Generalized Order Statistics with Random Indices in a Stationary Gaussian Sequence

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Abstract: In this paper we study the limit distributions of extreme, intermediate and central m-generalized order statistics (gos), as well as m-dual generalized order statistics (dgos), of a stationary Gaussian sequence (sGs) under equi-correlated set up, when the random sample size is assumed to converge weakly. Moreover, the result of extremes is extended to a wide subclass of gos (as well as dgos) which contains the most important models of ordered random variables (rv’s).

Keywords: Gaussian sequences, generalized order statistics, dual generalized order statistics, random sample size

1 Introduction

In testing the strength of materials, reliability analysis, lifetime studies, etc., the realizations of experiments arise in nondecreasing order and therefore we need to consider several models of ascendingly ordered rv’s. Kamps [17] introduced the concept of gos as a unification of several models of these ascendingly ordered rv’s.

Theoretically, many of the models of ascendingly ordered rv’s are contained in the gos model, such as ordinary order statistics (oos), order statistics with non-integral sample size, sequential order statistics (sos), record values, Pfeifer’s record model and progressive type II censored order statistics (pos). These models can be applied in reliability theory. For instance, the rth extreme order statistic represents the life-length of some r-out-of-n system, whereas the sos model is an extension of the oos model and serves as a model describing certain dependencies or interactions among the system components caused by failures of components and the pos model is an important method of obtaining data in lifetime tests. Live units removed early on can be readily used in other tests, thereby saving cost to the experimenter.

The concept of gos enables that known results in submodels can be subsumed, generalized, and integrated within a general framework. In [17] gos were introduced via a distributional approach. Namely, the gos $X(1, n, \tilde{m}, k)$, $X(2, n, \tilde{m}, k)$, ..., $X(n, n, \tilde{m}, k)$ are defined by

$$f_{1,2,...,n,n}(x_1, ..., x_n) = k \left( \prod_{i=1}^{n-1} \gamma_i \right) \left( \prod_{i=1}^{n-1} (1 - F(x_i))^{m_i}(1 - F(x_n))^{k-1} \right) \left( \prod_{i=1}^{n} f(x_i) \right),$$

on the cone $\{ (x_1, ..., x_n) : x_0 = F^{-1}(0) \leq x_1 \leq ... \leq x_n \leq F^{-1}(1) = \tilde{x}_0 \}$. The parameters $\gamma_1, ..., \gamma_n$ are defined by $\gamma_i = k > 0$ and $\gamma_{n,k} = k + n - s + M_k > 0$, $s = 1, 2, ..., n - 1$, where $\tilde{m} = (m_1, m_2, ..., m_{n-1})$, $M_k = \sum_{j=1}^{n-k} m_j$ and $m_1, ..., m_n \in \mathbb{R}$. If $m_1 = m_2 = ... = m_{n-1} = m$ (i.e., $\gamma_{n,k} = k + n - s$), then $s = 1, 2, ..., n - 1$, we get a wide subclass of gos, which is called m-gos, and write $X(s, n, m, k)$ instead of $X(s, n, \tilde{m}, k)$. The class of m-gos contains oos, k−records, sos, order statistics with non-integral sample size and pos, with special censoring schemes.

Nasiri-Roudsari [18] (see, also [2]) has derived the marginal df of the $s$th $m$−gos, $m \neq -1$, in the form $\Phi_{c,n}(\tilde{x},s,N-s+1)$, where $G_m(x) = 1 - (1 - F(x))^{m+1} = 1 - F^m(x)$, $N = \frac{1}{m+1} n + 1$ and $I_r(a,b) = \frac{1}{a^{b+1}} \frac{a^{-1}}{\Gamma(b+1)} \int_0^1 t^a(1-t)^{b-1} dt$ is the incomplete beta ratio function. By using the well-known relation $I_r(a,b) = 1 - I_r(b,a)$, where $\tilde{x} = 1 - x$, the marginal df of the $(n - s + 1)$th $m$−gos, $m \neq -1$, is given by $\Phi_{c,n}(\tilde{x},s,N-s+1)$, where $R_s = \frac{k}{m+1} + s - 1$. The possible non-degenerate limit df’s

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and the convergence rate of the upper extreme m-gos, are discussed in [19]. The necessary and sufficient conditions of the weak convergence, as well as the form of the possible limit df’s, of extreme, intermediate and central m-gos are derived in [2].

Burkschat et al. [8] introduced the concept of dgos to enable a common approach to descendingly ordered rv’s like reversed order statistics and lower records models. The dgos $X_d(1, n, \bar{m}, k), X_d(2, n, \bar{m}, k), \ldots, X_d(n, n, \bar{m}, k)$ based on a df $F$ are defined by their pdf

$$f_d(\bar{m}, k) = \binom{n-1}{i=1} \binom{m-1}{i=1} (F(x_i))^{m_i} (1-F(x_i))^{k-1},$$

where $x_d = F^{-1}(1) > x_2 > \cdots > x_n > F^{-1}(0) = x_0$. Moreover, we can write the df’s of $th$ lower m-dgos $X(n-s, m, k)$ and the $th$ upper m-dgos $X(n-s+1, m, k)$ in the forms $F_{d}^{(m,s)}(x) = \Gamma_{m}(N-s+1, s)$ and $F_{d}^{(m,s+1)}(x) = \Gamma_{m}(s, N-R-s+1)$, respectively, where $\Gamma_{m}(n, x)$ is the $th$ lower intermediate $m$-gos, respectively. A remarkable example of the $th$ lower intermediate $m$-gos, respectively. A remarkable example of the central order statistics is the $th$ sample quantile, with random index.

Theorem 2.1: Let $X_{d}(s, m, k)$ and $X_{d}(s, m, k)$ be a Gaussian sequence with zero expectation, unit variance and correlation $r_{n} = E(X_{d}(Y_{d})) \geq 0, \ i \neq j$. This sequence can be replaced, by the sequence $X_{d}(s, m, k)$, the iid standard normal variables $Y_{d}, Y_{d}, \ldots, Y_{d}$. Moreover, $X_{d}(s, m, k)$, for $n = 0$. Therefore, for any $0 \leq s \leq n$, we get

$$X_{d}(n, m, k) = \sqrt{n} Y_{d} + \sqrt{1-n} Y_{d}(s, m, k), \tag{2}$$

where $X_{d}(n, m, k)$ (or $X_{d}(s, m, k)$) and $Y_{d}(s, m, k)$ are the $th$ upper intermediate $m$-gos (or $m$-gos) based on the sequences $\{X_{d}\}_{j=1}^{n}$ and $\{Y_{d}\}_{j=1}^{n}$, respectively.

A sequence $\{X_{d}(s, n, m, k)\}$ (or $\{X_{d}(s, n, m, k)\}$) is called a sequence of $m$-gos (or $m$-gos) with variable rank $1 \leq s \leq n \leq m \leq n$, as $n \rightarrow \infty$. Here, we have the following two distinct cases:

1-If $\frac{s}{n} \rightarrow 0$ (or $\frac{s}{n} \rightarrow 1$), as $n \rightarrow \infty$, then $X_{d}(s, n, m, k)$ and $X_{d}(s, n, m, k)$ are called lower intermediate $m$-gos and lower intermediate $m$-gos (or upper intermediate $m$-gos and upper intermediate $m$-gos), respectively.

2-If $\frac{s}{n} \rightarrow \lambda (0 < \lambda < 1)$, as $n \rightarrow \infty$, then $X_{d}(s, n, m, k)$ and $X_{d}(s, n, m, k)$ are called central $m$-gos and central $m$-gos, respectively. A remarkable example of the central order statistics is the $th$ sample quantile, where $s_{n} = [np], 0 < p < 1$, and $[x]$ denotes the largest integer not exceeding $x$ (see [11]).

In many biological, agricultural and some quality control problems it is almost impossible to have a fixed sample size, because some observations always get lost for various reasons. Therefore, we often come across situations where the sample size $n$ in $X(s, n, m, k)$ and $X_{d}(s, n, m, k)$ is a rv $v_{n}$ following a given distribution function (df). The rv’s $X_{1|v_{n}} = X(1, v_{n}, 0, 1)$ and $X_{v_{n}} = X(1, v_{n}, 0, 1)$ arise naturally in reliability theory as the lifetimes of series and parallel systems, respectively, with $v_{n}$ identical components having lifetimes $X_{1}, X_{2}, \ldots, X_{v_{n}}$. Also, the rv $X_{v_{n}}$ arises naturally in transportation theory as the accident-free distance of a shipment of, say, explosives, where $v_{n}$ of them are defective, which may explode and cause an accident after $X_{1}, X_{2}, \ldots, X_{v_{n}}$ miles, respectively (cf. [20]). If one introduces the random sample size as an extension of a model (mainly for statistical inference), one can usually assume that it is independent of the underlying variables.

Many authors considered the limit theory of oos with random sample sizes when $r_{n} = 0$ (i.e., in the iid rv’s case) and $v_{n}$ is independent of the basic rv’s, where, the df of $\frac{\bar{X}}{n}$ converges weakly to a non-degenerate df. Among those authors are [1, 7, 12, 14, 15]. Kasdeepa and Vasudeva [21] studied the limit df of s th maxima of oos $X_{v_{n}}$, in the sGs (1), under a restrictive condition that the random correlation $r_{v_{n}}$ converges to a positive constant or infinity. The most recent contribution relevant to this topic is [22], in which it is obtained the limit theorems for the maxima of stationary Gaussian process, with random index.

In Section 2, we study the upper (or lower) extreme m-gos $X(s, v_{n}, m, k)$ (or $X(s, v_{n}, m, k)$) and the upper (or lower) extreme m-gos $X_{d}(s, v_{n}, m, k)$ (or $X_{d}(s, v_{n}, m, k)$) concerning the sequence (1) and (2), respectively, under mild conditions, where the restricted condition in [21] is got rid. Some of these results are extended to a wide subclass of gos, as well as dgos, when the parameters $\gamma_{1,n}, \gamma_{2,n}, \ldots, \gamma_{n,n}$ are assumed to be pairwise different. In Sections 3 and 4, we consider the parallel results for the central and intermediate m-gos and m-dgos, respectively.

Everywhere in what follows the symbols $\frac{\Sigma}{n}$, $\frac{\Sigma}{n}$ and $\frac{\Sigma}{n}$ stand for convergence, converge weakly and converge in probability, as $n \rightarrow \infty$, respectively. Moreover, for every $s, x \geq 0$, $\Gamma_{x}(x) = \frac{1}{\Gamma_{x}} \int_{0}^{\infty} t^{x-1} e^{-tdt}$ stands for the incomplete gamma ratio function, while $\Gamma_{x}(x) = 1 - \Gamma_{x}(x)$ denotes its survivor function. Finally, $\gamma_{x}(x)$ denotes the standard normal df.

2 Extreme m-gos (dgos) with random indices in a sGs

Theorem 2.2: Let $\{X((\nu_{n}), \nu_{m}, \bar{m}, k) - \bar{a}_{n,m}\}$ and $\{X((\nu_{n}), \nu_{m}, \bar{m}, k) - \bar{a}_{n,m}\}$, be investigated in Theorems (2.1) and (2.2), respectively, where $\bar{a}_{n,m} = \frac{1}{\nu_{m,n}} - \frac{1}{\nu_{n,m}} (\log n + \log \pi) + \frac{1}{\nu_{n,m}}$ and $b_{n,m} = \frac{1}{m^{1+\alpha}}$. Moreover, Theorems (2.3) and
Let \( \{a_{n,m}\} \) and \( \{b_{n,m}\} \), respectively, where \( a_{n,m} = \frac{1}{b_{n,m}} - \frac{1}{2} \), \( b_{n,m} = (2 \log n (m+1))^{\frac{1}{2}} \).

**Theorem 2.1.** Let \( v_n \) be a sequence of integer valued rv’s independent of \( X_1, \ldots, X_n \) and \( P(V_n < x) = A_n(x) \) is a non-degenerate df. Then

\[
P(B_n \sim n - \frac{1}{2} \sqrt{2 \log \log n}) \to \frac{1}{n^\alpha} \mathcal{N}(0, 1)
\]

and \((*)\) stands for the convolution operation. Moreover,

\[
P(B_n \sim n - \frac{1}{2} \sqrt{2 \log \log n}) \to \frac{1}{n^\alpha} \mathcal{N}(0, 1)
\]

Conversely, if \((B_n, V_n)\) hold, then the relation \((A)\) will be satisfied.

**Proof.** Let \( P_{ng} = P(V_n = g) \). Then, by the total variation rule, we get

\[
M_{s'(v_n); v_n}(a_{n,m} + b_{n,m}v_n) = P(X(s'(v_n), v_n, m, k) < a_{n,m} + b_{n,m}v_n)
\]

\[
= \sum_{q=1}^{\infty} M_{s'(v_n); v_n}(a_{n,m} + b_{n,m}v_n)P_{qg}.
\]

Assume that \( s = \frac{a}{n} \), thus the sum in \((3)\) is a Riemann sum of the integral

\[
M_{s'(v_n); v_n}(a_{n,m} + b_{n,m}v_n)
\]

\[
= \int_0^\infty \int_0^\infty M_{s'(v_n); v_n}(a_{n,m} + b_{n,m}x)\,dA_n(x).
\]

Now, consider the condition \((A)\) with \((C_1)\), by using \((1)\), we get

\[
X(s'(v_n), v_n, m, k) < a_{n,m} = U_{s'(v_n); v_n}(a_{n,m} + b_{n,m}v_n),
\]

\[
= Y_{s'(v_n); v_n}(a_{n,m} + b_{n,m}v_n) - U_{s'(v_n); v_n}(a_{n,m} + b_{n,m}v_n).
\]

Moreover, \( U_{s'(v_n); v_n} \) and \( V_{s'(v_n); v_n} \) are independent. Therefore,

\[
P(U_{s'(v_n); v_n} < x) \to \mathcal{N}(\frac{\sqrt{2}}{\sqrt{\pi} \tau}, \frac{1}{2})
\]

\[
P(U_{s'(v_n); v_n} > 0) \to \mathcal{N}(0, 1)
\]

On the other hand, we can write \( P(X(s'(v_n), v_n, m, k) < A_{n,m} + B_{n,m}v_n) \), where \( A_{n,m} \) and \( B_{n,m} = (1 - r_n) - \frac{1}{2} a_{n,m} \). By using Theorem 2.1 of \([2]\), we get \( P(Y(s'(v_n), v_n, m, k) < a_{n,m} + b_{n,m}v_n) \to \mathcal{N}(\frac{\sqrt{2}}{\sqrt{\pi} \tau}, \frac{1}{2}) \).

The latter is evident from the assumption \( r_n \log n (m+1) \to 0 \) and thus \( r_n \log n (m+1) \to 0 \). Hence, only the first relation needs proof. Applying that \((1)\) and \((2)\), we get

\[
P_{ng} = P(V_n = g) \to \frac{1}{n^\alpha} \mathcal{N}(0, 1)
\]

and \((*)\) holds. Thus, in view of Khinchin’s type theorem, we get

\[
P(B_n \sim n - \frac{1}{2} \sqrt{2 \log \log n}) \to \frac{1}{n^\alpha} \mathcal{N}(0, 1)
\]

By combining \((5)\) and \((6)\), Lemma 2.1 in \([13]\) thus yields

\[
M_{s'(v_n); v_n}(a_{n,m} + b_{n,m}v_n) \to \mathcal{N}(0, 1)
\]

uniformly with respect to \( x \) over any finite interval of \( z \) (the continuity of the limit in \( x \), implies that the convergence is uniform). Now, let \( c \) be a continuity point of \( A(x) \) such that \( 1 - A(c) < \epsilon \).

\[
\int_c^\infty H(x; \tau, z)\,dA(z) \leq 1 - A(c) < \epsilon
\]

Moreover, for sufficiently large \( n \), in view of condition \((A)\), we get

\[
\int_c^\infty M_{s'(v_n); v_n}(a_{n,m} + b_{n,m}v_n)\,dA_n(nz) \leq 1 - A_n(nc) < 2\epsilon
\]

For estimating the difference \( M_{s'(v_n); v_n}(a_{n,m} + b_{n,m}v_n) - \mathcal{N}(0, 1) \), we first estimate \( \int_0^c M_{s'(v_n); v_n}(a_{n,m} + b_{n,m}v_n)\,dA_n(nz) - \int_0^c H(x; \tau, z)\,dA(z) \). By using the triangle inequality

\[
\int_0^c M_{s'(v_n); v_n}(a_{n,m} + b_{n,m}v_n)\,dA_n(nz) - \int_0^c H(x; \tau, z)\,dA(z)
\]

\[
\leq \int_0^c M_{s'(v_n); v_n}(a_{n,m} + b_{n,m}v_n)\,dA_n(nz)
\]

\[
= (1 - r_n) - \frac{1}{2} a_{n,m} \]
Since, the convergence in (7) is uniform over the finite interval \([0, c]\). Therefore, for any arbitrary \(\varepsilon > 0\) and for sufficiently large \(n\), we get
\[
\left| \int_0^c H^{(m,k)}(x; \tau, z) dA_n(nz) \right| < \varepsilon, \quad \text{and} \quad \left| \int_0^c H^{(m,k)}(x; \tau, z) dA(z) - \int_0^c H^{(m,k)}(x; \tau, z) dA(z) \right| < \varepsilon. \tag{11}
\]

The third difference in (10) can be estimated by constructing Riemann sums, which are close to the integral there. Namely, let \(n_0\) be a fixed number and let \(0 = c_0 < c_1 < \ldots < c_{n_0} = c\) be the continuity points of \(A(x)\). Furthermore, let \(n_0\) and \(c_i\) be such that
\[
\left| \int_0^c H^{(m,k)}(x; \tau, z) dA_n(nz) - \sum_{i=0}^{n_0} H^{(m,k)}(x; \tau, c_i) A_n(nc_i) - A_n(nc_{i-1}) \right| < \varepsilon, \quad \text{and} \quad \left| \int_0^c H^{(m,k)}(x; \tau, z) dA(z) - \sum_{i=0}^{n_0} H^{(m,k)}(x; \tau, c_i) (A(c_i) - A(c_{i-1})) \right| < \varepsilon. \tag{14}
\]

where \(L_{n,m} = \frac{a_{n,m} - a_{n,m}}{\varepsilon} \), \(b_{n,m} = \frac{1}{\sqrt{2 \log n}} \), \(n \to 0\), we get
\[
\lim_{n \to \infty} L_{n,m} = \lim_{n \to \infty} \frac{-1}{2 \sqrt{\log n}} \left[ \log \frac{\log (n + \log z) + \log 4 \pi}{\log n} - \log \frac{\log n}{\log n} + \log 4 \pi \right] \tag{15}
\]

This completes the proof of the first part of the theorem.

Turning to the condition (A) with (C2), starting with the relation (4), we notice that
\[
\frac{X(s'(n), nz, m, k) - a_{n,m}}{\sqrt{T_{(m,k)}}} = \sqrt{\frac{r_n}{T_{(m,k)}}} Y_0 + T_{(m,k)}, \tag{12}
\]

Therefore, \(|T_{(m,k)}| \leq \sqrt{\frac{1}{r_n}} |Y(s'(n), nz, m, k) - a_{n,m}| \), for large \(n\), since \(0 \leq r_n \leq 1\) and \(a_{n,m} > 0\), for large \(n\).

Since \(\frac{r_n}{r_{n,m}} \to 1\) and \(\frac{r_{n,m}}{r_{n,m}} = \sqrt{\frac{2 \log n}{m+1}} \log nz \to n\), the relation (14) implies
\[
P \left( |T_{(m,k)}| \geq \varepsilon \right) \to 0. \tag{15}
\]
Combining (12), (13), (14) and (15), Lemma 2.2.1 in [13] thus yields $M_{i_{(t)}}(a_{m,n} + b_{n,m}) = \frac{w'}{n^\gamma} \mathcal{N}(x)$. The remaining part of this case follows exactly as the proof of the case $r_n \log n \rightarrow \infty$. 

Turning now to prove the converse part that (B1) and (C1) imply (A). Starting with the proof of the first part of theorem (when $\tau = 0$) for the subsequence $n^*$, with the exception that the point $c$ is chosen such that $A^*(\infty) = A^*(c) \leq \epsilon$, we get $M_{i_{(t)}}(a_{m,n} + b_{n,m}) = \frac{w'}{n^\gamma} \mathcal{N}(x)$ since, the two limits $\mathcal{N}(x)$ and $\tilde{\Gamma}(x)$ are for limits $t$, then $\mathcal{N}(x) = 1 = \int_0^\infty dA^*(x) = \mathcal{N}(0)$, $A^*$ is a df. Now, if $A_n(nz)$ did not converge weakly, then we could select two subsequences $n_1$ and $n_2$ such that $A_n(nz)$ does not coincide on some interval contained in $D$, for all real values of $x$. Thus by the uniqueness theory of analytic functions, $A_n(t)$ and $A_2(t)$ coincide on some interval contained in $D$, which means that $A_1(z) = A_2(z)$. This completes the proof of the theorem. 

**Theorem 2.2.** Let $v_n$ be a sequence of integer valued rv's independent of $X_1, \ldots, X_n$ and $P(v_n < x) = A_n(x)$. Furthermore, 

(A): let $A_n(nz) \rightarrow_{n \rightarrow \infty} A(x)$, where $A(0_+) = 0$ and $A(x)$ is a non-degenerate df. Then

\[
(B_1): \int_0^\infty H^{(m)}(x; \tau, \nu) dA(z), \text{ if } (C_1): r_n \log n \rightarrow \infty \quad \tau \geq 0, \quad \text{where}
\]

\[
H^{(m)}(x; \tau) = \left\{ \begin{array}{ll}
\tilde{\Gamma}(z) & \tau > 0, \\
\tilde{\Gamma}(z)^{\tau} & \tau = 0.
\end{array} \right.
\]

Moreover, 

\[
(B_2): \int_0^\infty H^{(m)}(x; \tau, \nu) dA(z), \text{ if } (C_2): r_n \log n \rightarrow \infty \text{ and } r_n \text{ is slowly varying function of } n.
\]

Conversely, if (B1) and (C1) (with $\tau = 0$) hold, then the relation (A) will be satisfied. 

**Proof.** By representation (2) and by using Theorem 1.1 of [5], it is easy to see that the proof of Theorem 2.2 is similar to the proof of Theorem 2.1, with only the exception of obvious changes. 

**Theorem 2.3.** Let $v_n$ be a sequence of integer valued rv's independent of $X_1, \ldots, X_n$ and $P(v_n < x) = A_n(x)$. Furthermore, 

(A): let $A_n(nz) \rightarrow_{n \rightarrow \infty} A(x)$, where $A(0_+) = 0$ and $A(x)$ is a non-degenerate df. Then

\[
(B_1): \int_0^\infty H^{(m)}(x; \tau, \nu) dA(z), \text{ if } (C_1): r_n \log n \rightarrow \infty \quad \tau \geq 0, \quad \text{where}
\]

\[
H^{(m)}(x; \tau) = \left\{ \begin{array}{ll}
\tilde{\Gamma}(z) & \tau > 0, \\
\tilde{\Gamma}(z)^{\tau} & \tau = 0.
\end{array} \right.
\]

Moreover, 

\[
(B_2): \int_0^\infty H^{(m)}(x; \tau, \nu) dA(z), \text{ if } (C_2): r_n \log n \rightarrow \infty \text{ and } r_n \text{ is slowly varying function of } n.
\]

Conversely, if (B1) and (C1) (with $\tau = 0$) hold, then the relation (A) will be satisfied. 

**Proof.** By representation (2) and by using Theorem 1.1 of [5], it is easy to see that the proof of Theorem 2.2 is similar to the proof of Theorem 2.1, with only the exception of obvious changes. 

Although, the above theorems provide a set-up, which includes many interesting models such as oos, sos and pos, with censoring scheme $(R_1, \ldots, R_M) \in \mathcal{N}^M$, a number of models of gos are excluded in this set-up, e.g., pos with general censoring scheme $(R_1, \ldots, R_M)$. The following two

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theorems extend Theorems 2.3 and 2.4 to a very wide subclass of gos in which the vector $\hat{m} = (m_1, m_2, \ldots, m_{n-1})$ is arbitrarily chosen such that $m_i > -1, i = 1, 2, \ldots, n-1$, and the parameters $\gamma_1, \gamma_2, \ldots, \gamma_n$ are pairwise different, i.e., $\gamma_n \neq \gamma_i, i \neq 1$, for all $i, j \in \{1, \ldots, n\}$. For instance, this assumption is no restriction on pos with general censoring scheme $(R_1, \ldots, R_M)$.

**Theorem 2.5.** Let $a_{\gamma_n} = \frac{1}{b_{\gamma_n}} - \frac{1}{b_{\gamma_n}} \log \log \gamma_n + \log 4\pi$, $b_{\gamma_n} = (2 \log \gamma_n)^{\frac{1}{2}}$ and $\gamma_n \to \infty$. Furthermore, let $\nu_n$ be a sequence of integer valued rv's independent of $X_1, \ldots, X_n$ and $P(\nu_n < x) = A_n(x)$. Furthermore, let $A_n(nx) \to A(x)$, where $A(+0) = 0$ and $A(x)$ is a non-degenerate df. Then

(B1): $P \left( \frac{X(s_n, n, m, k)}{\nu_n} < x \right) \to w_n \mathcal{N}(x)$, if $n \to \infty$ and $n^r \log \gamma_n \to \tau \geq 0$, where

$$H^{(\tilde{m}, k)}(x; \tau, z) = \left\{ \begin{array}{ll}
\mathcal{G}_\tau(z e^{-x}) & , \tau > 0 \\
\mathcal{G}_\tau(z e^x) & , \tau = 0
\end{array} \right.$$ 

Moreover, (B2): $P \left( \frac{X(s_n, n, m, k)}{\nu_n} < x \right) \to w_n \mathcal{N}(x)$, if

Conversely, if (B1) and (C1) (with $\tau = 0$) hold, then the relation (A) will be satisfied.

**Proof.** By representation (1) and by using Theorem 2.1 of [4], it is easy to see that the proof of Theorem 2.5 is similar to the proof of Theorem 2.3, with only the exception of obvious changes. □

**Theorem 2.6.** Let $\nu_n$ be a sequence of integer valued rv's independent of $X_1, \ldots, X_n$ and $P(\nu_n < x) = A_n(x)$. Furthermore, let $A_n(nx) \to A(x)$, where $A(+0) = 0$ and $A(x)$ is a non-degenerate df. Then

(B1): $P \left( \frac{X(s_n, n, m, k)}{\nu_n} < x \right) \to w_n \mathcal{N}(x)$, if $n \to \infty$ and $n^r \log \gamma_n \to \tau \geq 0$, where

$$H^{(\tilde{m}, k)}(x; \tau, z) = \left\{ \begin{array}{ll}
\mathcal{G}_\tau(z e^{-x}) & , \tau > 0 \\
\mathcal{G}_\tau(z e^x) & , \tau = 0
\end{array} \right.$$ 

Moreover, (B2): $P \left( \frac{X(s_n, n, m, k)}{\nu_n} < x \right) \to w_n \mathcal{N}(x)$, if

Conversely, if (B1) and (C1) (with $\tau = 0$) hold, then the relation (A) will be satisfied.

**Proof.** Proceeding exactly as the proof of (3), we get

$$M_{\nu_n, \nu_n}^{(\tilde{m}, k)}(x_0 + c_n x) = \mathbb{P}(X(s_n, \nu_n, n, m, k) < x_0 + c_n x)$$

$$= \int_0^\infty M_{\nu_n, \nu_n}^{(\tilde{m}, k)}(x; \tau, z) dA_n(nz),$$

(17)

where $M_{\nu_n, \nu_n}^{(\tilde{m}, k)}(x; \tau, z) = \mathbb{P}(X(s_n, \nu_n, n, m, k) < x_0 + c_n x)$. First, consider the condition (A) with (C1), by using (1), we get

$$X(s_n, \nu_n, n, m, k) - x_0 = \sqrt{\frac{\nu_n}{c_n}} Y_0$$

$$+ \sqrt{1 - \rho_n} F(s_n, \nu_n, n, m, k) - x_0 = U_n^{(m, k)} + \gamma_n^{(m, k)},$$

(18)
\[ U_{nc}^{(m,k)} = \frac{\max(-\sigma_n(z_{(n,m,k)}^{*}))}{\sqrt{\lambda(1-\lambda)}} Y_0 \] and \[ V_{nc}^{(m,k)} = \frac{\sqrt{1-\tau_n} \left( z_{(n,m,k)}^{*} - x_0 \right)}{c_n} \]

Moreover, \( U_{nc}^{(m,k)} \) and \( V_{nc}^{(m,k)} \) are independent. If \( nr_n \rightarrow \tau, \ 0 \leq \tau < \infty \), then

\[
P(U_{nc}^{(m,k)} < x) = \frac{w}{n} \mathcal{N} \left( \sqrt{\frac{\lambda(1-\lambda)}{\tau \Phi(x_0)}} \frac{c_{(m,k)}}{c_{\lambda}} (m+1)x, \right)
\]

if \( \tau > 0 \),

\[
U_{nc}^{(m,k)} \overset{P}{\rightarrow} 0, \text{ if } \tau = 0.
\]

On the other hand, we have

\[ P(V_{nc}^{(m,k)} < x) = P(Y(\sigma_n(z_{n,m,k}) < A_{nc,m} = B_{nc,m}) \), \( \overset{(20)}{\text{where}} \ A_{nc,m} = \frac{x_0}{\sqrt{r_n}}, \text{ and } B_{nc,m} = \frac{c_n}{\sqrt{1-r_n}} \]. Now, if \( nr_n \rightarrow \tau \geq 0 \), we get

\[ A_{nc,m} - x_0 = \frac{(1 + \frac{w}{n} (1 + o(1)))x_0 - x_0}{\lambda(1-\lambda)} \sqrt{r_n} \Phi(x_0)
\]

\[ \sim \frac{\sqrt{r_n} c_{(m,k)} \Phi(x_0)}{2 \lambda(1-\lambda)} \overset{w}{\rightarrow} 0.
\]

Moreover, \[ B_{nc,m} = \frac{(1 + \frac{w}{n} (1 + o(1)))}{\sqrt{r_n}} \overset{w}{\rightarrow} \sqrt{\tau}. \] Therefore, an application of Khinchin’s type theorem yields

\[ P(V_{nc}^{(m,k)} \leq x) \overset{w}{\rightarrow} \mathcal{N} \left( \sqrt{\frac{\lambda(1-\lambda)}{\tau \Phi(x_0)}} \frac{c_{(m,k)}}{c_{\lambda}} (m+1)x, \right)
\]

By combining (18),(19), (20) and (21), we get

\[ M_{(m,k)}^{(n,m,k)} = \mathcal{N} \left( \sqrt{\frac{\lambda(1-\lambda)}{\tau \Phi(x_0)}} \frac{c_{(m,k)}}{c_{\lambda}} (m+1)x, \right), \tau \geq 0, \]

uniformly with respect to \( x \) over any finite interval of \( z \) (the continuity of the limit in \( x \), implies that the convergence is uniform). The remaining part of the proof of the theorem, under the condition \( \infty > \tau \geq 0 \), follows now by using the relations (17) and (22) exactly as the proof of Theorem 2.1, under the same condition (i.e., \( r_n \log n \rightarrow \tau \geq 0 \)).

Turning now to the proof of the condition (A) with \( (C^*_2) \), by using (1), we get

\[ \delta_{(m,k)}^{(n,m,k)} = \sqrt{\frac{\lambda(1-\lambda)}{\tau \Phi(x_0)}} \frac{c_{(m,k)}}{c_{\lambda}} \]

\[ \leq \frac{\sqrt{\frac{\lambda(1-\lambda)}{\tau \Phi(x_0)}} \frac{c_{(m,k)}}{c_{\lambda}}} {\sqrt{n}} \left| Y(S_{nc,m,k} - x_0 \right| \leq \frac{\sqrt{\frac{\lambda(1-\lambda)}{\tau \Phi(x_0)}} \frac{c_{(m,k)}}{c_{\lambda}}} {\sqrt{n}} \left| Y(S_{nc,m,k}^{*}) - x_0 \right|. \]

Thus,

\[ P(\delta_{(m,k)}^{(n,m,k)} \geq \varepsilon) \leq P\left( \frac{\sqrt{\frac{\lambda(1-\lambda)}{\tau \Phi(x_0)}} \frac{c_{(m,k)}}{c_{\lambda}}} {\sqrt{\frac{\lambda(1-\lambda)}{\tau \Phi(x_0)}} \frac{c_{(m,k)}}{c_{\lambda}}} (m+1)x, \right) \geq \varepsilon
\]

\[ = P\left( \frac{\left| Y(S_{nc,m,k} - x_0 \right|}{c_{nc}} \geq \frac{\sqrt{\frac{\lambda(1-\lambda)}{\tau \Phi(x_0)}} \frac{c_{(m,k)}}{c_{\lambda}}} {\sqrt{\frac{\lambda(1-\lambda)}{\tau \Phi(x_0)}} \frac{c_{(m,k)}}{c_{\lambda}}} \varepsilon \right) \overset{w}{\rightarrow} 0.
\]

Lemma 2.2.1 in [13] thus yields \( M_{(m,k)}^{(n,m,k)} \overset{w}{\rightarrow} \mathcal{N} \left( \frac{C_{\lambda}^{(m,k)}}{C_{\lambda}^{(m)}} (m+1)x \right). \) The derivation of the limit df’s of central \( m \)-dgos of sGs (2) is proceeded exactly as the same as those of central \( m \)-gos of the sequence (1). The remaining part of the proof of the theorem, under the condition \( \tau = \infty \), follows now by using the relations (17) and the last relation (i.e., \( M_{(m,k)}^{(n,m,k)} ; c_n x \) \overset{w}{\rightarrow} \mathcal{N} \left( \frac{C_{\lambda}^{(m,k)}}{C_{\lambda}^{(m)}} (m+1)x \right), uniformly with respect to \( x \) exactly as the proof of Theorem 2.1, under the condition \( r_n \log n \rightarrow \infty \).

Turning now to prove the converse part that \( (B^*_1) \) and \( (C^*_2) \) imply (A). Starting with the relation (17), by the compactness of df’s, we can select a subsequence \( n^* \), such that \( A_{n^*}(n^*) \overset{w}{\rightarrow} A^*(z) \), where \( A^*(z) \) is an extended df (i.e., \( A^*(\infty) - A^*(0) \leq 1 \)). Therefore, by repeating the first part of theorem for the subsequence \( n^* \), with the exception that the point \( c \) is chosen such that \( A^*(\infty) - A^*(c) \leq \varepsilon \), we get \( M_{n^*,m}^{(m,k)} ; c_n x \) \overset{w}{\rightarrow} \mathcal{N}(x) = \int_0^\infty L(x; \tau ; d) A^*(z) \). Since, the two limits \( \Phi(x) \) and \( L(x; \tau ; d) \) are df’s, then \( \Phi(\infty) = 1 = \int_0^\infty d A^*(z) = A^*(\infty) - A^*(0) \). Thus, \( A^* \) is a df. Now, if \( A_0(\infty) \) did not converge weakly, then we could select two subsequences \( n_1 \) and \( n_2 \) such that \( A_{n_1}(z_{1}) \overset{w}{\rightarrow} A_1(z), \forall i = 1, 2 \). In this case, we have

\[ \Phi(x) = \int_0^\infty \mathcal{N}(\sigma \sqrt{x}) d A_1(z), \]

\[ = \int_0^\infty \mathcal{N}(\sigma \sqrt{x}) d A_2(z), \]

\[ \sigma = \sqrt{\frac{\lambda(1-\lambda)}{\tau \Phi(x_0)}} \frac{c_{(m,k)}}{c_{\lambda}} (m+1). \]

Let \( G_1(t) = \int_0^t \mathcal{N}(\tau \sqrt{x}) d A_1(z), i = 1, 2 \). If the functions \( G_1(t) \) and \( G_2(t) \) are determined in an interval \( t_1 < t < t_2 \), then in this interval both of them will be analytic. By differentiating \( G_1(t) \) and \( G_2(t) \) with respect to \( t \), in view of (23), we get \( \int_0^t e^{-\sigma x^2} \sqrt{x} d A_1(z) = \int_0^t e^{-\sigma x^2} \sqrt{x} d A_2(z) \). Put \( \sigma = \frac{\sigma_0}{\sqrt{\tau}} \), we get \( \int_0^t e^{-\sigma x^2} \sqrt{x} d A_1(z) = \int_0^t e^{-\sigma x^2} \sqrt{x} d A_2(z) \). Since, the Laplace transformations with respect to the measures \( \sqrt{x} A_1(z) \) and \( \sqrt{x} A_2(z) \) coincide, we deduce that \( A_1(z) = A_2(z) \). This completes the proof of the theorem.

The derivation of the limit df’s of central \( m \)-gos of Gaussian sequence (2) is proceeded exactly as the same as those of central \( m \)-gos of the sequence (1).

4 Intermediate \( m \)-gos (dgos) with random indices in a sGs

In this section we consider a general nondecreasing intermediate rank sequence \( s_n = o(n) \rightarrow 0 \), for which \( \frac{\log \lambda_n}{\log n} \rightarrow \beta, \ 0 \leq \beta \leq 1 \). Actually, the latter condition is very wide, e.g., it is easily to verify that this condition is satisfied, with \( \beta = 0 \), when \( s_n \sim \frac{\lambda_n}{2 \alpha n}, \ 0 < \alpha < 1 \) (the Chibisov rank sequence, see [10]). Also, this condition will be satisfied, with \( \beta = 0 \), when \( s_n = \log n \) (i.e., when

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the rank sequence is slowly varying function of \( n \). Finally, it will be satisfied, with \( \beta = 1 \), when \( s_n = \frac{\log n}{n} \) (i.e., when the rank sequence is rapidly varying function of \( n \)). The following results are indispensable for the study of the intermediate m-gos \( X(s^*(n), n, m, k) \), and m-dgos \( X_d(s_n, n, m, k) \), where \( s^*(n) = n - s_n + 1 \).

**Lemma 4.1.** Let \( Y_1, Y_2, \ldots, Y_n \), be iid rv’s, with common df \( \mathcal{N}(x) \). Furthermore, let \( Y(1, n, m, k) \leq Y(2, n, m, k) \leq \ldots \leq Y(n, n, m, k) \) be the corresponding m-gos. Then, \( P(Y(s^*(n), n, m, k) - x_{obs} < x) \rightarrow \mathcal{N}((m + 1)x) \), where \( d_n = \frac{1}{\sqrt{n}} \xi_n \), \( \mathcal{N}(x_n) = \frac{\log n}{n} \), \( x_n \sim \sqrt{2 \log \frac{m}{n}} \), as \( n \rightarrow \infty \), and \( \omega_n = \frac{s_n}{n} \) (remember that \( N = \frac{k}{m+1} + n - 1 \sim n \), thus \( \omega_n = \frac{o(n)}{n} \rightarrow 0 \)).

**Lemma 4.2.** Let \( Y_d(1, n, m, k) \geq Y_d(2, n, m, k) \geq \ldots \geq Y_d(n, n, m, k) \) be the corresponding m-gos to the rv’s \( Y_1, Y_2, \ldots, Y_n \). Then, \( P(Y_d(s_n, n, m, k) - x_{obs} \ll x) \rightarrow \mathcal{N}(x) \), where \( d_n = \frac{1}{\sqrt{n}} \xi_n \), \( \mathcal{N}(x_n) = \frac{\log n}{n} \), \( x_n \sim \sqrt{2 \log \frac{m}{n}} \), as \( n \rightarrow \infty \), and \( \sigma_n = (m + 1)N \).

**Theorem 4.1.** Let the condition (A) in the Theorems 2.1-2.6 satisfied, then

\[
(B^*): \quad P\left( \frac{X(s^*(n), n, m, k) - x_{obs}}{d_{obs}} < x \right) \xrightarrow{w} \mathcal{N}(\sqrt{\frac{(m+1)x}{2\tau(x)}}) \quad \text{as} \quad n \rightarrow \infty \quad \text{if} \quad 0 \leq \tau, \eta < \infty,
\]

\[
(C^*): \quad r_n\eta_n \log n \sim n \tau (1 + \log \frac{n}{m}) \quad \text{and} \quad \frac{\log r_n}{\log n} \sim \eta, n \rightarrow \infty.
\]

Moreover, \( B^* \): \( P\left( \frac{X(s^*(n), n, m, k) - x_{obs} \sqrt{1 - r_n}}{d_{obs}} < x \right) \xrightarrow{w} \mathcal{N}(x) \), if \( n \rightarrow \infty \).

**Remark 4.1.** Actually, the condition \( \frac{x_{obs}}{d_{obs}} \rightarrow \mathcal{N}(x) \) is not restrictive for being the sequence \( \frac{x_{obs}}{d_{obs}} \) is bounded, as \( n \rightarrow \infty \). This fact can be easily seen, since \( s_n \) is nondecreasing and \( \omega_n = \frac{o(n)}{n} \leq n \), for large \( n \). Moreover, it is easily to see that \( v = 0 \), for the two cases \( s_n \sim \xi_n^{\frac{1}{n}} \) and \( s_n \sim \frac{n}{\log n} \). Moreover, \( v = \frac{1}{m+1} \), if \( s_n \sim \log n \).

**Remark 4.2.** It is easily to see that \( \eta = \frac{1}{m+1} \), for \( s_n \sim \xi_n^{\frac{1}{n}} \) and \( \eta = \frac{1}{m+1} \), for \( s_n \sim \frac{n}{\log n} \). On the other hand, \( \eta = 0 \), when \( s_n \sim \log n \).

**Proof.** Proceeding exactly as the proof of (4), we get

\[
X(s^*(n), n, m, k) - x_{obs} = \sqrt{\frac{r_n}{d_{obs}}} Y_0 + \sqrt{\frac{r_n}{d_{obs}}} x_{obs}.
\]

Moreover, \( U_{nc}^{(m,k)} \) and \( V_{nc}^{(m,k)} \) are independent. Then, if \( s_n \log n \rightarrow \tau \), \( 0 \leq \tau < \infty \), and \( \frac{\log s_n}{n} \rightarrow \beta, 0 \leq \beta \leq 1 \), it is easily to check that \( s_n \log \omega_n \rightarrow \mathcal{N}(m\beta + 1) \). Consequently, we get

\[
x_{obs} \sqrt{X_{obs} R_n} \sim \sqrt{2s_n \log \omega_n} \sqrt{\frac{r_n}{s_n}} x_{obs} \sqrt{1 - \frac{\log s_n}{\log \omega_n}} \mathcal{N}(m\beta + 1). \tag{25}
\]

Thus,

\[
P\left( U_{nc}^{(m,k)} < x \right) \xrightarrow{w} \mathcal{N} \left( \sqrt{\frac{m+1}{2\tau(m+1)(1-\beta)}} x \right), \quad \text{if} \quad \tau > 0, \beta \neq 1, v > 0,
\]

\[
P\left( U_{nc}^{(m,k)} \xrightarrow{p} 0, \text{if} \quad \tau = 0, \text{or} \beta = 1, \text{or} v = 0 \right). \tag{26}
\]

On the other hand, we have

\[
P\left( V_{nc}^{(m,k)} < x \right) = P\left( Y(s^*(n), n, m, k) - A_{obs} + B_{obs} < x \right), \tag{27}
\]

where \( A_{obs} = \frac{x_{obs}}{d_{obs}} \) and \( B_{obs} = \frac{d_{obs}}{d_{obs}} \). It is clear that

\[
B_{obs} = \frac{(1-r_n)}{d_{obs}} = \frac{1}{r_n} \left( 1 + \frac{1}{\log n} \right) \frac{x_{obs}}{d_{obs}} \sqrt{\frac{r_n}{d_{obs}}} \rightarrow n \mathcal{N}(x_{obs} \sqrt{r_n} / d_{obs}) \rightarrow 0, \quad \text{since} \quad \sqrt{r_n} x_{obs} \rightarrow 0 \text{ and } \frac{x_{obs}}{d_{obs}} \leq 1, \text{for large } n. \quad \text{Therefore, in view of Lemma 4.1} \quad \text{and by applying Khinchin’s type theorem, we get}
\]

\[
P\left( V_{nc}^{(m,k)} < x \right) \xrightarrow{w} \mathcal{N} \left( \sqrt{\frac{m+1}{2\tau(m+1)}} x \right). \tag{28}
\]

Combining (25),(26), (27) and (28), we get \( B^* \).

Turning now to the proof of the condition (A) with \( \mathcal{N}(x) \), by using (1), we get

\[
X(s^*(n), n, m, k) - x_{obs} \mathcal{N} \left( \frac{r_n}{d_{obs}} \right) \rightarrow \mathcal{N}(Y_0 + Y_{nc}^{(m,k)}) \quad \text{where} \quad \left| Y_{nc}^{(m,k)} \right| \leq \sqrt{\frac{1}{r_n}} Y(s^*(n), n, m, k) - x_{obs} \quad \text{and} \quad \sqrt{t_{nc} x_{obs}} \sim \frac{t_{nc}}{r_n} \rightarrow 1. \quad \text{Thus,}
\]

\[
P\left( T_{nc}^{(m,k)} \geq \epsilon \right) \leq P\left( \frac{\left| Y(s^*(n), n, m, k) - x_{obs} \right|}{\sqrt{r_n}} \right) \geq \epsilon
\]

\[
= P\left( \frac{\left| Y(s^*(n), n, m, k) - x_{obs} \right|}{d_{obs}} \right) \geq \epsilon
\]

\[
\geq \sqrt{s_{nc} x_{obs}} \frac{t_{nc}}{s_{nc}} \sqrt{\frac{r_n}{f_{nc}}} \mathcal{N}(\epsilon) \rightarrow 0.
\]

Therefore, Lemma 2.2.1 in [13] implies \( B^* \). The remaining part of the proof of the theorem, under the condition \( \tau = \infty \), follows now by using the relations (24) and the relation \( M_{s_{nc}^{(m,k)}}(x_{obs} + d_{obs} x) \xrightarrow{w} \mathcal{N}(x) \) exactly as...
The proof of Theorem 2.1, under the condition $s_n r_n \log n \to \infty$. This completes the proof of the theorem. □

**Theorem 4.2.** Let the condition (A) in Theorems 2.1-2.6 satisfied. Then,

$$(B^*_1): \quad P \left( \frac{X_n(y_n, y_{n}, m, k)}{s_n} - x_{\alpha m} s_n \right) < x \quad \frac{w}{n^\alpha} \quad \mathcal{N} \left( \frac{\xi}{\sqrt{1 + 2 \mu \tau^2 (1 - \beta)}} x, 0 \leq \mu, \xi < \infty, \text{if} \right)$$

$$(C^*_1): \quad r_n s_n \log n \to \tau \geq 0, \quad \frac{\log n}{s_n^2} \to \mu \quad \text{and} \quad \frac{1}{s_n^2} \to \xi^2 \text{. Moreover,}$$

$$(B^*_2): \quad P \left( \frac{X_n(y_n, y_{n}, m, k)}{s_n} - x_{\alpha m} s_n \sqrt{1 - r_n} \right) < x \quad \frac{w}{n^\alpha} \quad \mathcal{N} (x), \text{if}$$

$$(C^*_2): \quad r_n s_n \log n \to \infty \text{and} \quad r_n \text{ is slowly varying function of } n \text{.}$$

**Proof.** The proof of theorem follows along the same line of the proof of Theorem 4.1, by combining the result of [9] with the result of Lemma 3.1 of [6]. Note that in this case $\mu = (m + 1)^{\alpha}$ and $\xi = \alpha$ when $s_n \sim \ell^2 n^\alpha$, while $\mu = m + 1$ and $\xi = 1$ when $s_n \sim \frac{n}{\log n}$. Moreover, $\mu = 1$ and $\xi = 0$ when $s_n \sim \log n$. □

5 Conclusion

Often the realizations of experiments in reliability analysis and lifetime studies are stationary and arise in nondecreasing (or nonincreasing) order, where in such experiments it is almost impossible to have a fixed sample size, because some observations always get lost for various reasons. Hence, the need arises to establish the limiting distributions of several models of ascendingly (or descendingly) ordered stationary random variables with random number. In this paper we obtained the limit distributions of extreme, intermediate and central generalized order statistics, as well as dual generalized order statistics, of a stationary Gaussian sequence of random variables under equi-correlated set up, when the sample size is itself a random variable.

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References


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