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Block Convergence of Series in Topological Vector Spaces

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Abstract: A block convergence of a series is the convergence of a subsequence of the sequence of partial sums of the series. Generalized bases are studied in this article, based on this type of convergence.

Keywords: Schauder basis, Block basic sequence, Topological Vector Space

1 Introduction

Basic sequences and block basic sequences are known concepts in the theory of bases (see: [5]). These concepts are studied in this article with a very slight variation. The series $\sum_{n=1}^{\infty}(-1)^n$ does not converge in the real line. The series $\sum_{n=1}^{\infty}\left((-1)^{2n+1}+(-1)^{2n}\right)$ converges to zero. The series $-1+\sum_{n=1}^{\infty}\left((-1)^{2n}+(-1)^{2n+1}\right)$ converges to -1. So, importance on subsequential limits of partial sums are not taken into account, when it is stated that $\sum_{n=1}^{\infty}(-1)^n$ does not converge. This article gives importance to these subsequential limits and few fundamental results have been derived.

Block convergence is studied in the next section so that a concept of block generalized basis can be introduced and studied. The terminology "block basis" was used in the article [2]. I. Singer [4] suggested that it may be called " block basic sequence". This concept is different from the concept of block generalized basis to be studied. The book [5] of I. Singer may be used for an introduction to block basic sequences.

All topologies in this article are Hausdorff topologies. All vector spaces are over the real field. 'Topological vector space' is abbreviated as 'TVS', and definitions may be found in [3].

2 Block series

A strictly increasing sequence of natural numbers $n_1<n_2<n_3<\ldots$ is fixed, and concepts are introduced with respect to this fixed sequence in this article. The particular case $n_i=i$, $\forall\, i$, leads to classical concepts.

Definition 2.1. A sequence $(x_n)_{n=1}^{\infty}$ or (x_n) in a topological space X is said to block-converge to x in X, if the subsequence $(x_{n_i})_{i=1}^{\infty}$ converges to x in X. A sequence (x_n) in a metric space (X,d) is said to be block-Cauchy , if $d\left(x_{n_i},x_{n_j}\right)\to 0$, as $i,j\to+\infty$ By following [1], let us call $d:X\times X\to[0,\infty)$ as a semimetric, if $d(x,x)=0, d(x,y)=d(y,x)$, and $d(x,z)\le d(x,y)+d(y,z); \forall x,y,z\in X$. A sequence (x_n) in a uniform space $\left(X,(d_k)_{k\in I}\right)$, in which the uniformity is induced by a family of semimetrics $(d_k)_{k\in I}$, is said to be block-Cauchy, if $d_k\left(x_{n_i},x_{n_j}\right)\to 0$ as $i,j\to\infty$; for every fixed $k\in I$. Similarly, a sequence (x_n) in a TVS (X,τ) is said to be block-Cauchy, if $(x_{n_i})_{i=1}^{\infty}$ is Cauchy.

Note that block-convergence of a sequence implies block-Cauchyness in a uniform space.

Definition 2.2. A series $\sum_{n=1}^{\infty}x_n$ or $\sum x_n$ in a TVS X is said to block-converge if $\left(\sum_{n=1}^{n_i}x_i\right)_{i=1}^{\infty}$ converges to some x in X. In this case, let us write $x=b-\sum_{n=1}^{\infty}x_n$ or $x=b-\sum x_n$. Similarly, $b-\sum_{n=1}^{\infty}x_n$ or $b-\sum x_n$ is used for block

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convergence. Other variations will have unambiguous meanings .

Remark 2.3. Block convergence of a series $\sum x_n$ in a TVS implies the following: with $j > i$, $\sum_{k=n_i+1}^{n_j} x_k \to 0$ as $i, j \to \infty$, and $\sum_{k=n_i+1}^{n_{i+1}} x_k \to 0$ as $i \to \infty$.

Theorem 2.4. Let (x_n) be a sequence in a TVS (X, τ_X) such that $\{x_{n_m+1}, x_{n_m+2}, \ldots, x_{n_{m+1}}\}$ is linearly independent, $\forall\ m = 0, 1, 2, \ldots$, with $n_0 = 0$. Under natural coordinatewise addition and scalar multiplication, let A be a linear space of sequences of scalars, which is defined by

$$A = \left\{ (\alpha_n) : \left(\sum_{k=1}^{n_i} \alpha_k x_k \right)_{i=1}^{\infty} \textit{ is Cauchy in } X \right\}.$$

To each $U \in \tau_X$ such that $0 \in U$, let

$$A_U = \left\{ (\alpha_n) \in A : \sum_{k=1}^{n_i} \alpha_k x_k \in U,\ \forall\ i = 1, 2, \ldots \right\}.$$

Then there is a vector topology τ_A on A that makes (A, τ_A) into a TVS with a local base $\{A_U : 0 \in U \in \tau_X\}$ at 0. If (X, τ_X) is metrizable, then (A, τ_A) is metrizable. If (X, τ_X) is locally convex, then (A, τ_A) is also locally convex. If (X, τ_X) is locally bounded, then (A, τ_A) is also locally bounded. If (X, τ_X) is normable, then (A, τ_A) is also normable. Above all, (A, τ_A) is complete.

*Proof.*If U is an open neighbourhood of 0 in X, and V is an open neighbourhood of 0 in X such that $V + V \subseteq U$, then, by definition, $A_V + A_V \subseteq A_U$.

Let us fix $(\alpha_n) \in A$ and a scalar α. Again, for a given open neighbourhood U of 0 in X, find an open balanced neighbourhood V of 0 such that $V + |\alpha| V + V \subseteq U$. Since $\left(\sum_{k=1}^{n_i} \alpha_k x_k \right)_{i=1}^{\infty}$ is Cauchy in X, there is a $\delta \in (0, 1)$ such that $\beta \sum_{k=1}^{n_i} \alpha_k x_k \in V, \forall\, i = 1, 2, \ldots$, and whenever $|\beta| < \delta$. Thus, if $(\beta_n) \in A_V$ and $|\beta| < \delta$, then

$$\begin{aligned}(\alpha + \beta) \left(\sum_{k=1}^{n_i} \alpha_k x_k + \sum_{k=1}^{n_i} \beta_k x_k \right) &\in \sum_{k=1}^{n_i} \alpha \alpha_k x_k + V + |\alpha| V + V \\ &\subseteq \sum_{k=1}^{n_i} \alpha \alpha_k x_k + U, \forall\, i = 1, 2, \ldots.\end{aligned}$$

So, $(\alpha + \beta)((\alpha_n) + (\beta_n)) \in (\alpha \alpha_n) + A_U$, whenever $|\beta| < \delta$ and $(\beta_n) \in A_V$. So, addition and scalar multiplication are continuous in (A, τ_A), when τ_A has a basis $\{(\alpha_n) + A_U : 0 \in U \in \tau_X, (\alpha_n) \in A\}$.

To verify Hausdorffness of (A, τ_A), let us fix (α_n) in A such that $(\alpha_n) \in A_U$, whenever $0 \in U \in \tau_X$. So, $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{n_1} x_{n_1} = 0$. Since $x_1, x_2, x_3, \ldots, x_{n_1}$ are independent, then $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_{n_1} = 0$. So, $\alpha_{n_1+1} x_{n_1+1} + \cdots + \alpha_{n_2} x_{n_2} = 0$. Since $x_{n_1+1}, x_{n_1+2}, \ldots, x_{n_2}$ are linearly independent, then $\alpha_{n_1+1} = 0, \alpha_{n_1+2} = 0, \ldots, \alpha_{n_2} = 0$. Proceeding in this way, it can be concluded that $\alpha_i = 0$, $\forall\ i = 1, 2, \ldots$, so that (α_n) is the zero element of A. This proves that (A, τ_A) is a Hausdorff space and hence it is a TVS.

Suppose (X, τ_X) is locally bounded, and U be a balanced bounded neighbourhood of 0 in (X, τ_X). Let V be another balanced neighbourhood of 0 in X. Then there is a $k > 0$ such that $tU \subseteq V$, whenever $|t| \le k$. Then $tA_U \subseteq A_V$ whenever $|t| \le k$. Thus A_U is a bounded neighbourhood of the zero element in A, and hence (A, τ_A) is also locally bounded.

If (X, τ_X) is metrizable, and it has a countable local base $\{V_n : n = 1, 2, \ldots\}$ of open sets at 0 in X, then $\{A_{V_n} : n = 1, 2, \ldots\}$ is a countable local base in (A, τ_A) and hence it is metrizable.

If U is a convex open neighbourhood of 0 in X, then A_U is a convex neighbourhood of zero in A. Thus (A, τ_A) is locally convex, when (X, τ_X) is locally convex. If (X, τ_X) is normable, then (A, τ_A) is also normable, because a TVS is normable if and only if it is locally bounded and locally convex.

To prove the final part, let us consider a Cauchy sequence $\left(\left(\alpha_n^{(i)} \right)_{n=1}^{\infty} \right)_{i=1}^{\infty}$ in (A, τ_A), for simplicity in notation. Let U be a given open neighbourhood of 0 in X. Let us find a balanced open neighbourhood V of 0 such that $\overline{V} + \overline{V} \subseteq U$, where $\overline{V}$ is the closure of V in X. Then there is an integer p such that $\left(\alpha_n^{(i)} - \alpha_n^{(j)} \right) \in A_V$, $\forall\, i, j \ge p$. Then

$$\sum_{k=1}^{n_m} \left(\alpha_k^{(i)} - \alpha_k^{(j)} \right) x_k \in V, \tag{2.1}$$

$\forall\ m = 1, 2, \ldots$, and $\forall\ i, j \ge p$. So, with $n_0 = 0$,

$$\sum_{k=n_m+1}^{n_{m+1}} \left(\alpha_k^{(i)} - \alpha_k^{(j)} \right) x_k \in V + V \subseteq U,\ \ \forall\, i, j \ge p$$

and $\forall\ m = 0, 1, 2, \ldots$. So, to each fixed m, the sequence

$$\left(\sum_{k=n_m+1}^{n_{m+1}} \alpha_k^{(i)} x_k \right)_{i=1}^{\infty}$$

is a Cauchy sequence in X. Now, the linear independence of $\{x_{n_m+1}, \ldots, x_{n_{m+1}}\}$, and continuity of linear functionals on finite dimensional TVSs imply that $\left(\alpha_k^{(i)} \right)_{i=1}^{\infty}$ is Cauchy, $\forall\, k = 1, 2, \ldots$.

Let $\alpha_k^{(i)} \to \alpha_k$ as $i \to \infty$, for every fixed k. If the earlier relation (2.1) is used again, it can be concluded that

$$\sum_{k=1}^{n_m} \left(\alpha_k^{(i)} - \alpha_k \right) x_k \in \overline{V} \subseteq U,$$

$\forall\, i \ge p$, and $\forall\, m = 1, 2, \ldots$, and hence it can be concluded that $\left(\left(\alpha_k^{(i)} \right)_{k=1}^{\infty} \right)_{i=1}^{\infty}$ converges to $(\alpha_k)_{k=1}^{\infty}$ in (A, τ_A), provided $(\alpha_k)_{k=1}^{\infty} \in A$. Let us next show that $(\alpha_k)_{k=1}^{\infty} \in A$.

For fixed U,V and p as above, find q such that $\sum_{k=n_m+1}^{n_r} \alpha_k^{(p)} x_k \in V$ whenever $r > m \geq q$. Then

$$\sum_{k=n_m+1}^{n_r} \alpha_k x_k = \sum_{k=n_m+1}^{n_r} \left(\alpha_k - \alpha_k^{(p)}\right) x_k + \sum_{k=n_m+1}^{n_r} \alpha_k^{(p)} x_k \in \overline{V} + \overline{V} \subseteq U,$$

whenever $r > m \geq q$. So, $\left(\sum_{k=1}^{n_i} \alpha_k x_k\right)_{i=1}^{\infty}$ is a Cauchy sequence in (X,τ_X). This proves that $(\alpha_k)_{k=1}^{\infty} \in A$. So, (A,τ_A) is sequentially complete. In the previous arguments, one may consider a Cauchy net in (A,τ_A), instead of a Cauchy sequence, and verify that it converges in (A,τ_A). Thus (A,τ_A) is actually a complete TVS. This proves the theorem.

Remark 2.5. If (X,τ_X) in the previous theorem 2.4 is normable with a norm $\| \ \|$, then a norm $||| \ |||$ on (A,τ_A) that induces τ_A can be written explicitely as it follows:

$$|||(\alpha_n)||| = \sup \left\{ \left\| \sum_{k=1}^{n_i} \alpha_k x_k \right\| : i = 1,2,\ldots \right\}$$

This follows from the definition of the Minkowski functional.

3 Block generalized bases

Definition 3.1. Let (x_n) be a sequence in a TVS (X,τ_X). Then (x_n) is said to be a block generalized basis , if to each $x \in X$, there are unique scalars $\alpha_n, n = 1,2,\ldots,$ such that $x = b - \sum_{n=1}^{\infty} \alpha_n x_n$. It is said to be a Schauder block generalized basis, if, in addition, each coefficient functional is continuous on X. If $x = b - \sum \alpha_n x_n$ and $f_j(x) = \alpha_j$, then the sequence (f_n) of functionals is called associated sequence of coefficient functionals, and simply written as a.s.c.f. Let us observe again that the particular case $n_i = i, \ \forall i$, leads to the definition of a basis.
The first aim is to prove that every block generalized basis in an $F-$ space (that is, a complete metrizable TVS) is a Schauder block generalized basis.

Theorem 3.2. Let (X,τ_X) be a sequentially complete TVS with a block generalized basis (x_n). Let (A,τ_A) be a defined as in the theorem 2.4. Then the mapping $T : A \to X$ defined by $T((\alpha_n)) = b - \sum \alpha_n x_n$ is a continuous bijective linear transformation.

*Proof.*Let (α_n) be fixed in A. Then $\left(\sum_{k=1}^{n_i} \alpha_k x_k\right)_{i=1}^{\infty}$ is a Cauchy sequence in (X,τ_X) and it block-converges to a unique element $x = b - \sum \alpha_n x_n$ in (X,τ_X). So, T is defined. If $T((\alpha_n)) = 0$, then $b - \sum \alpha_n x_n = 0$, and hence $\alpha_n = 0, \forall n$, because (x_n) is a block generalized basis in (X,τ_X). So, T is injective. If $x \in X$ and $x = b - \sum \alpha_n x_n$, then $(\alpha_n) \in A$ and $T((\alpha_n)) = x$ so that T is bijective.

To prove continuity of T, consider an open neighbourhood U of 0 in (X,τ_X). Find a balanced open neighbourhood V of 0 in (X,τ_X) such that $\overline{V} \subseteq U$; where $\overline{V}$ is the closure of V in (X,τ_X). If $(\alpha_n) \in A_V$ then $\sum_{k=1}^{n_i} \alpha_k x_k \in V, \forall i$, and hence $b - \sum \alpha_n x_n \in \overline{V} \subseteq U$. Thus $T(A_V) \subseteq \overline{V} \subseteq U$. This proves that T is continuous, and the proof is complete.

Corollary 3.3. Suppose further in the theorem 3.2 that (X,τ_X) is an $F-$space. Then T is a homeomorphism.

*Proof.*It is a consequence of the open mapping theorem.

Corollary 3.4. Suppose further in the theorem 3.2 that the given TVS $(X, \| \ \|)$ is a Banach space. Then the norm $||| \ |||$ on X defined by

$$|||x||| = \sup \left\{ \left\| \sum_{k=1}^{n_i} f_k(x) x_k \right\| : i = 1,2,\ldots \right\}$$

is equivalent to the norm $\| \ \|$, when (f_n) is the a.s.c.f. for (x_n).

Lemma 3.5. Let (X,τ_X) be a sequentially complete TVS with a block generalized basis (x_n) with a.s.c.f. (f_n). Let T and (A,τ_A) be as in the theorem 3.2. If $0 \in V \in \tau_X$, V is balanced, $V + V \subseteq U$, and if $x \in T(A_V)$, then $\sum_{k=n_i+1}^{n_{i+1}} f_k(x) x_k \in T(A_U), \forall i = 0,1,2,\ldots,$ with $n_0 = 0$.

*Proof.*If $x \in T(A_V)$, then

$$\sum_{k=1}^{n_i} f_k(x) x_k \in T(A_V), \forall i = 1,2,\ldots$$

Then, for $i \geq 1$,

$$\sum_{k=n_i+1}^{n_{i+1}} f_k(x) x_k = \sum_{k=1}^{n_{i+1}} f_k(x) x_k - \sum_{k=1}^{n_i} f_k(x) x_k \in T(A_V) + T(A_V) \subseteq T(A_U).$$

Also,

$$\sum_{k=1}^{n_1} f_k(x) x_k \in T(A_V) \subseteq T(A_U).$$

Thus

$$\sum_{k=n_i+1}^{n_{i+1}} f_k(x) x_k \in T(A_U), \forall i = 0,1,2,\ldots,$$

with $n_0 = 0$. This proves the lemma.

Theorem 3.6. Suppose (X,τ_X) be an $F-$ space with a block generalized basis (x_n) with a.s.c.f. (f_n). Then each f_n is continuous.

*Proof.*Lemma 3.5 and corollary 3.3 imply that for each $i=0,1,2\dots$, the mapping $x\longmapsto\sum_{k=n_i+1}^{n_{i+1}} f_k(x)x_k$ is a continuous linear transformation from (X,τ_X) into (X,τ_X), with $n_0=0$. Since each linear functional on a finite dimensional TVS is continuous, each f_k is continuous on (X,τ_X), for $k=1,2,\dots$.

For a TVS X, the dual linear space of all continuous linear functionals on X is denoted by X^*. Let us recall that a sequence of pairs (x_n,f_n) with $x_n\in X$ and $f_n\in X^*$ is called a biorthogonal system, if $f_n(x_m)=\delta_{m,n}$, the Kronecker delta, $\forall\, m,n$. A biorthogonal system (x_n,f_n) of a TVS X is called a block regular biorthogonal system, if (x_n) is a block generalized basis and (f_n) is its a.s.c.f..

Theorem 3.7. Let (x_n,f_n) be a biorthogonal system in an $F-$space X such that (linear span of $\{x_k:k=1,2,\dots\}$ =) $\left\{\sum_{k=1}^{n_i}\alpha_k x_k : i=1,2,\dots,\ \textit{and}\ \alpha_k\ \textit{are scalars}\right\}$ is dense in X. To each $x\in X$, let $s_n(x)=\sum_{i=1}^{n} f_i(x)x_i, \forall\, n$. Then the following are equivalent.

(i)(x_n,f_n) is a block regular biorthogonal system
(ii)$\lim_{i\to\infty} s_{n_i}(x)=x, \forall\, x\in X$
(iii)$\{s_{n_i}(x):i=1,2,\dots\}$ is bounded, $\forall\, x\in X$
(iv)$\{s_{n_i}:i=1,2,\dots\}$ is an equicontinuous family on X.

Proof.(i)⇒(ii)⇒(iii) are trivial implications. The uniform boundedness principle gives the implication (iii)⇒(iv). The implication (ii)⇒(i), follows from the continuity of f_n, $\forall\, n$. Let us now assume that (iv) is true. Observe that $s_{n_i}(p)=p$ for $p=\sum_{k=1}^{j}\alpha_k x_k$ with $j\le n_i$ and scalars α_k. So, the equicontinuity of $\{s_{n_i}:i=1,2,\dots\}$ implies that $\lim_{i\to\infty} s_{n_i}(x)=x, \forall\ x\in X$, because $\left\{\sum_{k=1}^{n_i}\alpha_k x_k : i=1,2,\dots,\ \textit{and}\ \alpha_k\ \textit{are scalars}\right\}$ is dense in X. Thus (iv)⇒(ii) is proved, and this completes the proof.

Note that the application of the uniform boundedness principle for the implication (iii)⇒ (iv) requires complete metrizability in the proof of the previous theorem.

Proposition 3.8. Let (x_n,f_n) be a biorthogonal system in a TVS X. Let

$$X_0=\left\{x\in X:\lim_{i\to\infty} s_{n_i}(x)=x\right\}$$

and $X_1=\left\{x\in X:\lim_{i\to\infty} s_{n_i}(x)\ \textit{exists}\right\}$, where $s_n=\sum_{i=1}^{n} f_i(x)x_i, \forall\, n,\ \forall\, x\in X$. Then $\{x\in X: f_n(x)=0, \forall\ n=1,2,\dots\}=\{0\}$, if and only if $X_0=X_1$.

*Proof.*Suppose $X_0=X_1$, and let $x\in X$ be such that $f_n(x)=0,\ \forall\ n$. Then $\lim_{n\to\infty} s_n(x)=0$ so that $x\in X_1=X_0$. Then $x=\lim_{i\to\infty} s_{n_i}(x)$ so that $x=0$. This proves one part.

Conversely assume that $\{x\in X: f_n(x)=0,\ \forall\, n\}=\{0\}$. Let $x\in X_1$. Then

$$f_k\left(x-\lim_{i\to\infty} s_{n_i}(x)\right) = f_k(x)-f_k(x) = 0,\ \forall\, k=1,2,\dots$$

So, $x-\lim_{i\to\infty} s_{n_i}(x)=0$ so that $x\in X_0$. This proves that $X_1=X_0$.

Definition 3.9. To a given sequence (x_n) in a TVS X, let $[x_n]$ denote the closure of $\left\{\sum_{k=1}^{n_i}\alpha_k x_k : i=1,2,\dots\ \textit{and}\ \alpha_k\ \textit{are scalars}\right\}$ in X. A sequence (x_n) in a TVS X is called a block generalized basic sequence, if (x_n) is a block generalized basis in $[x_n]$, the closure of linear span of $\{x_1,x_2,\dots\}$.

Theorem 3.10. Let (x_n) be a block generalized basis of a Banach space $(X,\|\ \|)$ and let (f_n) be the a.s.c.f.. Then (f_n) is a block generalized basic sequence in X^*, and the following relation is true for every $f\in[f_n]: f=b-\sum_{i=1}^{\infty} f(x_i)f_i$.

*Proof.*Let $s_n(x)=\sum_{i=1}^{n} f_i(x)x_i, \forall\, n=1,2,\dots$, and $\forall\, x\in X$. Let s_n^* be the adjoint of s_n. Then, for $g\in X^*, n=1,2\dots$, and $x\in X$, it is true that

$$(s_n^*(g))(x) = g\left(\sum_{i=1}^{n} f_i(x)x_i\right) = \left(\sum_{i=1}^{n} g(x_i)f_i\right)(x).$$

Thus

$$s_n^*(g)=\sum_{i=1}^{n} g(x_i)f_i,$$

$\forall\, g\in X^*$, and $\forall\, n=1,2,\dots$. For $n\ge m$ and $g=\sum_{j=1}^{m}\beta_j f_j$, with scalars β_j, it is true that $s_n^*(g)=g$. Let $M=\sup_{i=1,2,\dots}\|s_{n_i}\|<\infty$. Let $f\in[f_n]$ and $\varepsilon>0$ be fixed. Find $g=\sum_{j=1}^{n_k}\beta_j f_j$ for some k and for some scalars β_j, such that $\|f-g\|<\frac{\varepsilon}{M+1}$. Then

$$\left\|s_{n_i}^*(f)-f\right\| \le \left\|s_{n_i}^*(f)-s_{n_i}^*(g)\right\|+\left\|s_{n_i}^*(g)-g\right\|+\|g-f\| \le M\frac{\varepsilon}{M+1}+\frac{\varepsilon}{M+1} = \varepsilon$$

for $i \geq k$. Thus $s^*_{n_i}(f) \to f$ in X^* as $i \to \infty$. On the other hand, if $b - \sum_{i=1}^{\infty} \alpha_i f_i = 0$ in X^*, then $\alpha_j = b - \sum_{i=1}^{\infty} \alpha_i f_i(x_j) = 0, \forall\ j = 1,2,\ldots$ Thus (f_n) is a block generalized basis of $[f_n]$.

Definition 3.11. Let (x_n) be a block generalized basis in TVS X with a.s.c.f. (f_n), and (y_n) be a block generalized basis in a TVS Y with a.s.c.f. (g_n). Let us write $(x_n) \prec (y_n)$, if $\sum_{n=1}^{\infty} \alpha_n x_n$ block-converges in X, whenever $\sum_{n=1}^{\infty} \alpha_n y_n$ block-converges in Y.

Proposition 3.12. Let (x_n) be a block generalized basis in a Banach space X with a.s.c.f. (f_n), and let (y_n) be a block generalized basis in a Banach space Y with a.s.c.f. (g_n). Suppose $(y_n) \prec (x_n)$ in X and Y. Then $(f_n) \prec (g_n)$ in $[g_n]$ and $[f_n]$.

*Proof.*Let the mapping $T : X \to Y$ be defined by

$$T\left(b - \sum \alpha_n x_n\right) = b - \sum \alpha_n y_n.$$

By the closed graph theorem, this mapping T is continuous. So, for the adjoint mapping T^* of T,

$$\begin{aligned}(T^*(g_i))(x_j) &= g_i(T(x_j)) \\ &= g_i(y_j) \\ &= f_i(x_j), \quad \forall\ i,j\end{aligned}$$

So, $T^*(g_i) = f_i,\ \forall\ i = 1,2,3,\ldots$ This, of course, proves that $(f_n) \prec (g_n)$ in $[g_n]$ and $[f_n]$. This proves the result.

The next theorem provides a method to transfer a block generalized basis into a basis. Let us recall the important fact that every block generalized basis in an $F-$space is a Schauder block generalized basis (Theorem 3.6).

Theorem 3.13. Let (x_n) be a block generalized basis in an $F-$space X with a.s.c.f. (f_n). Define

$$N = \left\{ x \in X : \sum_{j=n_i+1}^{n_{i+1}} f_j(x) = 0, \forall\ i = 0,1,2,\ldots,\ with\ n_0 = 0 \right\}$$

Let

$$e_{i+1} = \alpha_{n_i+1} x_{n_i+1} + \alpha_{n_i+2} x_{n_i+2} + \cdots + \alpha_{n_{i+1}} x_{n_{i+1}}$$

satisfying

$$\alpha_{n_i+1} + \alpha_{n_i+2} + \cdots + \alpha_{n_{i+1}} = 1,\ \forall\ i = 0,1,2,\ldots.$$

for some scalars α_j. Then N is a closed linear subspace of X, and $(e_i + N)_{i=1}^{\infty}$ is a basis in the quotient space X/N

*Proof.*Since the coefficient functionals f_j are continuous on X, to each $i = 0,1,2,\ldots$ the functional $y \longmapsto \sum_{j=n_i+1}^{n_{i+1}} f_j(y)$ is continuous on X, with $n_0 = 0$. So, if $\left(x^{(n)}\right)_{n=1}^{\infty}$ is a sequence in N that converges to some x in X, then

$$0 = \sum_{j=n_i+1}^{n_{i+1}} f_j\left(x^{(n)}\right) \to \sum_{j=n_i+1}^{n_{i+1}} f_j(x),$$

as $n \to \infty, \forall\ i = 0,1,2\ldots$ This proves that N is a closed linear subspace of X. Let $Y = X/N$, d_X be an addition invariant metric on X that induces the topology on X, and d_Y be the addtion invariant quotient metric defined by

$$d_Y(0, x+N) = \inf\{d_X(0,y) : y+N = x+N\}.$$

If

$$y_{i+1} = \beta_{n_i+1} x_{n_i+1} + \beta_{n_i+2} x_{n_i+2} + \cdots + \beta_{n_{i+1}} x_{n_{i+1}}$$

such that $\beta_{n_i+1} + \beta_{n_i+2} + \cdots + \beta_{n_{i+1}} = 1$, then $e_i - y_i \in N$ so that $e_{i+1} + N = y_{i+1} + N, \forall\ i = 0,1,2,\ldots$ In particular $e_{i+1} + N = x_j + N$ for $n_i + 1 \leq j \leq n_{i+1}, \forall\, i = 0,1,2,\ldots.$ Let $\Pi : X \to X/N$ be the natural continuous quotient mapping. Then for every $x \in X$, it is true that (with $n_0 = 0$)

$$\begin{aligned}\Pi\left(b - \sum_{i=1}^{\infty} f_i(x) x_i\right) &= b - \sum_{i=1}^{\infty} f_i(x)\, \Pi(x_i) \\ &= \sum_{i=1}^{\infty}\left(\sum_{j=n_{i-1}+1}^{n_i} f_j(x)\right)(e_i + N).\end{aligned}$$

To complete the proof, it is to be proved that $\beta_i = 0,\ \forall\ i,$ when

$$\sum_{n=1}^{\infty} \beta_n (e_n + N) = 0.$$

Suppose $\sum_{n=1}^{\infty} \beta_n(e_n + N) = 0$ in Y for some scalars β_n. To each $x \in X$, and each $n = 1,2,\ldots$, let $s_n(x) = \sum_{j=1}^{n} f_j(x) x_j$. Then, by theorem 3.7, $\{s_{n_i} : i = 1,2,\ldots\}$ is an equicontinuous family on X.

Fix a positive integer m and a fix a sequence $\varepsilon_1 > \varepsilon_2 > \ldots$ of positive numbers such that $\sum_{i=1}^{\infty} \varepsilon_i < \frac{1}{m}$, and such that

$$d_X\left(0, s_{n_i}(x) - s_{n_j}(x)\right) < \frac{1}{m 2^{k+1}},$$

$\forall\ j = 1,2,\ldots$ and $\forall\ i = 1,2,\ldots,$ whenever $d_X(0,x) < \varepsilon_k$ in X, for every $k = 1,2,\ldots.$ This is possible, because $\{s_{n_i} - s_{n_j} : i,j = 1,2,\ldots\}$ is equicontinuous.

Find a sequence $i(1) < i(2) < \ldots$ of positive integers such that

$$d_Y\left(0, \sum_{j=i(k)+1}^{i(k+1)} \beta_j (e_j + N)\right) < \varepsilon_{k+1},\ \forall\ k = 0,1,2,\ldots$$

with $i(0) = 0$. Find a sequence $z_1, z_2, \ldots$ in X such that $d_X(0, z_{k+1}) < \varepsilon_{k+1}$ and such that $\Pi(z_{k+1}) = \sum_{j=i(k)+1}^{i(k+1)} \beta_j(e_j + N), \forall k = 0,1,2,\ldots$ Then

$$d_X\left(0, s_{n_i}(z_{k+1}) - s_{n_j}(z_{k+1})\right) < \frac{1}{m 2^{k+1}},\ \forall\ k = 0,1,2,\ldots.$$

$\forall\ i=1,2,\ldots$ and $\forall\ j=1,2,\ldots$. To each $k=0,1,2,\ldots,$ write

$$y_{k+1}=s_{n_{i(k+1)}}(z_{k+1})-s_{n_{i(k)}}(z_{k+1})$$

so that

$$\Pi(y_{k+1})=\sum_{j=i(k)+1}^{i(k+1)}\beta_j(e_j+N).$$

Then $\sum_{k=1}^{\infty}y_k$ converges to some $w_m\in X$ and

$$\begin{aligned}d_X(0,w_m)&\le\sum_{k=1}^{\infty}d_X(0,y_k)\\&\le\frac{1}{m}.\end{aligned}$$

Moreover,

$$\begin{aligned}\Pi(w_m)&=\Pi\left(\sum_{k=1}^{\infty}y_k\right)\\&=\sum_{j=1}^{\infty}\beta_j(e_j+N)\\&=0.\end{aligned}$$

Thus $\sum\limits_{j=n_i+1}^{n_{i+1}}f_j(w_m)=0,\quad\forall\ i=0,1,2,\ldots$ Since $s_{n_{i(k+1)}}(w_m)-s_{n_{i(k)}}(w_m)=y_{k+1},$ since $\Pi(y_{k+1})=\sum\limits_{j=i(k)+1}^{i(k+1)}\beta_j(e_j+N)$ and since each β_j, with $i(k)+1\le j\le i(k+1)$, is a fixed finite sum of the form $\sum\limits_{l=n_p+1}^{n_{p+1}}f_l(w_m)$, then $\beta_j=0$, for every $j=1,2,3\ldots$. This proves the theorem.

4 Perturbation of block generalized bases

It is not being possible to say that all results for bases can be extended to block generalized bases through theorem 3.13. There is no immediate application of theorem 3.13 to perturbation of block basis. Two classical results for perturbation are modified to block generalized bases. The first result is of Paley-Wiener type.

Definition 4.1. Two block gereralized bases (x_n) and (y_n) of a TVS X is said to be equivalent, if there is a bijective linear transformation $T:X\to X$ such that it is a homeomorphism and such that $Tx_n=y_n,\ \forall\ n$.

Theorem 4.2. Let (x_n) be a block generalized basis for a Banach space $(X,\|\ \|)$. Let (y_n) be a sequence in X and $0\le\lambda<1$ be a constant such that

$$\left\|\sum_{k=1}^{n_i}c_k(x_k-y_k)\right\|\le\lambda\left\|\sum_{k=1}^{n_i}c_kx_k\right\|,$$

for any i, and for any scalars c_k. Then (y_n) is a block gerenalized basis for X that is equivalent to (x_n).

*Proof.*Let (f_n) be a.s.c.f. of (x_n). Given $x=b-\sum f_n(x)x_n$, by our assumption, it is true that

$$\left\|\sum_{k=n_i+1}^{n_j}f_k(x)(x_k-y_k)\right\|\le\lambda\left\|\sum_{k=n_i+1}^{n_j}f_k(x)x_k\right\|,$$

$\forall\ i,j$, whenever $j>i$, and hence $\sum\limits_{k=1}^{\infty}f_k(x)(x_k-y_k)$ block-converges. Define a bounded linear transformation $T:X\to X$ by

$$T(x)=b-\sum_{k=1}^{\infty}f_k(x)(x_k-y_k),\ \forall\ x\in X$$

Also,

$$\begin{aligned}\|Tx\|&=\left\|b-\sum_{k=1}^{\infty}f_k(x)(x_k-y_k)\right\|\\&\le\lambda\left\|b-\sum_{k=1}^{\infty}f_k(x)x_k\right\|\\&=\lambda\|x\|,\quad x\in X.\end{aligned}$$

Thus $\|T\|=\lambda<1$ so that $(I-T):X\to X$ is invertible. So, there is a bijective homeomorphism $(I-T):X\to X$, which is also a linear transformation such that $(I-T)(x_n)=y_n,\ \forall\ n$. This proves the theorem.

Corollary 4.3. Let (x_n) be a block generalized basis for a Banach space $(X,\|\ \|)$ with a.s.c.f. (f_n) in X^*. Suppose (y_n) is a sequence in X such that $\lambda=\sum\limits_{n=1}^{\infty}\|f_n\|\|x_n-y_n\|<1$. Then (y_n) is a block generalized basis for X that is equivalent to (x_n).

*Proof.*If $x=\sum\limits_{k=1}^{n}c_k(x_k-y_k)$, then

$$\begin{aligned}\left\|\sum_{k=1}^{n}c_k(x_k-y_k)\right\|&=\left\|\sum_{k=1}^{n}f_k(x)(x_k-y_k)\right\|\\&\le\sum_{k=1}^{n}|f_k(x)|\,\|x_k-y_k\|\\&\le\|x\|\sum_{k=1}^{n}\|f_k\|\|x_k-y_k\|\\&\le\lambda\|x\|\\&=\lambda\left\|\sum_{k=1}^{n}c_k(x_k-y_k)\right\|.\end{aligned}$$

Theorem 4.4. Let (x_n) be a block generalized basis for a Banach space $(X,\|\ \|)$, with a.s.c.f. (f_n) in X^*. Let (y_n) be a X-complete sequence in X (in the sense that span $\{y_1,y_2,\ldots\}$ is dense in X) such that $\lambda=\sum\limits_{n=1}^{\infty}\|f_n\|\|x_n-y_n\|<\infty$. Then (y_n) is a block generalized basis that is equivalent to (x_n).

Proof. Define a bounded linear operator $T : X \to X$ by

$$T(x) = b - \sum_{n=1}^{\infty} f_n(x)(x_n - y_n), \ \forall \ x \in X,$$

when

$$\begin{aligned} \|Tx\| &= \left\| b - \sum_{n=1}^{\infty} f_n(x)(x_n - y_n) \right\| \\ &\le \sum_{n=1}^{\infty} |f_n(x)| \|x_n - y_n\| \\ &\le \|x\| \sum_{n=1}^{\infty} \|f_n\| \|x_n - y_n\| \\ &\le \lambda \|x\|, \quad x \in X. \end{aligned}$$

To each i, define a compact linear operator $T_i : X \to X$ by

$$T_i(x) = \sum_{k=1}^{n_i} f_k(x)(x_k - y_k), \quad \forall x \in X.$$

Then $\|T - T_i\| \to 0$ as $i \to \infty$ under operator norm. So, T is also compact so that $(T - I) : X \to X$ has closed range. Since $(I - T)(x_n) = y_n, \ \forall \ n$, and since span $\{y_1, y_2 \dots\}$ is dense in X, then $(I - T)(X) = X$. To prove that $I - T$ is $1 - 1$, consider an integer j such that

$$\sum_{k=n_j+1}^{\infty} \|f_k\| \|x_k - y_k\| < 1.$$

Then, by the previous corollary, $\{x_1, x_2, \dots, x_{n_j}, y_{n_j+1}, y_{n_j+2}, \dots\}$ is a block generalized basis of X that is equivalent to (x_n). Define

$$X_1 = span \ \{x_1, x_2, \dots x_{n_j}\},$$

and

$$Y_1 = closure\ of\ span \ \{y_{n_j+1}, y_{n_j+2}, \dots\}.$$

Then $X = X_1 + Y_1$, $X_1 \bigcap Y_1 = \{0\}$, and X_1 and Y_1 are closed subspaces of X. Consider a relation $b - \sum_{k=1}^{\infty} c_k y_k = 0$ in X for some scalars c_k. If $c_k \neq 0$ for some $k \le n_j$, then

$$y_k = b - \left(-\frac{1}{c_k} \left(\sum_{i=1}^{\infty} z_i \right) \right)$$

with $z_i = c_i y_i$ for $i \neq k$, and $z_k = 0$. Thus $X = Z_1 + Y_1$, when

$$Z_1 = span \{y_1, y_2, \dots, y_{k-1}, y_{k+1}, \dots y_{n_j}\},$$

because $Z_1 + Y_1$ is closed in X, and $span\{y_1, y_2, \dots\}$ is dense in X. Since $dim Z_1 < dim X_1$, $X = X_1 + Y_1 = Z_1 + Y_1$ and $X_1 \bigcap Y_1 = \{0\}$, there is a contradiction. So, $c_k = 0$ for any $k \le n_j$. Thus (with a natural sense) $b - \sum_{k=n_j+1}^{\infty} c_k y_k = 0$. Since $\{x_1, x_2, \dots, x_{n_j}, y_{n_j+1}, y_{n_j+2}, \dots\}$ is a block generalized basis, $c_k = 0, \ k \ge n_j + 1$. Thus, if $b - \sum_{k=1}^{\infty} c_k y_k = 0$, then $c_k = 0, \ \forall \ k = 1, 2, \dots$. In particular, if $(I - T)(x) = 0$, then $b - \sum_{n=1}^{\infty} f_n(x) y_n = 0$ and then $f_n(x) = 0, \ \forall \ n$, and hence $x = 0$. Thus $(I - T) : X \to X$ is a bijective homeomorphism which is a linear transformation such that $(I - T)(x_n) = y_n, \ \forall \ n$. This proves the theorem.

Definition 4.5. A subset A of $\mathbb{N}$, the set of all natural numbers, is called a block subset of $\mathbb{N}$, if $k \in A$, whenever $l \in A$ and $n_i + 1 \le k, l \le n_{i+1}$, for some $i = 0, 1, 2, \dots$ (with $n_0 = 0$).

Let (x_n) be a sequence in a TVS X; and let A be a block subset of $\mathbb{N}$. Put $y_n = x_n$ if $n \in A$, and zero otherwise. Let us say that a partial series $\sum_{n \in A} x_n$ block-converges to some element $x \left(= b - \sum_{n \in A} x_n \right)$ in X, if $\sum_{n=1}^{\infty} y_n$ block-converges to x in X.

Suppose further that (x_n) is a block generalized basis in a TVS X. Let us say that $(x_n)_{n \in A}$ is a partial block generalized basis in a linear space Y of X, if $x_n \in Y, \ \forall \ n \in A$ and if to each $x \in Y$, there are unique scalars α_n, for $n \in A$ such that $x = b - \sum_{n \in A} \alpha_n x_n$.

Theorem 4.6. Let (x_n) be a block generalized basis in an $F-$space X. Let A be an infinite block subset of $\mathbb{N}$ and $B = \mathbb{N} - A$. Let Y be the closure of linear span of $\{x_n : n \in A\}$. Let Z be the quotient space X/Y, and let $\Pi : X \to Z$ be the natural quotient mapping, $\Pi(x) = x + Y$. Then $(x_n)_{n \in A}$ is a partial block generalized basis in Y(relative to A). The collection $(\Pi(x_n))_{n \in B}$ is also a partial block generalized basis of Z(relative to B), when B is infinite.(If B is finite, $\{\Pi(x_n) : n \in B\}$ is a Hamel basis of Z).

Proof. Let (f_n) be the a.s.c.f. of the block generalized basis (x_n) in X. To each $n = 1, 2, \dots,$ let $s_n(x) = \sum_{i=1}^{n} f_i(x) x_i \ \ \forall \ \ x \in X$. By theorem 3.7, $\{s_{n_i} : i = 1, 2, \dots\}$ is equicontinuous on X. To each i, let $\tilde{s}_{n_i}$ denote the restriction of s_{n_i} to Y; with codomain Y. Then $\{\tilde{s}_{n_i} : i = 1, 2, 3 \dots\}$ is also equicontinuous on Y. So, a variation of the theorem 3.7 implies that $(x_n)_{n \in A}$ is a partial block generalized basis of Y, with a biorthogonal system $(x_n, f_n)_{n \in A}$. To each $n \in B$, define a linear functional φ_n on Z by $\varphi_n(x + Y) = f_n(x), \ \forall x \in X$. Then φ_n is a well defined continuous linear functional on Z such that $\varphi_n(\Pi(x_m))$ is 1 if $n = m$; and it is zero if $n \neq m \in B$. Then $(\Pi(x_n), \varphi_n)_{n \in B}$ is a biorthogonal system for Z. If $x + Y \in Z$, then $\lim_{i \to \infty} s_{n_i}(x) = x$, and hence $\lim_{i \to \infty} \Pi(s_{n_i}(x)) = x + Y$. So, by a variation of theorem 3.7, $(\Pi(x_n))_{n \in B}$ is a partial block generalized basis of Z. This completes the proof.

Remark 4.7. When $(x_n)_{n \in A}$ is written as a sequence such that order among suffices is preserved, a block generalized basis to Y is obtained with respect to a different (n_i). Similarly, Z has a block generalized basis with respect to a different (n_i). These facts can be extended to unconditional block generalized bases, when they are introduced in the following way.

Definition 4.8. Let $\Pi : \mathbb{N} \to \mathbb{N}$ be a bijective mapping. It is said to be a block permutation, if $\Pi(\{n_i+1, n_i+2, \dots, n_{i+1}\})$ is a finite sequence of successive elements of $\mathbb{N}$. In this case, Π is said to induce a new block sequence $0 = m_0 < m_1 < m_2 < \dots$, when

$$\Pi(\{n_0, n_1, n_2, \dots\}) = \{m_0, m_1, m_2, \dots\}$$

with $n_0 = 0$.

Let (x_n) be a sequence in a TVS X. Then $\sum\limits_{n=1}^{\infty} x_n$ is said to block-converge unconditionally, if $\sum\limits_{i=0}^{\infty}\left(\sum\limits_{j=m_i+1}^{m_{i+1}} x_j\right)$ converges whenever the block sequence $0 = m_0 < m_1 < m_2 < \dots$ is obtained from $0 = n_0 < n_1 < n_2 < \dots$ by means of a block permutation.

A block generalized basis (x_n) in TVS X is called an unconditional block generalized basis, whenever $\sum \alpha_n x_n$ block-converges unconditionally in X, for every $b - \sum \alpha_n x_n \in X$

Remark 4.9. A series $\sum_{n=1}^{\infty} x_n$ of scalars block-converges unconditionally if and only if $\sum\limits_{i=0}^{\infty}\left|\sum\limits_{j=n_i+1}^{n_i} x_j\right|$ converges, when $n_0 = 0$. In this case,

$$\sum_{i=0}^{\infty}\sum_{j=n_i+1}^{n_{i+1}} x_j = \sum_{i=0}^{\infty}\sum_{j=m_i+1}^{m_{i+1}} x_j$$

for any block sequence $0 = m_0 < m_1 < m_2 < \dots$ obtained from $0 = n_0 < n_1 < n_2 < \dots$ through any block permutation.

This fact implies the following result.

Proposition 4.10. Suppose a series $\sum\limits_{n=1}^{\infty} x_n$ block-converges unconditionally in a TVS X, for which the points are separated by its dual X^*. Then

$$\sum_{i=0}^{\infty}\sum_{j=n_i+1}^{n_{i+1}} x_j = \sum_{i=0}^{\infty}\sum_{j=m_i+1}^{m_{i+1}} x_j$$

for any block sequence $0 = m_0 < m_1 < m_2 < \dots$ obtained from $0 = n_0 < n_1 < n_2 < \dots$ through any block permutation.

Example 4.11. To each sequence of scalars (α_n), let us define

$$\|(\alpha_n)\| = \sup\left\{\left|\sum_{k=1}^{m}\alpha_k\right| : m = 1, 2, \dots\right\}.$$

Let $X = \{(\alpha_n) : \|(\alpha_n)\| < \infty$ and α_k are scalars $\}$. Then $(X, \|\ \|)$ is a Banach space with respect to natural coordinatewise addition and scalar multiplication. To each $n = 1, 2, \dots$, let $e^{(n)} = (\alpha_k)$, where $\alpha_k = 1$ for $k = n$, and zero otherwise. To each $n = 1, 2, \dots$, and $(\alpha_k) \in X$, let

$$s_n((\alpha_k)) = \sum_{i=1}^{n}\alpha_i e^{(i)} = (\alpha_1, \alpha_2, \dots \alpha_n, 0, 0, \dots),$$

and $t_n((\alpha_k)) = (0, 0, \dots 0, \alpha_{n+1}, \alpha_{n+2}, \dots)$, when zeros occupy the first n-coordinates for t_n. Define

$$Y = \{(\alpha_k) \in X : \|t_{n_i}((\alpha_k))\| \to 0 \ as \ i \to \infty\}.$$

Then $(Y, \|\ \|)$ is a Banach space with a block generalized basis $\left(e^{(n)}\right)_{n=1}^{\infty}$. If $n_i = 2^i, \ \forall\, i$, then $\left((-1)^k\right)_{k=1}^{\infty} \in Y$.

5 A non-trivial example

Example 4.11 is a natural example of a block generalized basis. Lemma 3.5 assures convergence of $\left(\sum_{k=n_i+1}^{n_{i+1}} f_k(x)x_k\right)_{i=0}^{\infty}$ to 0 in F-spaces; and convergence of $(f_k(x)x_k)_{k=1}^{\infty}$ to 0 fails to be true in the space $(Y, \|\ \|)$ given in Example 4.11, with $x_k = e^{(k)}$. So, some results for classical bases need modifications for extensions. This section provides a non-trivial example, which also gives a natural motivation to study block generalized bases.

Example 5.1. When a pair (i, j) is used in this example, it is assumed that i and j are natural numbers and $1 \le j \le 2^{i-1}$. Let us write $(i, j_i) \le (i+1, j_{i+1})$, if $j_{i+1} = 2j_i$ or $2j_i - 1$. To each sequence $\alpha = (\alpha_{(1,1)}, \alpha_{(2,1)}, \alpha_{(2,2)}, \alpha_{(3,1)}, \alpha_{(3,2)}, \alpha_{(3,3)}, \alpha_{(3,4)}, \alpha_{(4,1)}, \dots, \alpha_{(4,8)}, \alpha_{(5,1)}, \dots, \alpha_{(5,16)}, \dots)$ of real scalars, let us define $\|\alpha\|_1$ by

$$\|\alpha\|_1 = \sup\{\sup\left\{\left|\sum_{i=1}^{n}\alpha_{(i,j_i)}\right| : (1, j_1) \le (2, j_2) \le \dots \le (n, j_n)\right\}$$

$$: 1 \le j_n \le 2^{n-1},\ n = 1, 2, \dots\}.$$

Let X denote the collection of all sequences $\alpha = \left(\alpha_{(1,1)}, \dots\right)$ for which $\|\alpha\|_1 < \infty$, and

$$\sup\left\{\left|\sum_{i=m}^{n}\alpha_{(i,j_i)}\right| : (m, j_m) \le (m+1, j_{m+1}) \le \dots \le (n, j_n)\right\} \to 0$$

as $m, n \to \infty$ with $m \le n$. Then $(X, \|\ \|_1)$ is a Banach space with respect to natural coordinatewise addition and scalar multiplication. This space has a block generalized basis $\{(1, 0, 0, 0, \dots), (0, 1, 0, 0, \dots), (0, 0, 1, 0, \dots), \dots\}$ with $n_1 = 1, n_2 = 1 + 2 = 3, n_3 = 1 + 2 + 2^2 = 7, n_4 = 1 + 2 + 2^2 + 2^3 = 15, \dots$. Let $I_{(1,1)} = [0, 1]$, $I_{(2,1)} = [0, \frac{1}{3}]$, $I_{(2,2)} = [\frac{2}{3}, 1]$, $I_{(3,1)} = [0, \frac{1}{3^2}]$, $I_{(3,2)} = [\frac{2}{3^2}, \frac{3}{3^2}]$, $I_{(3,3)} = [\frac{6}{3^2}, \frac{7}{3^2}]$, $I_{(3,4)} = [\frac{8}{3^2}, 1], \dots$ be the closed intervals associated with a construction of classical Cantor ternary set S over the interval $[0, 1]$. Then $S = \bigcap\limits_{n=1}^{\infty} J_n$, where $J_1 = I_{(1,1)}$, $J_2 = I_{(2,1)} \bigcup I_{(2,2)}$, $J_3 = I_{(3,1)} \bigcup I_{(3,2)} \bigcup I_{(3,3)} \bigcup I_{(3,4)}, \dots\dots$. Let $C(S)$ denote the Banach space of all continuous real valued functions on S with the supremum norm defined by $\|f\|_\infty = \sup\{|f(x)| : x \in S\}$, for $f \in C(S)$. To each (i, j), let $F_{(i,j)}$ denote the characteristic function $\chi_{I_{(i,j)}}$ defined on S, that is restricted to S. Then

$$\{F_{(1,1)}, F_{(2,1)}, F_{(2,2)}, F_{(3,1)}, F_{(3,2)}, F_{(3,3)}, F_{(3,4)}, F_{(4,1)}, \ldots, F_{(4,8)}$$

$$, F_{(5,1)}, \ldots, F_{(5,16)}, \ldots\}$$

is not a block generalized basis in $(C(S), \| \ \|_\infty)$ with $n_1 = 1, n_2 = 1 + 2 = 3, n_3 = 1 + 2 + 2^2 = 7, n_4 = 1 + 2 + 2^2 + 2^3 = 15, \ldots$. Define $T : (X, \| \ \|_1) \to (C(S), \| \ \|_\infty)$ by

$$T\left((\alpha_{(1,1)}, \ldots)\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{2^{i-1}} \alpha_{(i,j)} F_{(i,j)}$$
$$= \sum_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}} \alpha_{(i,j)} F_{(i,j)},$$

when the limit exists with respect to $\| \ \|_\infty$. Since $\|T(\alpha)\|_\infty \le \|\alpha\|_1$, for every $\alpha \in X$, T is a continuous linear transformation. Fix $f \in C(S)$. To each (i, j), let $\beta_{(i,j)} = \max\left\{f(x) : x \in S \bigcap I_{(i,j)}\right\}$. Then, define $\alpha_{(1,1)} = \beta_{(1,1)}$, $\alpha_{(2,1)} = \beta_{(2,1)} - \alpha_{(1,1)}$, $\alpha_{(2,2)} = \beta_{(2,2)} - \alpha_{(1,1)}$, $\alpha_{(3,1)} = \beta_{(3,1)} - \alpha_{(2,1)} - \alpha_{(1,1)}$, $\alpha_{(3,2)} = \beta_{(3,2)} - \alpha_{(2,1)} - \alpha_{(1,1)}$, $\alpha_{(3,3)} = \beta_{(3,3)} - \alpha_{(2,2)} - \alpha_{(1,1)}$, $\alpha_{(3,4)} = \beta_{(3,4)} - \alpha_{(2,2)} - \alpha_{(1,1)}$, $\alpha_{(4,1)} = \beta_{(4,1)} - \alpha_{(3,1)} - \alpha_{(2,1)} - \alpha_{(1,1)}$, $\alpha_{(4,2)} = \beta_{(4,2)} - \alpha_{(3,1)} - \alpha_{(2,1)} - \alpha_{(1,1)}$, $\alpha_{(4,3)} = \beta_{(4,3)} - \alpha_{(3,2)} - \alpha_{(2,1)} - \alpha_{(1,1)}$, $\alpha_{(4,4)} = \beta_{(4,4)} - \alpha_{(3,2)} - \alpha_{(2,1)} - \alpha_{(1,1)}$, $\alpha_{(4,5)} = \beta_{(4,5)} - \alpha_{(3,3)} - \alpha_{(2,2)} - \alpha_{(1,1)}$, $\alpha_{(4,6)} = \beta_{(4,6)} - \alpha_{(3,3)} - \alpha_{(2,2)} - \alpha_{(1,1)}$, $\alpha_{(4,7)} = \beta_{(4,7)} - \alpha_{(3,4)} - \alpha_{(2,2)} - \alpha_{(1,1)}$, $\alpha_{(4,8)} = \beta_{(4,8)} - \alpha_{(3,4)} - \alpha_{(2,2)} - \alpha_{(1,1)}, \ldots$. Then $(\alpha_{(1,1)}, \ldots) \in X$, and $f = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{2^{i-1}} \alpha_{(i,j)} F_{(i,j)}$ in $C(S)$. Thus T is surjective. The above representation of f is true, even if "max" is replaced by "min" in the definition of $\beta_{(i,j)}$. So, T is not injective. Conclusion: Continuous image of a block generalized basis need not be a block generalized basis.

6 Conclusion

A systematic study has been presented on block convergence and block generalized bases. A few classical results for bases have been generalized to block generalized bases. This includes a generalization: Every block generalized basis in an F - space is a Schauder block generalized basis. It is expected that many results for classical bases can be generalized to block generalized bases

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