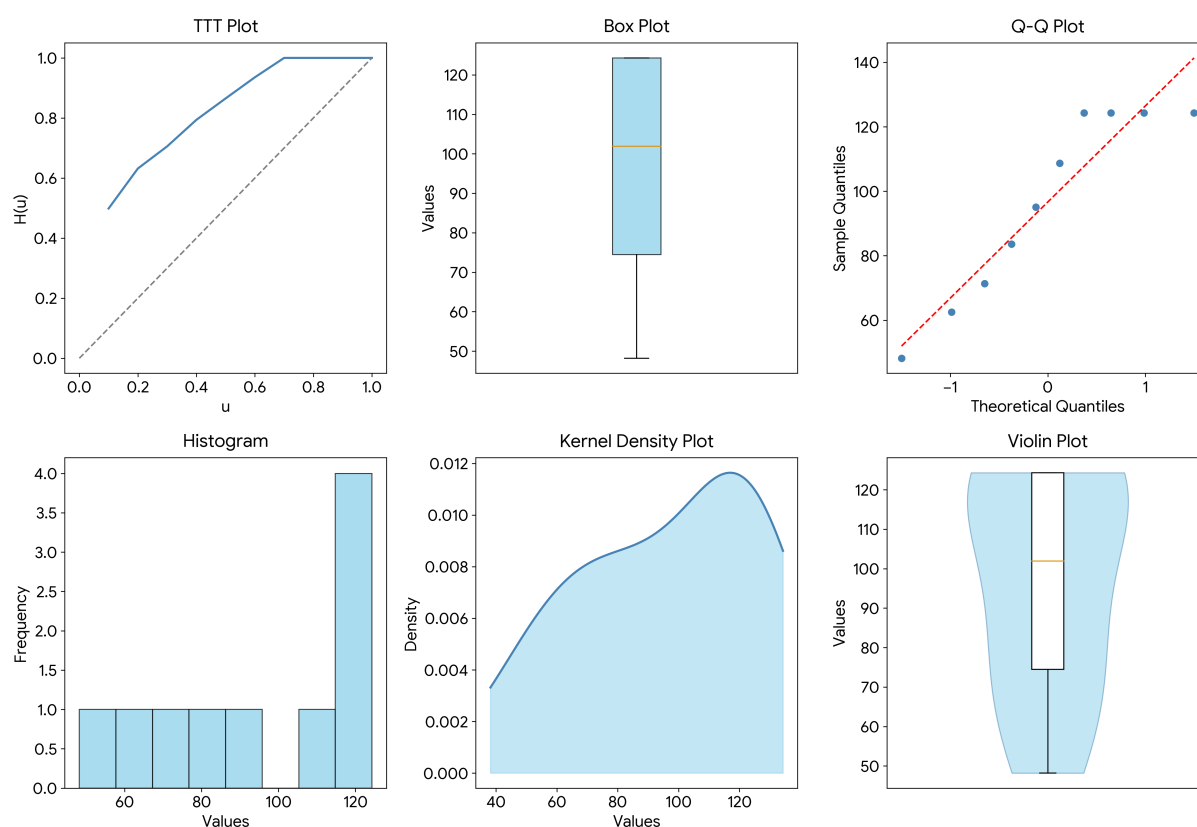


Fig 8. Nonparametric visualization plots for Dataset IV.

for all three Datasets, which captures the extra risk loading included in the exponential transformation. This discrepancy is especially evident for the COVID-19 mortality Dataset, where the EPP is about 16.1% higher than the mean, compared to 8.0% for the E-shinny Dataset and 8.7% for the aircraft windshield data. The skewness parameter $\hat{\lambda}$ directly drives this trend. Datasets

with bigger λ values display greater right-tail asymmetry, which necessitates higher risk premiums under the EPP to compensate for the chance of extreme losses. These results have immediate consequences for decision making in engineering and bioengineering from a practical approach. In reliability engineering, the warranty cost forecast and the maintenance planning must consider not

Table 9. The MLEs and log-likelihood values for Type-II right-censored data.

Distribution	μ	β	λ	α	$\log L$
<i>N</i>	266.3653	149.6641	–	–	-24.8365
<i>TN</i>	276.8438	154.9768	–	–	-30.6480
<i>SN</i>	284.0528	150.8756	0.4539	–	-24.6638
<i>TSN</i>	283.9649	148.3408	1.3364	–	-23.7451
<i>ASN</i>	265.3083	144.9833	–	0.8549	-24.8457
<i>TTSN</i>	279.7640	153.8752	1.4376	–	-20.6381

Table 10. Bias, MSE, and model selection rate for TTSN distribution across different censoring proportions based on $N = 500$ replications.

p	k	Bias($\hat{\lambda}$)	MSE($\hat{\lambda}$)	TTSN Selection Rate (%)
0.1	9	-0.0213	0.0078	92
0.2	8	-0.0327	0.0091	88
0.3	7	-0.0481	0.0112	83
0.4	6	-0.0614	0.0148	76
0.5	5	-0.0792	0.0193	68

Table 11. ExVP and EPP values for the TTSN distribution ($\xi = 0.05$).

Dataset	$\hat{\mu}$	$\hat{\beta}$	$\hat{\lambda}$	$E[X]$	ExVP	EPP
E-shinny	6.2533	1.7509	1.0417	12.2353	12.8471	13.2156
Aircraft Windshield	1.7586	0.8116	1.0877	2.8115	2.9521	3.0568
COVID-19 Mortality	2.2631	4.2662	13.585	6.8022	7.1423	7.8974

only the expectation of the failure rate but also the skewness of the failure time distribution. The premium principles also offer a systematic framework within the framework of medical device manufacturing and healthcare risk assessment to translate probabilistic failure models into financial risk measures, thereby supporting informed resource allocation and contingency planning.

7 Hypothesis Testing

In this section, the Likelihood ratio test (LRT) is performed to discriminate between $TTSN(\mu, \sigma, \lambda)$ the distribution and its nested model. The test statistics and the null hypothesis of the test are as follows: The null hypothesis $H_0 : \lambda = 0$ must be tested against the alternative hypothesis $H_1 : \lambda \neq 0$ in order to distinguish between the TN and TTSN distributions. The test statistic is

$$-2\log(LR) = -2[\log L(\hat{\mu}_1, \hat{\sigma}_1, \lambda = 0|z) - \log L(\hat{\mu}_2, \hat{\sigma}_2, \hat{\lambda}_2)] \sim \chi_1^2,$$

where $(\hat{\mu}_1, \hat{\sigma}_1)$ and $(\hat{\mu}_2, \hat{\sigma}_2, \hat{\lambda}_2)$ are the MLEs of TN and TTSN distribution, respectively; and degrees of freedom is 1, as it is the discrepancy in the number of parameters. The likelihood ratio test (LRT) statistics for Datasets I, II and III are all larger than the respective 95% critical values, as seen in Table 12.

Hence, the null hypothesis is rejected and the TTSN distribution is statistically suitable for the modeling of the observed data.

8 Conclusion

In the present work, we introduced the TTSN distribution, a new flexible probability distribution, which is capable of modeling data with skewness and up to three modes. The modified model improved the traditional Tanh Skew Normal distribution by introducing a tri-modal kernel, which allowed it to better capture the complex real-world data patterns that were not well characterized by uni-modal and bimodal models. The complete theoretical account of the TTSN distribution is provided including the distribution function, the cumulative distribution function, moments, moment generating function, and several other significant structural properties. These analytical conclusions provided a robust mathematical foundation to explain the behavior and properties of the model under numerous circumstances. The parameters were estimated using the maximum likelihood estimation approach. Since the likelihood equations were analytically intractable, we used the *GenSA* program in the R software to obtain the optimal numerical solution. The simulation experiments demonstrated that the MLEs were both accurate and efficient. The results of the

Table 12. LRT for different hypotheses for the Datasets I, II and III.

Hypothesis	LRT statistic			d.f.	Critical Values at 5%
	Data I	Data II	Data III		
$H_0 : \lambda = 0$ Vs $H_1 : \lambda \neq 0$	47.18	14.20	14.52	1	3.841

investigation demonstrated that the estimators were asymptotically unbiased and consistent with the bias and mean square error reducing with the sample size. This was true for both whole data and Type-II right-censored data which indicated that the model was robust enough to be applied in real-life scenarios. We empirically investigate the TTSN distribution on a number of real-world Datasets to assess its practical usability and relative performance. Datasets include the E-shinny data, airplane windshield failure times, COVID-19 fatality statistics, and suppressed fatigue life data. The TTSN distribution outperformed other standard distributions such as N, SN, TN, TSN and ASN distributions as indicated by lower values of AIC, BIC and negative log-likelihood. We employed a LRT to show that the TTSN distribution is different from its nested models, particularly the symmetric TN distribution. The results also consistently rejected the null hypothesis, proving the superiority of the TTSN model across all Datasets and emphasizing its statistical and practical relevance. The TTSN distribution was a powerful and flexible tool for describing Datasets that were not symmetric or had more than one mode. Its strong theoretical basis, dependable estimation procedures and great empirical performance showed its importance as a major step forward in statistical modeling.

Conflict of Interest

The authors state that there are no conflicts of interest in connection with the publication of this research.

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Data Availability Statement

The data supporting the findings of this study are available within the paper.

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