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Some Fixed Point Theorems for Two Hybrid Pairs of Mappings in Partial Metric Spaces

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Abstract: In this paper some common fixed point results for two hybrid pairs of non-self mappings in the framework of partial metric spaces are established. The main result, in particular, generalizes the metric fixed point results due to Khan and Imdad, Ćirić and Ume, and Rhoades to partial metric spaces.

Keywords: partial metric spaces, metrically convex spaces, weakly compatible mappings

1 Introduction

The study of common fixed points for pair(s) of single-valued and multi-valued mappings, also called hybrid fixed point theory, is a landmark in the development of fixed point theory as the existing literature of the theory contains numerous results for the pair(s) of mappings. Assad and Kirk [6] initiated the study of fixed points for non-self mappings of metric spaces, by proving a fixed point theorem for multi-valued mappings of complete metrically convex metric spaces. Their results have been extended by several researchers including Rhoades [1]. Khan and Imdad [11] established some metric fixed point results for two hybrid pairs of non-self mappings of complete metrically convex metric spaces, which generalize partially or completely, in particular, fixed point results due to Rhoades [1] and Ćirić and Ume [4].

Partial metric spaces are one of the generalizations of the notion of metric spaces such that the distance of a point from itself is not necessarily zero [8]. Partial metric spaces were first introduced and studied by Matthews while studying denotational semantics of computer programming languages, showing that the essential tools of metric spaces like the Banach contraction principle can be generalized to partial metric spaces [7,8]. Aydi, Abbas and Vetro [2] introduced and studied the notion of partial Hausdorff metric, and established the Nadler's fixed point theorem [9] in the setup of partial metric spaces.

Recently, there has been several studies on possible generalizations of the existing metric fixed point results to partial metric spaces. This paper forms a part of the studies for metric fixed point results for two hybrid pairs of weakly compatible non-self mappings of complete metrically convex metric spaces. The purpose of this paper is to generalize a metric fixed point theorem due to Khan and Imdad [11] to partial metric spaces.

2 Preliminaries

The following definitions and preliminary results are necessary to establish the results.

Definition 2.1. Let X be a non-empty set. Let $T : X \to 2^X$, where 2^X denotes the collection of all non-empty subsets of X, be a multi-valued mapping and $f : X \to X$ be a single-valued mapping, then a point $t \in X$ is called a common fixed point of T and f if $t = ft \in Tt$.

Definition 2.2. *([8])* Let X be non-empty set. A partial metric space is a pair (X, p), where p is a function $p : X \times X \to [0,\infty)$, called the partial metric, such that for all $x,y,z \in X$:

(P1) $x = y \Leftrightarrow p(x,y) = p(x,x) = p(y,y)$;
(P2) $p(x,x) \le p(x,y)$;
(P3) $p(x,y) = p(y,x)$; and
(P4) $p(x,y) + p(z,z) \le p(x,z) + p(z,y)$.

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Clearly, by (P1) - (P3), $p(x,y)=0$ implies $x=y$. But, the converse is in general not true.

A classical example of partial metric spaces is the pair $([0,\infty),p)$ where $p(x,y)=\max\{x,y\}$ for all $x,y\in[0,\infty)$. For more examples of partial metric spaces, we refer the reader to [5,8].

Each partial metric p on X generates a T_0 topology τ_p on X whose basis is the collection of all open p-balls $\{B_p(x,\varepsilon):x\in X,\ \varepsilon>0\}$ where $B_p(x,\varepsilon)=\{y\in X:p(x,y)<p(x,x)+\varepsilon\}$ for all $x\in X$ and ε is a real number.

Let (X,p) be a partial metric space, B any non-empty subset of the set X and x an element of the set X. It is well known [10] that $x\in\bar{B}$, where $\bar{B}$ is the closure of B, if and only if $p(x,B)=p(x,x)$. Also, the set B is said to be closed in (X,p) if and only if $B=\bar{B}$.

Definition 2.3.(*[8]*)

(i) A sequence $\{x_n\}$ in a partial metric space (X,p) is said to converge to some $x\in X$ if and only if $p(x,x)=\lim_{n\to\infty}p(x,x_n)$.
(ii) A sequence $\{x_n\}$ in a partial metric space (X,p) is a Cauchy sequence if and only if $\lim_{n,m\to\infty}p(x_n,x_m)$ exists and is finite.
(iii) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to the topology τ_p to a point $x\in X$ such that $p(x,x)=\lim_{n,m\to\infty}p(x_n,x_m)$.

Lemma 2.1.(*[8]*) Let (X,p) be a partial metric space, then the mapping $p^s:X\times X\to[0,\infty)$ given by

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y)$$

for all $x,y\in X$ defines a metric on X.

Lemma 2.2. (*[8]*)

(i)A sequence $\{x_n\}$ in a partial metric space (X,p) is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).
(ii)A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete.

Let (X,d) be a metric space and $CB(X)$ denotes the collection of all non-empty bounded closed subsets of X. For $A,B\in CB(X)$, define

$$H(A,B)=\max\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(b,A)\}$$

where $d(x,A)=\inf\{d(x,a):a\in A\}$ is the distance from a point $x\in X$ to the set $A\in CB(X)$. It is well known [9] that H is a metric, called the Pompeiu-Hausdorff metric, on $CB(X)$ induced by the metric d. The metric space $(CB(X),H)$ is complete whenever (X,d) is complete.

Definition 2.4. (*[2]*) Let (X,p) be a partial metric space and $CB^p(X)$ denotes the collection of all non-empty bounded and closed subsets of X. For $A,B\in CB^p(X)$ and $x\in X$, define $H_p(A,B)=\max\{\delta_p(A,B),\delta_p(B,A)\}$ where
$p(x,A)=\inf\{p(x,a):a\in A\}$,
$\delta_p(A,B)=\sup\{p(a,B):a\in A\}$
and $\delta_p(B,A)=\sup\{p(b,A):b\in B\}$. Then the mapping H_p is a partial metric, called the *partial Hausdorff metric*, on $CB^p(X)$ induced by the partial metric p.

Lemma 2.3.(*[2]*) Let (X,p) be a partial metric space. Let $A,B\in CB^p(X)$ and $q>1$. Then for any $a\in A$, there exists $b\in B$ that depends on a such that

$$p(a,b)\le qH_p(A,B).$$

Definition 2.5. (*[4]*) Let (X,d) be a metric space and K be a non-empty subset of X. The mappings $F:K\to CB(X)$ and $f:K\to X$ are said to be weakly compatible on K if, for any sequences $\{x_n\}$ and $\{y_n\}$ in K such that $fx_n\in K$, $Fx_n\cap K\neq\varnothing$, the following limits exist and satisfy:

(i)$\limsup_{n\to\infty}D(fy_n,Ffx_n)\le\limsup_{n\to\infty}H(Ffx_n,Fx_n)$, and
(ii)$\limsup_{n\to\infty}d(fy_n,fx_n)\le\limsup_{n\to\infty}H(Ffx_n,Fx_n)$, whenever $\{y_n\}\in Fx_n\cap K$ and $\lim_{n\to\infty}d(y_n,fx_n)=0$.

Definition 2.6.(*[6]*) Let (X,d) be a metric space. If for each $x,y\in X$, $x\neq y$ there exists $z\in X$, $x\neq y\neq z$, such that $d(x,z)+d(z,y)=d(x,y)$, then X is said to be a metrically convex metric space.

Lemma 2.4.(*[6]*) Let (X,d) be a metrically convex metric space and K a non-empty closed subset of X. Then for given $x\in K$, $y\notin K$, there exists $z\in\partial K$, boundary of K, such $d(x,z)+d(z,y)=d(x,y)$.

Analogous to Definition 2.6. we have the following definition.

Definition 2.7. (*[12]*) Let (X,p) be a partial metric space . If for each $x,y\in X$, $x\neq y$ there exists $z\in X$, $x\neq y\neq z$ such that $p(x,z)+p(z,y)=p(x,y)+p(z,z)$, then X is said to be a metrically convex partial metric space.

Lemma 2.4. extends to partial metric spaces.

Lemma 2.5.(*[12]*) Let (X,p) be a metrically convex partial metric space and K a non-empty closed subset of X. Then for given $x\in K$, $y\notin K$, there exists $z\in\partial K$, boundary of K, such $p(x,z)+p(z,y)=p(x,y)+p(z,z)$.

Definition 2.5. extends to partial metric spaces as follows:

Definition 2.8. Let (X,p) be a partial metric space and K be a non-empty subset of X. Let $F:K\to CB^p(X)$ and $f:K\to X$ be mappings. If for any sequences $\{x_n\}$ and $\{y_n\}$ in K such that $fx_n\in K$, $Fx_n\cap K\neq\varnothing$, the following hold:

(i) $\limsup_{n\to\infty}p(fy_n,Ffx_n)\le\limsup_{n\to\infty}H_p(Ffx_n,Fx_n)$, and
(ii) $\limsup_{n\to\infty}p(fy_n,fx_n)\le\limsup_{n\to\infty}H_p(Ffx_n,Fx_n)$, whenever $\{y_n\}\in Fx_n\cap K$ and $\lim_{n\to\infty}p(y_n,fx_n)=0$.

Then the mappings $\{F,f\}$ are said to be weakly compatible on K.

Khan and Imdad [11] established the following common fixed point theorem.

Theorem 2.1. Let (X,d) be a complete and metrically convex metric space and K be a nonempty closed subset of X. Let the mappings $F,G : K \to CB(X)$ and $g, f : K \to X$ satisfy the following condition:

$$H(Fx,Gy) \leq h\max\left\{\frac{1}{a}d(fx,gy), D(fx,Fx), D(gy,Gy),\right.$$
$$\left.\frac{1}{a+h}[D(fx,Gy)+D(gy,Fx)]\right\} \tag{1}$$

for all $x,y \in X$ with $x \neq y$,
where $0 < h < \frac{2}{3}$, $a \geq 1+\frac{2h^2}{1+h}$, and

(i) $\partial K \subseteq gK \cap fK$, $FK \cap K \subseteq fK$, $GK \cap K \subseteq gK$,
(ii) $fx \in \partial K \Rightarrow Fx \subseteq K$, $gx \in \partial K \Rightarrow Gx \subseteq K$,
(iii) $\{F, f\}$ and $\{G,g\}$ are weakly compatible mappings, and
(iv) g and f are continuous on K.

Then there exists a point s in K such that $s = fs = gs \in Fs \cap Gs$.

In this paper Theorem 2.1. is generalized to partial metric spaces.

3 Main Results

We now present a generalization of Theorem 2.1. to partial metric spaces.

Theorem 3.1. Let (X, p) be a metrically convex complete partial metric space. Let K be a non-empty closed subset of X. Let the mappings $F,G : K \to CB^p(X)$ and $g, f : K \to X$ satisfy the following condition:
$H(Fx,Gy)$

$$\leq h\max\left\{\frac{1}{a}p(fx,gy), p(fx,Fx), p(gy,Gy),\right.$$
$$\left.\frac{1}{a+h}[p(fx,Gy)+p(gy,Fx)]\right\} \tag{2}$$

for all $x,y \in X$ with $x \neq y$,
where $0 < h < \frac{2}{3}$, $a \geq 1+\frac{2h^2}{1+h}$, and

(i) $\partial K \subseteq gK \cap fK$, $FK \cap K \subseteq fK$, $GK \cap K \subseteq gK$,
(ii) $fx \in \partial K \Rightarrow Fx \subseteq K$, $gx \in \partial K \Rightarrow Gx \subseteq K$,
(iii) $\{F, f\}$ and $\{G,g\}$ are weakly compatible mappings, and
(iv) g and f are continuous on K.

Then there exists a point s in K such that $s = gs = fs \in Fs \cap Gs$.

Proof. We construct two sequences $\{x_n\}$ and $\{y_n\}$ in K as follows: Let $y_0 \in X$ be an arbitrary point in ∂K. Since $y_0 \in \partial K$ and $\partial K \subseteq gK \cap fK$, we can find a point $x_0 \in K$ such that $fx_0 = y_0 \in \partial K$. By Theorem 3.1. (ii) we have $Fx_0 \subseteq K$, and by Theorem 3.1. (i) we can find a point $x_1 \in K$ such that $fx_1 \in Fx_0 \subseteq K$. Set $y_1 = fx_1$. Let q be any real number such that

$$q > 1 \quad \text{and} \quad qh = t < \frac{2}{3} \quad (\text{for instance}\, q = \frac{3}{2} - h). \tag{3}$$

Since $y_1 \in Fx_0$, by Lemma 2.3., there exists a point $y_2 \in Gx_1$ such that

$$p(y_1,y_2) \leq qH_p(Fx_0,Gx_1).$$

We proceed as follows:

(i) $y_{2n} \in Gx_{2n-1}$, $y_{2n+1} \in Fx_{2n}$
(ii) $y_{2n} \in K \Rightarrow y_{2n} = fx_{2n}$ or $y_{2n} \notin K \Rightarrow fx_{2n} \in \partial K$, and $p(gx_{2n-1}, fx_{2n}) + p(fx_{2n},y_{2n}) = p(gx_{2n-1},y_{2n}) + p(fx_{2n}, fx_{2n})$
(iii) $y_{2n+1} \in K \Rightarrow y_{2n+1} = gx_{2n+1}$ or $y_{2n+1} \notin K \Rightarrow gx_{2n+1} \in \partial K$, and $p(fx_{2n},gx_{2n+1}) + p(gx_{2n+1},y_{2n+1}) = p(fx_{2n},y_{2n+1}) + p(gx_{2n+1},gx_{2n+1})$
(iv) $p(y_{2n-1},y_{2n}) \leq qH_p(Gx_{2n-1},Fx_{2n-2})$
(v) $p(y_{2n},y_{2n+1}) \leq qH_p(Fx_{2n},Gx_{2n-1})$

We denote
$P_0 = \{fx_{2i} \in \{fx_{2n}\} : fx_{2i} = y_{2i}\}$,
$P_1 = \{fx_{2i} \in \{fx_{2n}\} : fx_{2i} \neq y_{2i}\}$,
$Q_0 = \{gx_{2i+1} \in \{gx_{2n+1}\} : gx_{2i+1} = y_{2i+1}\}$,
$Q_1 = \{gx_{2i+1} \in \{gx_{2n+1}\} : gx_{2i+1} \neq y_{2i+1}\}$.

Remark 3.1. $(fx_{2n},gx_{2n+1}) \notin P_1 \times Q_1$ and $(gx_{2n-1}, fx_{2n}) \notin Q_1 \times P_1$.

We now consider three possible cases.

Case 1. If $(fx_{2n},gx_{2n+1}) \in P_0 \times Q_0$, then:

$$\begin{aligned}
p(fx_{2n},gx_{2n+1}) &\leq qH_p(Fx_{2n},Gx_{2n-1})\\
&\leq t\max\left\{\frac{1}{a}p(fx_{2n},gx_{2n-1}), p(fx_{2n},Fx_{2n}),\right.\\
&\quad p(gx_{2n-1},Gx_{2n-1}),\\
&\quad \left.\frac{1}{a+h}[p(fx_{2n},Gx_{2n-1})+p(gx_{2n-1},Fx_{2n})]\right\}\\
&\leq t\max\left\{\frac{1}{a}p(fx_{2n},gx_{2n-1}), p(fx_{2n},gx_{2n+1}),\right.\\
&\quad p(gx_{2n-1}, fx_{2n}),\\
&\quad \left.\frac{1}{a+h}[p(gx_{2n-1}, fx_{2n})+p(fx_{2n},gx_{2n+1})]\right\}\\
&\leq t\max\{p(fx_{2n},gx_{2n-1}), p(fx_{2n},gx_{n+1})\}\\
&\leq tp(fx_{2n},gx_{2n-1})
\end{aligned}$$

Similarly, if $(gx_{2n-1}, fx_{2n}) \in Q_0 \times P_0$, then $p(gx_{2n-1}, fx_{2n}) \leq tp(fx_{2n-2},gx_{2n-1})$.

Case 2. If $(fx_{2n},gx_{2n+1}) \in P_0 \times Q_1$, then
$p(fx_{2n},gx_{2n+1}) + p(gx_{2n+1},y_{2n+1}) = p(fx_{2n},y_{2n+1}) + p(gx_{2n+1},gx_{2n+1})$, which implies that $p(fx_{2n},gx_{2n+1}) \leq p(fx_{2n},y_{2n+1}) = p(y_{2n},y_{2n+1})$.

By Remark 3.1. we have
$p(fx_{2n},gx_{2n+1}) \leq p(y_{2n},y_{2n+1}) \leq qH_p(Fx_{2n},Gx_{2n-1})$, and hence from Case 1. we have
$p(fx_{2n},gx_{2n+1}) \leq tp(fx_{2n},gx_{2n-1})$.
Also, if $p(gx_{2n-1},fx_{2n}) \in Q_1 \times P_0$, then $p(gx_{2n-1},fx_{2n}) \leq tp(gx_{2n-1},fx_{2n-2})$.

Case 3. If $(fx_{2n},gx_{2n+1}) \in P_1 \times Q_0$, then $gx_{2n-1} = y_{2n-1}$ and
$p(gx_{2n-1},fx_{2n}) + p(fx_{2n},y_{2n})$
$= p(gx_{2n-1},y_{2n}) + p(fx_{2n},fx_{2n})$.
If we assume that $p(fx_{2n},gx_{2n+1}) \leq p(gx_{2n-1},y_{2n})$, then

$$p(fx_{2n},gx_{2n+1} \leq p(gx_{2n-1},fx_{2n-2})). \tag{4}$$

We suppose now that $p(fx_{2n},gx_{2n+1}) > p(gx_{2n-1},y_{2n})$. Clearly from (4) we have two possibilities:
$p(fx_{2n},gx_{2n-1}) < \frac{1}{2}p(gx_{2n-1},y_{2n})$ or
$p(fx_{2n},y_{2n}) < \frac{1}{2}p(gx_{2n-1},y_{2n})$.
Proceeding as in Case 1. gives:
$p(fx_{2n},gx_{2n+1})$

$$\begin{aligned}
&\leq qH_p(Fx_{2n},Gx_{2n-1})\\
&\leq t\max\Big\{\frac{1}{a}p(fx_{2n},gx_{2n-1}),p(fx_{2n},Fx_{2n}),\\
&\quad p(g_{2n-1},Gx_{2n-1}),\frac{1}{a+h}[p(fx_{2n},Gx_{2n-1})\\
&\quad +p(gx_{2n-1},Fx_{2n})]\}\\
&\leq t\max\Big\{\frac{1}{a}p(fx_{2n},y_{2n-1}),p(fx_{2n},y_{2n+1}),\\
&\quad p(gx_{2n-1},y_{2n}),\\
&\quad \frac{1}{a+h}[p(gx_{2n-1},y_{2n})+p(gx_{2n-1},y_{2n+1})]\Big\}
\end{aligned}$$

Since
$p(fx_{2n},gx_{2n-1}) < p(gx_{2n-1},y_{2n}) < p(fx_{2n},gx_{2n+1})$,
then:
$\frac{1}{a+h}[p(gx_{2n-1},y_{2n}) + p(gx_{2n-1},y_{2n+1})]$

$$\begin{aligned}
&= \frac{1}{a+h}[p(gx_{2n-1},y_{2n}) + p(gx_{2n-1},fx_{2n})\\
&\quad + p(fx_{2n},gx_{2n+1})]\\
&\leq \frac{1}{a+h}[p(fx_{2n},gx_{2n+1}) + p(fx_{2n},gx_{2n+1})\\
&\quad + p(fx_{2n},gx_{2n+1})]\\
&< p(fx_{2n},gx_{2n+1}).
\end{aligned}$$

and therefore we have $p(fx_{2n},gx_{2n+1}) \leq tp(fx_{2n},gx_{2n-1})$.
Now proceeding as earlier, we also obtain
$p(gx_{2n-1},fx_{2n}) \leq tp(gx_{2n-1},fx_{2n-2})$.
Thus, if we set $fx_{2n} = z_{2n}$, $gx_{2n+1} = z_{2n+1}$, then,
$p(z_n,z_{n+1}) \leq t\max\{p(z_{n-1},z_n),p(z_{n-2},z_{n-1})\}$ for $n \geq 2$.

Using the approach of Ćirić [3] it can be shown that $\{z_n\}$ is a Cauchy sequence, and therefore converges to a point s in K. It follows that $\{y_{2n}\}$ converges to $s \in K$.
For each y_{2n} denote by Y_{2n} one of the subsets $\{Gx_{2n-1}\}$ which contains y_{2n} and y_{2n+1} denote by Y_{2n+1} one of the subsets $\{Fx_{2n}\}$ which contains y_{2n+1}. Then

$$H_p(Y_n,Y_{n+1}) \leq hp(z_{2n-1},z_{2n}),$$

Also,

$$H_p(Y_{n-1},Y_{n+1}) \leq h\max\{p(z_{n-2},z_{n-1}),p(z_{n-1},z_n)\}.$$

Since $\{z_n\}$ is a Cauchy sequence so is $\{Y_n\}$. Therefore, $\{Y_n\}$ converges
to Y, for some Y in $CB^p(X)$, as $(CB^p(X),H_p)$ is a complete partial metric space. So we have

$$\begin{aligned}
p(s,Y) &\leq p(s,z) \quad \text{for some} \quad z \in Y\\
&\leq p(s,y_n) + p(y_n,z) - p(y_n,y_n)\\
&\leq p(s,y_n) + H_p(Y_n,Y) - H_p(Y_n,Y_n)\\
&\leq p(s,s) \quad \text{as} \quad n \quad \text{approaches} \quad \infty.
\end{aligned}$$

Therefore,

$$p(s,Y) = p(s,s) \Rightarrow s \in Y, \quad \text{since} \quad Y \quad \text{is closed.} \tag{5}$$

By the construction of the sequence there exists at least one subsequence $\{fx_{2n_k}\}$ or $\{gx_{2n_k+1}\}$ which is contained in P_0 or Q_0 respectively. Consequently subsequence $\{gx_{2n_k+1}\}$ which is contained in Q_0 for each $k \in \mathbb{N}$, converges to z. Since $gx_{2n+1} = y_{2n+1}$, $\{ggx_{2n_k+1}\}$ and $\{Ggx_{2n_k-1}\}$ are well defined.
Set

$$L_j = p(gx_{2n_k+1},Ggx_{2n_k-1})$$

and

$$R_j = H_p(Gx_{2n_k-1},Ggx_{2n_k-1}),$$

then

$$\begin{aligned}
R_j &\leq H_p(Y_{2n_k},Y_{2n_k+1}) + H_p(Fx_{2n_k},Ggx_{2n_k-1})\\
&\leq H_p(Y_{2n_k},Y_{2n_k+1}) + h\max\{p(fx_{2n_k},ggx_{2n_k-1}),\\
&\quad p(fx_{2n_k},Fx_{2n_k}),p(ggx_{2n_k-1},Ggx_{2n_k-1}),\\
&\quad [p(fx_{2n_k},Ggx_{2n_k-1})+p(Fx_{2n_k},ggx_{2n_k-1})]\}\\
&\leq H_p(Y_{2n_k},Y_{2n_k+1}) + h\max\{p(y_{2n_k},ggx_{2n_k-1}),\\
&\quad p(y_{2n_k},y_{2n_k+1}),p(ggx_{2n_k-1},Gx_{2n_k-1})\\
&\quad +H_p(Gx_{2n_k-1},Ggx_{2n_k-1}),\\
&\quad [H_p(Gx_{2n_k-1},Ggx_{2n_k-1})+p(Fx_{2n_k},ggx_{2n_k-1})]\}\\
&\leq H_p(Y_{2n_k},Y_{2n_k+1}) + h\max\{p(gg_{2n_k-1},Ggx_{2n_k-1})\\
&\quad +H_p(Gg_{2n_k-1},Gx_{2n_k-1}),p(y_{2n_k},y_{2n_k+1}),\\
&\quad p(ggx_{2n_k-1},Gx_{2n_k-1}))+R_j,p(ggx_{2n_k-1},y_{2n_k+1})\\
&\quad +R_j\}.
\end{aligned}$$

Hence

$$Rj \leq H_p(Y_{2n_k}, Y_{2n_k+1}) + \frac{2}{3}\max\{p(ggx_{2n_k-1}, Ggx_{2n_k-1}) + R_j, p(y_{2n_k}, y_{2n_k+1}), p(ggx_{2n_k-1}, Gx_{2n_k-1}) + R_j, p(ggx_{2n_k-1}, y_{2n_k+1}) + R_j\}. \quad (6)$$

Therefore,
Since $Y_n \to Y$, $gx_{2n_k+1} \to s$ and as g is continuous, then the sequence $\{R_j\}$ is bounded and hence $\{\sup_j R_j\}$ is a convergent sequence. Also, since G and g are weakly compatible mappings (see Definition 2.8.), and as $gx_{2n_k+1} = y_{2n_k+1} \in K$, $y_{2n_k+1} \in GK \cap K$ and $\lim_{n\to\infty} p(gx_{2n_k+1}, y_{2n_k}) = 0$, we have

$$\limsup_{n\to\infty} L_j \leq \limsup_{n\to\infty} R_j, \quad (7)$$

and

$$\limsup_{n\to\infty} p(ggx_{2n_k+1}, gx_{2n_k-1}) \leq \limsup_{n\to\infty} R_j. \quad (8)$$

We denote $\limsup_{j\to\infty} R_j$ by R. Taking the upper limit in (6) and using (7), (8) yields

$$R \leq \frac{2}{3}\max\{R, R, H_p(Y, Y), R\}.$$

So we have $R \leq \frac{2}{3}R$, since $H_p(Y,Y) \leq R$. Therefore, $R = 0$. Using (8), we have $p(gs, s) = 0$. This implies $s = gs$ (see Definition 2.2.) (P3).
In the similar way one can obtain $fs = s$.
We now consider,

$$p(y_{2n_k+1}, Gs) \leq H_p(Fx_{2n_k}, Gs) \leq h\max\{p(fx_{2n_k}, gs), p(fx_{2n_k}, Fx_{2n_k}), p(gs, Gs), [p(fx_{2n_k}, Gs) + p(gs, Fx_{2n_k})]\}$$

Taking the limit as $j \to \infty$ and using (5) gives,
$p(s, Gs)$
$\leq \frac{2}{3}\max\{p(s,s), p(s,s), p(s,Gs), p(s,Gs) + p(s,s)\}$.
So we have $p(s,Gs) \leq \frac{2}{3}[p(s,Gs) + p(s,s)]$, since $p(s,s) \leq p(s,Gs)$.
Hence, $p(s,Gs) = 0 \Rightarrow p(s,Gs) = p(s,s)$, and therefore $s \in Gs$.
We next consider:

$$p(Fs, s) \leq H_p(Fs, Gs) \leq h\max\{p(s,s), p(s,Fs), p(s,s), p(s,s) + p(s,Fs)\}$$

and hence,

$$p(s, Fs) = 0 \Rightarrow p(s, Fs) = p(s,s) \Rightarrow s \in Fs.$$

Therefore, $s = fs = gs \in Fs \cap Gs$, as desired.

For $f = g$, Theorem 3.1. reduces to the following result, a generalization of a fixed point theorem due to Ćirić and Ume [4] to partial metric spaces.

Corollary 3.1. Let (X, p) be a metrically convex complete partial metric space. Let K be a non-empty closed subset of X. Let the mappings $F, G : K \to CB^p(X)$ and $f : K \to X$ satisfy the following condition:
$H(Fx, Gy)$

$$\leq h\max\left\{\frac{1}{a}p(fx, fy), p(fx, Fx), p(fy, Gy), \frac{1}{a+h}[p(fx, Gy) + p(fy, Fx)]\right\} \quad (9)$$

for all $x, y \in X$ with $x \neq y$,
where $0 < h < \frac{2}{3}$, $a \geq 1 + \frac{2h^2}{1+h}$, and

(i) $\partial K \subseteq fK, FK \cap K \subseteq fK, GK \cap K \subseteq fK$,
(ii) $fx \in \partial K \Rightarrow Fx \subseteq K, fx \in \partial K \Rightarrow Gx \subseteq K$,
(iii) $\{F, f\}$ and $\{G, f\}$ are weakly compatible mappings, and
(iv) f are continuous on K.

Then there exists a point s in K such that $s = fs \in Fs \cap Gs$.

For $F = G$ and $f = g = I_K$, Theorem 3.1. reduces to the following result, a generalization of a fixed point theorem due to Rhoades [1] to partial metric spaces.

Corollary 3.2. Let (X, p) be a metrically convex complete partial metric space. Let K be a non-empty closed subset of X. Let the mappings $F : K \to CB^p(X)$ satisfy the following condition:
$H(Fx, Gy)$

$$\leq h\max\left\{\frac{1}{a}p(x,y), p(x,Fx), p(y,Fy), \frac{1}{a+h}[p(x,Fy) + p(y,Fx)]\right\} \quad (10)$$

for all $x, y \in X$ with $x \neq y$,
where $0 < h < \frac{2}{3}$, $a \geq 1 + \frac{2h^2}{1+h}$, and $x \in \partial K \Rightarrow Fx \subseteq K$,
Then there exists a point s in K such that $s \in Fs$.

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