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Characterization of Γ-Semigroup by Intuitionistic N-Fuzzy Set (INFS) and its Level Set

Muhammad Akram[1,2,*], *Jacob Kavikumar*[1] *and Zaffar Iqbal*[2]

[1] Department of Mathematics and Statistics, Faculty of Science, Technology and Human Development, University Tun Hussein Onn Malaysia, Malaysia
[2] Department of Mathematics, University of Gujrat, Pakistan

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Abstract: Some characterizations of Γ-Semigroup by intuitionistic N-fuzzy sets have been given here. The concept of intuitionistic N-fuzzy set (INFS) and its level set has been applied to Γ-semigroup. The notions of intuitionistic N-fuzzy Γ-subsemigroup and intuitionistic N-fuzzy Γ- ideals (left, right, lateral, quasi, and bi) have been introduced and characterized by intuitionistic N-fuzzy sets.

Keywords: Γ-smigroup, intuitionistic N-fuzzy set, intuitionistic N-fuzzy Γ-subsmigroup, intuitionistic N-fuzzy Γ-ideal.

1 Introduction

The notion of Γ-semigroup was introduced by Sen [14] in 1981 as a generalization of semigroup and ternary semigroup. This concept was modified by Sen and Saha [15] in 1986. Saha [12,13], extended many classical notions of semigroups to Γ-semigroups. Chinram [5,6], worked on bi-Γ-ideals and quasi-Γ-ideals of Γ-semigroups. Many researchers worked on this structure and proposed many theories related to Γ-semigroups.

The concept of fuzzy set was given by Zadeh [17] in 1965. The applications of fuzzy sets have been found very useful in the domain of mathematics and elsewhere. The concept of fuzzy algebraic structure was given by Rosenfeld [11] in 1971, when he applied the fuzzy set theory to the algebraic structures of subgroup (subgroupoid) and its ideals. In 1986, Atanassov [3] introduced the notion of intuitionistic fuzzy set as a generalization of fuzzy set. Biswas [4], introduced the notion of intuitionistic fuzzy subgroupoids. Kim and Jun [10] developed the theory of intuitionistic fuzzy ideals of semigroups. A number of authors have applied the concept of fuzzy set and intuitionistic fuzzy set to many other algebraic structures.

A fuzzy set is a function $\mu : X \to [0,1]$, whereas an intuitionistic fuzzy set is a pair of two fuzzy sets i.e. an intuitionistic fuzzy set A is written as $A = (\mu_A, \nu_A)$, where, $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ with $0 \le \mu_A(x) + \nu_A(x) \le 1$, for all $x \in X$. Both fuzzy set and intuitionistic fuzzy set relied on spreading the positive meaning of information but there is no any suggestions for use of the negative meanings of informations.

In 2009, Jun et al. [8] constructed a new function which is called negative-valued function (or negative fuzzy set, briefly, N-fuzzy set) and constructed N-structures. Jun et al. [9] applied the concept of coupled N-structures to BCK/BCI -algebras in 2013, which was extended to d-algebras and BH-algebras by Ahan et al. and Seo et al. The notion of intuitionistic N-fuzzy set was applied to biΓ-ternary semigroup by Akram et al. [2].

In this paper the concept of intuitionistic N-fuzzy set has been applied to Γ-semigroup. The notions of intuitionistic N-fuzzy Γ-subsemigroup, intuitionistic N-fuzzy Γ-left (right, quasi and bi) ideals have been defined. The relationship between these ideals have been discussed and the characterizations of Γ-semigroup by these ideals have been investigated here.

2 Preliminary Concepts

Definition 1.[15] Let $S = \{x, y, z, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two nonempty sets. Then S is called a Γ-semigroup if it satisfies:

* Corresponding author e-mail: makram_69@yahoo.com

(i) $x\alpha y \in S$
(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$, for all $x,y,z \in S$ and $\alpha,\beta \in \Gamma$.

Example 1. *Let $S = \{-i,0,i\}$ and $\Gamma = S$. Then S is a Γ-semigroup under the multiplication of complex numbers, while S is not a semigroup under multiplication of complex numbers.*

Example 2.[15] *Let $S = \{4n+3, n \in N\}$ and $\Gamma = \{4n+1, n \in N\}$. Define the mapping $S \times \Gamma \times S \to S$ as $x\alpha y = x+\alpha+y$. Then S is a Γ-semigroup.*

Example 3. *Let $S = Z^-$ and $\Gamma \subseteq Z^-$. Define $x\alpha y$, for $x,y \in S$ and $\alpha \in \Gamma$ as the usual multiplication of integers. Then S is a Γ-semigroup but not a semigroup.*

Example 4. *Let $S = iR$, where, $i = \sqrt{-1}$ and R is the set of real numbers. If $\Gamma = [i,-i]$ and $x\alpha y$ is defined as the usual multiplication of complex numbers. Then S is a Γ-semigroup but not a semigroup.*

Definition 2.[15] A nonempty subset A of a Γ-semigroup S is called a Γ-subsemigroup of S if, $A\Gamma A \subseteq A$.

Definition 3.[15] A nonempty subset A of a Γ-semigroup S is called a Γ-left ideal of S if, $S\Gamma A \subseteq A$.

Definition 4.[15] A nonempty subset A of a Γ-semigroup S is called a Γ-right ideal of S if, $A\Gamma S \subseteq A$.

Definition 5.[15] A nonempty subset A of a Γ-semigroup S is called a Γ-ideal of S if it is a Γ-left and a Γ-right ideal of S.

Definition 6.[5] Let S be a Γ-semigroup. A nonempty subset Q of S is called Γ-quasi-ideal of S if $Q\Gamma S \cap S\Gamma Q \subseteq Q$.

Definition 7.[6] Let S be a Γ-semigroup. A Γ-subsemigroup B of S is called a Γ-bi-ideal of S if $B\Gamma S\Gamma B \subseteq B$.

Proposition 1.[7] Let S be a Γ-semigroup and $\phi \neq X \subseteq S$, then:

(i) $S\Gamma X$ is a Γ-left ideal of S.
(ii) $X\Gamma S$ is a Γ-right ideal of S.
(iii) $S\Gamma X\Gamma S$ is a Γ-ideal of S.
(iv) $S\Gamma X \cap X\Gamma S$ is a Γ-quasi-ideal of S.

3 Institutionistic N-fuzzy set

Definition 8.[8] A negative fuzzy set (briefly, N-fuzzy set) in a nonempty set X is a function $\overline{\mu} : X \to [-1,0]$. Here we are using $''-''$ for the negative fuzzy function.

Jun et al. [8] used the term negative-valued function and N-function for negative fuzzy set and N-fuzzy set.

Definition 9.[2] An intuitionistic N-fuzzy set (briefly, INFS) A in a nonempty set X is an object of the form $A = \{\langle x; \overline{\mu}_A, \overline{\gamma}_A : x \in X\rangle\}$, where $\overline{\mu}_A : X \to [-1,0]$ and $\overline{\gamma}_A : X \to [-1,0]$ such that $-1 \leq \overline{\mu}_A(x) + \overline{\gamma}_A(x) \leq 0$ for all $x \in X$. An intuitionistic N-fuzzy set $A = \{\langle x; \overline{\mu}_A, \overline{\gamma}_A : x \in X\rangle\}$ in X can be identified to an ordered pair $(\overline{\mu}_A, \overline{\gamma}_A)$ in $F(X,[-1,0]) \times F(X,[-1,0])$, where $F(X,[-1,0])$ denotes the set of all functions from X to $[-1,0]$. For the sake of simplicity, we shall use the notation $A = (\overline{\mu}_A, \overline{\gamma}_A)$ instead of $A = \{\langle x; \overline{\mu}_A, \overline{\gamma}_A : x \in X\rangle\}$.

Definition 10.[2] Let $A = (\overline{\mu}_A, \overline{\gamma}_A)$ be an INFS in X. Then the set $N\{(\overline{\mu}_A, \overline{\gamma}_A);(t,s)\} = \{x \in X | \overline{\mu}_A(x) \leq t, \overline{\gamma}_A(x) \geq s\}$ where $t,s \in [-1,0]$ with $t+s \geq -1$ is called an $N(t,s)$-level set of A. An $N(t,t)$-level set of $A = (\overline{\mu}_A, \overline{\gamma}_A)$ is called an N-level set of A.
For simplicity, we shall use the notation $N_A(t,s)$ instead of $N\{(\overline{\mu}_A, \overline{\gamma}_A);(t,s)\}$ for $N(t,s)$-level set of $A = (\overline{\mu}_A, \overline{\gamma}_A)$, *i.e.* $N_A(t,s) = \{x \in X | \overline{\mu}_A(x) \leq t, \overline{\gamma}_A(x) \geq s\}$.

Definition 11.[2] Let $A = (\overline{\mu}_A, \overline{\gamma}_A)$ and $B = (\overline{\mu}_B, \overline{\gamma}_B)$ be two INFSs in X. If for all $x \in X$, $\overline{\mu}_A(x) \geq \overline{\mu}_B(x)$ and $\overline{\gamma}_A(x) \leq \overline{\gamma}_B(x)$, then A is called an intuitionistic N-fuzzy subset (INFSS) of B and is written as $A \overline{\subseteq} B$. We say $A = B$ if and only if $A \overline{\subseteq} B$ and $B \overline{\subseteq} A$.

Definition 12.[2] Let $A = (\overline{\mu}_A, \overline{\gamma}_A)$ and $B = (\overline{\mu}_B, \overline{\gamma}_B)$ be two INFSs in X. Then their union and intersection is also an intuitionistic N-fuzzy set in X, defined as for all $x \in X$

$$\begin{aligned}(A \overline{\cup} B) &= \{x,\ \min\{\overline{\mu}_A(x), \overline{\mu}_B(x)\}, \max\{\overline{\gamma}_A(x), \overline{\gamma}_B(x)\}\} \\ &= (\overline{\mu}_A(x) \wedge \overline{\mu}_B(x),\ \overline{\gamma}_A(x) \vee \overline{\gamma}_B(x)), \\ \text{and } (A \overline{\cap} B) &= \{x,\ \max\{\overline{\mu}_A(x), \overline{\mu}_B(x)\}, \min\{\overline{\gamma}_A(x), \overline{\gamma}_B(x)\}\} \\ &= (\overline{\mu}_A(x) \vee \overline{\mu}_B(x),\ \overline{\gamma}_A(x) \wedge \overline{\gamma}_B(x)).\end{aligned}$$

Example 5. *Let $X = \{w,x,y,z\}$ be a nonempty set. Define $\overline{\mu}_A : X \to [-1,0]$ and $\overline{\gamma}_A : X \to [-1,0]$ as, $\overline{\mu}_A(w) = -0.6$, $\overline{\mu}_A(x) = -0.2$, $\overline{\mu}_A(y) = -0.5$, $\overline{\mu}_A(z) = -0.4$ and $\overline{\gamma}_A(w) = -0.3$, $\overline{\gamma}_A(x) = -0.7$, $\overline{\gamma}_A(y) = -0.3$, $\overline{\gamma}_A(z) = -0.6$, then $A = \{< w,-0.6,-0.3 >, < x,-0.2,-0.7 >, < y,-0.5,-0.3 >, < z,-0.4,-0.6 >\}$. It is easy to verify that $A = (\overline{\mu}_A, \overline{\gamma}_A)$ is an intuitionistic N-fuzzy set in X.*

Example 6. *Let X is as in Example 5 and*
$A = \{< w,-.9,-.1 >, < x,-0.5,-0.4 >, < y,-0.4,-0.5 >, < z,-0.3,-0.6 >\}$,
$B = \{< w,-1,0 >, < x,-0.9,-0.1 >, < y,-0.7,-0.2 >, < z,-0.6,-0.3 >\}$.
Then $A = (\overline{\mu}_A, \overline{\gamma}_A)$ and $B = (\overline{\mu}_B, \overline{\gamma}_B)$ are intuitionistic N-fuzzy sets in X. Easily we can verify that $A \overline{\subseteq} B$.

Example 7. *Let X is as in Example 5 and*
$A = \{< w,-0.8,-0.1 >, < x,-0.6,-0.2 >, < y,-0.5,-0.6 >, < z,-0.4,-0.5 >\}$
$B = \{< w,-0.6,-0.3 >, < x,-0.7,-0.3 >, < y,-0.4,-0.5 >, < z,-0.5,-0.3 >\}$.

Then $A = (\overline{\mu}_A, \overline{\gamma}_A)$ and $B = (\overline{\mu}_B, \overline{\gamma}_B)$ are intuitionistic N-fuzzy sets in X and

$A \bigcup B = \{< w, -0.8, -0.1 >, < x, -0.7, -0.2 >, < y, -0.5, -0.5 >, < z, -0.5, -0.3 >\}$

$A \bigcap B = \{< w, -0.6, -0.3 >, < x, -0.6, -0.3 >, < y, -0.4, -0.6 >, < z, -0.4, -0.5 >\}$.

Obviously, $A \bigcup B$ and $A \bigcap B$ are intuitionistic N-fuzzy sets in X.

Definition 13.[2] Let C be a nonempty subset of X. Then the intuitionistic N-fuzzy characteristic function of C is a function $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C}, \overline{\gamma}_{\overline{\chi}_C})$ defined as, for any $x \in X$,

$$\overline{\mu}_{\overline{\chi}_C}(x) = \begin{bmatrix} -1, \text{ if } x \in C \\ 0, \text{ if } x \notin C \end{bmatrix} \text{ and } \overline{\gamma}_{\overline{\chi}_C}(x) = \begin{bmatrix} 0, \text{ if } x \in C \\ -1, \text{ if } x \notin C \end{bmatrix}.$$

We denotes the intuitionistic N-fuzzy characteristic function of X by $\overline{\chi}_X = (\overline{\mu}_X, \overline{\gamma}_X)$.

4 Intuitionistic N-fuzzy Γ-semigroups

In what fallows, let S denotes a Γ-semigroup unless otherwise specified.

Definition 14. *Let $A = (\overline{\mu}_A, \overline{\gamma}_A)$ and $B = (\overline{\mu}_B, \overline{\gamma}_B)$ be the two INFSs in S. Then their product is defined as, $A \circ B = (\overline{\mu}_A \circ_\Gamma \overline{\mu}_B, \overline{\gamma}_A \circ_\Gamma \overline{\gamma}_B) = (\overline{\mu}_{A \circ_\Gamma B}, \overline{\gamma}_{A \circ_\Gamma B})$, where for any $s \in S$,*

$$(\overline{\mu}_{A \circ_\Gamma B})(s) =$$

$$\begin{bmatrix} \bigwedge_{s=a\alpha b} \{\overline{\mu}_A(a) \vee \overline{\mu}_B(b)\}, \text{ if } s = a\alpha b, \text{ for } a, b \in S, \alpha \in \Gamma \\ 0, \qquad \text{otherwise.} \end{bmatrix}$$

and

$$(\overline{\gamma}_{A \circ_\Gamma B})(s) =$$

$$\begin{bmatrix} \bigvee_{s=a\alpha b} \{\overline{\gamma}_A(a) \wedge \overline{\gamma}_B(b)\}, \text{ if } s = a\alpha b, \text{ for } a, b \in S, \alpha \in \Gamma \\ -1, \qquad \text{otherwise.} \end{bmatrix}$$

Note 1. For any two INFSs $A = (\overline{\mu}_A, \overline{\gamma}_A)$ and $B = (\overline{\mu}_B, \overline{\gamma}_B)$ in S, $\overline{\mu}_A \circ_\Gamma \overline{\mu}_B = \overline{\mu}_{A \circ_\Gamma B}$ and $\overline{\gamma}_A \circ_\Gamma \overline{\gamma}_B = \overline{\gamma}_{A \circ_\Gamma B}$.

Definition 15. An INFS, $A = (\overline{\mu}_A, \overline{\gamma}_A)$ in S is called an intuitionistic N-fuzzy Γ-subsemigroup of S if,

$$\overline{\mu}_A(x\alpha y) \leq \max\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} \text{ and}$$
$$\overline{\gamma}_A(x\alpha y) \geq \min\{\overline{\gamma}_A(x), \overline{\gamma}_A(y)\}, \text{ for all } x, y \in S, \alpha \in \Gamma.$$

Definition 16. An INFS, $L = (\overline{\mu}_L, \overline{\gamma}_L)$ in S is called an intuitionistic N-fuzzy Γ-left ideal of S if,

$$\overline{\mu}_L(x\alpha y) \leq \overline{\mu}_L(y) \text{ and } \overline{\gamma}_L(x\alpha y) \geq \overline{\gamma}_L(y),$$

for all $x, y \in S, \alpha \in \Gamma$.

Definition 17. An INFS, $R = (\overline{\mu}_R, \overline{\gamma}_R)$ in S is called an intuitionistic N-fuzzy Γ-right ideal of S if,

$$\overline{\mu}_R(x\alpha y) \leq \overline{\mu}_R(x), \text{ and } \overline{\gamma}_R(x\alpha y) \geq \overline{\gamma}_R(x),$$

for all $x, y \in S, \alpha \in \Gamma$.

Definition 18. An INFS, $A = (\overline{\mu}_A, \overline{\gamma}_A)$ in S is called an intuitionistic N-fuzzy Γ-ideal of S if it is an intuitionistic N-fuzzy Γ-left and an intuitionistic N-fuzzy Γ-right ideal of S.

Example 8. *Let $S = Z^-$ and $\Gamma \subseteq Z^-$. Then S is a Γ-semigroup but not a semigroup under the operation defined as, for $x, y \in S$, $\alpha \in \Gamma$, $(x\alpha y) = x\alpha y$, the usual multiplication of integers. Now define, $\overline{\mu}_A : S \to [-1, 0]$ and $\overline{\gamma}_A : S \to [-1, 0]$ as, for $x \in S$,*

$$\overline{\mu}_A(x) = \begin{bmatrix} -0.5, \text{ if } x \text{ is even} \\ -0.1, \text{ otherwise} \end{bmatrix} \text{ and}$$

$$\overline{\gamma}_A(x) = \begin{bmatrix} -0.3, \text{ if } x \text{ is even} \\ -0.7, \text{ otherwise} \end{bmatrix}.$$

Then $A = (\overline{\mu}_A, \overline{\gamma}_A)$ is an intuitionistic N-fuzzy set in S. By simple calculations we can verify that A is an intuitionistic N-fuzzy Γ-subsemigroup of S.

Note 2. In a Γ-semigroup S,

(i) An intuitionistic N-fuzzy Γ-left (right) ideal of S is an intuitionistic N-fuzzy Γ-subsemigroup of S but the converse is not true.

(ii) An intuitionistic N-fuzzy Γ-left ideal of S may not be an intuitionistic N-fuzzy Γ-right ideal of S and vice versa.

Example 9. *Let $S = \{a, b, c\}$ and $\Gamma = \{\alpha\}$. Then S is a Γ-semigroup under the operation defined in the following table,*

α	a	b	c
a	a	a	a
b	a	b	b
c	a	c	c

Define, $\overline{\mu}_A : T \to [-1, 0]$ and $\overline{\gamma}_A : T \to [-1, 0]$ such that $A = (\overline{\mu}_A, \overline{\gamma}_A) = \{< a, -0.3, -0.1 >, < b, -0.7, -0.2 >, < c, -0.5, -0.5 >\}$. Then A is an intuitionistic N-fuzzy set in S which is an intuitionistic N-fuzzy Γ-subsemigroup of S. Now $\overline{\mu}_A(c\alpha b) = \overline{\mu}_A(c) = -0.5 \nleq -0.7 = \overline{\mu}_A(b)$. Hence A is not an intuitionistic N-fuzzy Γ-left ideal of S. If we take $B = (\overline{\mu}_B, \overline{\gamma}_B) = \{< a, -0.7, -0.2 >, < b, -0.5, -0.4 >, < c, -0.5, -0.4 >\}$, then B is an intuitionistic N-fuzzy Γ-left and a Γ-right of S, hence an intuitionistic N-fuzzy Γ-ideal of S. Obviously it is an intuitionistic N-fuzzy Γ-subsemigroup of S.

Example 10. *Let $S = \{2n, n \in N\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$. Define $(x\alpha y) = 2x + y$, for $x, y \in S, \alpha \in \Gamma$,then S is a Γ-semigroup. Now define, $\overline{\mu}_A : S \to [-1, 0]$ and $\overline{\gamma}_A : S \to [-1, 0]$ as*

$$\overline{\mu}_A(x) = \begin{bmatrix} -0.3, \text{ if } x = 4n \text{ for some } n \in N \\ -0.6, \qquad \text{otherwise} \end{bmatrix} \text{ and}$$

$$\overline{\gamma}_A(x) = \begin{bmatrix} -0.6, & \text{if } x = 4n \text{ for some } n \in N \\ -0.1, & \text{otherwise} \end{bmatrix}.$$

Then $A = (\overline{\mu}_A, \overline{\gamma}_A)$ is an intuitionistic N-fuzzy Γ-left ideal of S.
Now, for $2,4 \in S, \alpha \in \Gamma$, $2\alpha 4 = 2(2)+4 = 8 = 4(2)$ $\Rightarrow \overline{\mu}_A(2\alpha 4) = \overline{\mu}_A(4(2)) = -0.3$ and $\overline{\mu}_A(2) = -0.6$, which implies that $\overline{\mu}_A(2\alpha 4) \nleq \overline{\mu}_A(2)$ also $\overline{\gamma}_A(2\alpha 4) \ngeq \overline{\gamma}_A(2)$. Hence $A = (\overline{\mu}_A, \overline{\gamma}_A)$ is not an intuitionistic N-fuzzy Γ-right ideal of S. If we define $(x\alpha y) = x+2y$ then $A = (\overline{\mu}_A, \overline{\gamma}_A)$ is an intuitionistic N-fuzzy Γ-right ideal of S but not an intuitionistic N-fuzzy Γ-left ideal of S. Further if $(x\alpha y) = 2x+2y$, then $A = (\overline{\mu}_A, \overline{\gamma}_A)$ is an intuitionistic N-fuzzy Γ-ideal of S.

Lemma 1. Let S be a Γ-semigroup then,
(i) The intersection of any collection of intuitionistic N-fuzzy Γ-subsemigroups of S is an intuitionistic N-fuzzy Γ-subsemigroup of S.
(ii) The intersection of any collection of intuitionistic N-fuzzy Γ-left (right) ideals of S is an intuitionistic N-fuzzy Γ-left (right) ideal of S.

Proof.(i) Let $\{A_i = (\overline{\mu}_{Ai}, \overline{\gamma}_{Ai}), i \in I\}$ be a collection of intuitionistic N-fuzzy Γ-subsemigroups of S. Then $\overline{\mu}_{Ai}(x\alpha y) \leq \max\{\overline{\mu}_{Ai}(x), \overline{\mu}_{Ai}(y)\}$

$$\text{and } \overline{\gamma}_{Ai}(x\alpha y) \geq \min\{\overline{\gamma}_{Ai}(x), \overline{\gamma}_{Ai}(y)\},$$

for all $i \in I$, $x,y \in S, \alpha \in \Gamma$.
Now, for all for $i \in I$, $x,y \in S, \alpha \in \Gamma$.

$$\begin{aligned}(\bigcap_{i\in I}\overline{\mu}_{Ai})(x\alpha y) &= \bigcap_{i\in I}(\overline{\mu}_{Ai}(x\alpha y)) \leq \bigcap_{i\in I}(\max\{\overline{\mu}_{Ai}(x), \overline{\mu}_{Ai}(y)\}) \\ &= \max\{\bigcap_{i\in I}\overline{\mu}_{Ai}(x), \bigcap_{i\in I}\overline{\mu}_{Ai}(y)\}. \\ &= \max\{(\bigcap_{i\in I}\overline{\mu}_{Ai})(x), (\bigcap_{i\in I}\overline{\mu}_{Ai})(y)\}. \text{ Also,}\end{aligned}$$

$$\begin{aligned}(\bigcap_{i\in I}\overline{\gamma}_{Ai})(x\alpha y) &= \bigcap_{i\in I}(\overline{\gamma}_{Ai}(x\alpha y)) \geq \bigcap_{i\in I}(\min\{\overline{\gamma}_{Ai}(x), \overline{\gamma}_{Ai}(y)\}) \\ &= \min\{\bigcap_{i\in I}\overline{\gamma}_{Ai}(x), \bigcap_{i\in I}\overline{\gamma}_{Ai}(y)\}. \\ &= \min\{(\bigcap_{i\in I}\overline{\gamma}_{Ai})(x), (\bigcap_{i\in I}\overline{\gamma}_{Ai})(y)\}.\end{aligned}$$

Hence $\bigcap_{i\in I} A_i$ is an intuitionistic N-fuzzy Γ-subsemigroup of S.
(ii) proof is similar as (i).

Proposition 2. Let $A = (\overline{\mu}_A, \overline{\gamma}_A)$ be an INFS in S then
(i) A is an intuitionistic N-fuzzy Γ-subsemigroup of S if and only if

$$\overline{\mu}_A \circ_\Gamma \overline{\mu}_A \geq \overline{\mu}_A \text{ and } \overline{\gamma}_A \circ_\Gamma \ \overline{\gamma}_A \leq \ \overline{\gamma}_A.$$

(ii) A is an intuitionistic N-fuzzy Γ-left ideal of S if and only if

$$\overline{\mu}_T \circ_\Gamma \overline{\mu}_A \geq \overline{\mu}_A \text{ and } \overline{\gamma}_T \circ_\Gamma \ \overline{\gamma}_A \leq \ \overline{\gamma}_A.$$

(iii) A is an intuitionistic N-fuzzy Γ-right ideal of S if and only if

$$\overline{\mu}_A \circ_\Gamma \overline{\mu}_T \geq \overline{\mu}_A \text{ and } \ \overline{\gamma}_A \circ_\Gamma \overline{\gamma}_T \leq \ \overline{\gamma}_A.$$

Proof. (i) Let A is an intuitionistic N-fuzzy Γ-subsemigroup of S and $x \in S$.
Case1. If $x \neq a\alpha b$, for $\alpha \in \Gamma$, $a,b \in S$, then

$$(\overline{\mu}_{A\circ_\Gamma A})(x) = 0 \geq (\overline{\mu}_A)(x) \text{ and } (\overline{\gamma}_{A\circ_\Gamma A})(x) = -1 \leq (\overline{\gamma}_A)(x).$$

Case2. If $x = a\alpha b$, for $\alpha \in \Gamma$ and $a,b \in S$, then

$$\begin{aligned}(\overline{\mu}_{A\circ_\Gamma A})(x) &= \min_{x=a\alpha b}\{\max((\overline{\mu}_A)(a), (\overline{\mu}_A)(b))\} \\ &\geq \min_{x=a\alpha b}(\overline{\mu}_A)(a\alpha b) \\ &\geq (\overline{\mu}_A)(x), \forall x \in S. \text{ Also}\end{aligned}$$

$$\begin{aligned}(\overline{\gamma}_{A\circ_\Gamma A})(x) &= \max_{x=a\alpha b}\{\min((\overline{\gamma}_A)(a), (\overline{\gamma}_A)(b))\} \\ &\leq \max_{x=a\alpha b}(\overline{\gamma}_A)(a\alpha b) \\ &\leq (\overline{\gamma}_A)(x), \ \forall x \in S.\end{aligned}$$

This implies that $\overline{\mu}_A \circ_\Gamma \overline{\mu}_A \geq \overline{\mu}_A$ and $\overline{\gamma}_A \circ_\Gamma \overline{\gamma}_A \leq \overline{\gamma}_A$.
Conversely, we suppose that $\overline{\mu}_A \circ_\Gamma \overline{\mu}_A \geq \overline{\mu}_A$ and $\overline{\gamma}_A \circ_\Gamma \overline{\gamma}_A \leq \overline{\gamma}_A$. Let, $a,b \in S$, $\alpha \in \Gamma$ and $x = a\alpha b$ then

$$\begin{aligned}(\overline{\mu}_A)(a\alpha b) = (\overline{\mu}_A)(x) &\leq (\overline{\mu}_{A\circ_\Gamma A})(x) \\ &= \min_{x=u\delta v}\{\max((\overline{\mu}_A)(u), (\overline{\mu}_A)(v))\} \\ &\leq \max((\overline{\mu}_A)(a), (\overline{\mu}_A)(b)). \text{ Also,}\end{aligned}$$

$$\begin{aligned}(\overline{\gamma}_A)(a\alpha b) = (\overline{\gamma}_A)(x) &\geq (\overline{\gamma}_{A\circ_\Gamma A})(x) \\ &= \max_{x=u\delta v}\{\min((\overline{\gamma}_A)(u), (\overline{\gamma}_A)(v))\} \\ &\geq \min((\overline{\gamma}_A)(a), (\overline{\gamma}_A)(b)).\end{aligned}$$

Hence A is an intuitionistic N-fuzzy Γ-subsemigroup of S. Similarly, we can prove (ii) and (iii).

Lemma 2. Let $A = (\overline{\mu}_A, \overline{\gamma}_A)$ be an INFS in S then

(i) $S \circ_\Gamma A$ is an intuitionistic N-fuzzy Γ-left ideal of S.
(ii) $A \circ_\Gamma S$ is an intuitionistic N-fuzzy Γ-right ideal of S.

Proof. (i) Let $L = S \circ_\Gamma A = (\overline{\mu}_{S\circ_\Gamma A}, \overline{\gamma}_{S\circ_\Gamma A}) = (\overline{\mu}_L, \overline{\gamma}_L)$. Now as

$$\begin{aligned}\overline{\mu}_S \circ_\Gamma \overline{\mu}_L &= \overline{\mu}_{S\circ_\Gamma L} = \overline{\mu}_{S\circ_\Gamma S\circ_\Gamma A} \ \ (\text{since} S \circ_\Gamma S \subseteq S) \\ &\geq \overline{\mu}_{S\circ_\Gamma A} = \overline{\mu}_L \text{ implies that } \overline{\mu}_S \circ_\Gamma \overline{\mu}_L \geq \overline{\mu}_{L.}\end{aligned}$$

Similarly, $\gamma_T \circ_\Gamma \gamma_L \leq \overline{\gamma}_L$. Hence $L = S \circ_\Gamma A$ is an intuitionistic N-fuzzy Γ-left ideal of S. Similarly, we can prove (ii).

Theorem 1. Let $A = (\overline{\mu}_A, \overline{\gamma}_A)$ be an INFS in S. Then A is an intuitionistic N-fuzzy Γ-subsemigroup of S if and only if $N_A(t,s)$ is a Γ-subsemigroup of S, for all $t,s \in [-1,0]$ with $t+s \geq -1$.

Proof. Let $A = (\overline{\mu}_A, \overline{\gamma}_A)$ be an intuitionistic N-fuzzy Γ-subsemigroup of S. Let $x,y \in N_A(t,s)$, where $t,s \in [-1,0]$ with $t+s \geq -1$ then $\overline{\mu}_A(x) \leq t, \overline{\gamma}_A(x) \geq s$ and $\overline{\mu}_A(y) \leq t, \overline{\gamma}_A(y) \geq s$. Now for $\alpha \in \Gamma$

$$\overline{\mu}_A(x\alpha y) \leq \max\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} \leq t, \text{ and}$$
$$\overline{\gamma}_A(x\alpha y) \geq \min\{\overline{\gamma}_A(x), \overline{\gamma}_A(y)\} \geq s.$$

This implies that $x\alpha y \in N_A(t,s)$, for all $x,y \in N_A(t,s)$ and $\alpha \in \Gamma$ implies that $N_A(t,s)\Gamma N_A(t,s) \subseteq N_A(t,s)$. Hence $N_A(t,s)$ is a Γ-subsemigroup of S.

Conversely, we suppose that $N_A(t,s)$ is a Γ-subsemigroup of S, for all $t,s \in [-1,0]$ with $t+s \geq -1$. Let $x,y \in S$ such that $\overline{\mu}_A(x) = t_x, \overline{\gamma}_A(x) = s_x$ and $\overline{\mu}_A(y) = t_y, \overline{\gamma}_A(y) = s_y$ with $-1 \leq t_x + s_x \leq 0$ and $-1 \leq t_y + s_y \leq 0$ then $x \in N_A(t_x,s_x)$ and $y \in N_A(t_y,s_y)$. We may assume that $t_x \leq t_y$ and $s_x \geq s_y$ then $N_A(t_x,s_x) \subseteq N_A(t_y,s_y)$, which implies that $x,y \in N_A(t_y,s_y)$. Since, $N_A(t_y,s_y)$ is a Γ-subsemigroup of S implies that $x\alpha y \in N_A(t_y,s_y)$, for $\alpha \in \Gamma$. Then

$$\overline{\mu}_A(x\alpha y) \leq t_y = \max(t_x,t_y) = \max(\overline{\mu}_A(x), \overline{\mu}_A(y)), \text{ and}$$
$$\overline{\gamma}_A(x\alpha y) \geq s_y = \min(s_x,s_y) = \min(\overline{\gamma}_A(x), \overline{\gamma}_A(y)),$$

for all $x,y \in S$ and $\alpha \in \Gamma$. Hence $A = (\overline{\mu}_A, \overline{\gamma}_A)$ is an intuitionistic N-fuzzy Γ-subsemigroup of S.

Theorem 2. Let $A = (\overline{\mu}_A, \overline{\gamma}_A)$ be an INFS in S. Then A is an intuitionistic N-fuzzy Γ-left (right) ideal of S if and only if $N_A(t,s)$ is a Γ-left (right) ideal of S, for all $t,s \in [-1,0]$ with $t+s \geq -1$.

Proof. Straightforward.

Theorem 3. A nonempty subset C of S is a Γ-subsemigroup of S if and only if $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C}, \overline{\gamma}_{\overline{\chi}_C})$ is an intuitionistic N-fuzzy Γ-subsemigroup of S.

Proof. Let C be a Γ-subsemigroup of S then $C\Gamma C \subseteq C$. Let $x,y \in S$, $\alpha \in \Gamma$ then we have following cases.

Case1. If $x,y \in C$ then $x\alpha y \in C$ and hence $\overline{\mu}_{\overline{\chi}_C}(x) = \overline{\mu}_{\overline{\chi}_C}(y) = \overline{\mu}_{\overline{\chi}_C}(x\alpha y) = -1$ implies that $\overline{\mu}_{\overline{\chi}_C}(x\alpha y) = \max\{\overline{\mu}_{\overline{\chi}_C}(x), \overline{\mu}_{\overline{\chi}_C}(y)\}$. Also, $\overline{\gamma}_{\overline{\chi}_C}(x) = \overline{\gamma}_{\overline{\chi}_C}(y) = \overline{\gamma}_{\overline{\chi}_C}(x\alpha y) = 0$ implies that $\overline{\gamma}_{\overline{\chi}_C}(x\alpha y) = \min\{\overline{\gamma}_{\overline{\chi}_C}(x), \overline{\gamma}_{\overline{\chi}_C}(y)\}$.

Case2. If either $x \notin C$ or $y \notin C$ then either $\overline{\mu}_{\overline{\chi}_C}(x) = 0, \overline{\gamma}_{\overline{\chi}_C}(x) = -1$ or $\overline{\mu}_{\overline{\chi}_C}(y) = 0, \overline{\gamma}_{\overline{\chi}_C}(y) = -1$. This implies that, $\max\{\overline{\mu}_{\overline{\chi}_C}(x), \overline{\mu}_{\overline{\chi}_C}(y)\} = 0$ but $\overline{\mu}_{\overline{\chi}_C}(x\alpha y) \leq 0$ implies that $\overline{\mu}_{\overline{\chi}_C}(x\alpha y) \leq \max\{\overline{\mu}_{\overline{\chi}_C}(x), \overline{\mu}_{\overline{\chi}_C}(y)\}$. Also, either $\overline{\gamma}_{\overline{\chi}_C}(x) = -1$or $\overline{\gamma}_{\overline{\chi}_C}(y) = -1$, which implies that, $\min\{\overline{\gamma}_{\overline{\chi}_C}(x), \overline{\gamma}_{\overline{\chi}_C}(y)\} = -1$ but $\overline{\gamma}_{\overline{\chi}_C}(x\alpha y) \geq -1$ implies that $\overline{\gamma}_{\overline{\chi}_C}(x\alpha y) \geq \min\{\overline{\gamma}_{\overline{\chi}_C}(x), \overline{\gamma}_{\overline{\chi}_C}(y)\}$. This implies that $\overline{\mu}_{\overline{\chi}_C}(x\alpha y) \leq \max\{\overline{\mu}_{\overline{\chi}_C}(x), \overline{\mu}_{\overline{\chi}_C}(y)\}$ and $\overline{\gamma}_{\overline{\chi}_C}(x\alpha y) \geq \min\{\overline{\gamma}_{\overline{\chi}_C}(x), \overline{\gamma}_{\overline{\chi}_C}(y)\}$, for all $x,y \in T$, $\alpha \in \Gamma$.

Case3. When $x \notin C$ and $y \notin C$ gives the same result as in Case2.

Hence $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C}, \overline{\gamma}_{\overline{\chi}_C})$ is an intuitionistic N-fuzzy Γ-subsemigroup of S.

Conversely, we suppose that $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C}, \overline{\gamma}_{\overline{\chi}_C})$ is an intuitionistic N-fuzzy Γ-subsemigroup of S. Let $x,y \in C$ and $\alpha \in \Gamma$ then $x\alpha y \in C\Gamma C$. By definition of $\overline{\chi}_C$, $\overline{\mu}_{\overline{\chi}_C}(x) = \overline{\mu}_{\overline{\chi}_C}(y) = -1$ implies that $\max\{\overline{\mu}_{\overline{\chi}_C}(x), \overline{\mu}_{\overline{\chi}_C}(y)\} = -1$. Since $\overline{\chi}_C$ is an intuitionistic N-fuzzy biΓ-ternary subsemigroup of T then $\overline{\mu}_{\overline{\chi}_C}(x\alpha y) \leq \max\{\overline{\mu}_{\overline{\chi}_C}(x), \overline{\mu}_{\overline{\chi}_C}(y)\} = -1$ implies that $\overline{\mu}_{\overline{\chi}_C}(x\alpha y) \leq -1$ but by definition $\overline{\mu}_{\overline{\chi}_C}(x\alpha y) \geq -1$, which implies that $\overline{\mu}_{\overline{\chi}_C}(x\alpha y) = -1$. Similarly, we can show that $\overline{\gamma}_{\overline{\chi}_C}(x\alpha y) = 0$. This gives that $x\alpha y \in C$ implies that $C\Gamma C \subseteq C$. Hence C is a Γ-subsemigroup of S.

Theorem 4. A nonempty subset C of S is a Γ-left (right) ideal of S if and only if $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C}, \overline{\gamma}_{\overline{\chi}_C})$ is an intuitionistic N-fuzzy Γ-left (right) ideal of C.

Proof. Straightforward.

Definition 19.[2] Let C be a nonempty subset of S and $a,b \in [-1,0]$ with $a \leq b$. Define an intuitionistic N-fuzzy set in S as $C_a^b = (\overline{\mu}_{C_a^b}, \overline{\gamma}_{C_a^b})$, where,

$$\overline{\mu}_{C_a^b}(x) = \begin{bmatrix} b \text{ if } x \notin C \\ a \text{ if } x \in C \end{bmatrix} \text{ and } \overline{\gamma}_{C_a^b}(x) = \begin{bmatrix} a \text{ if } x \notin C \\ b \text{ if } x \in C \end{bmatrix}.$$

Lemma 3. A nonempty subset C of S is a Γ-subsemigroup (Γ-left ideal, Γ-right ideal) of S if and only if C_a^b is an intuitionistic N-fuzzy Γ-subsemigroup (Γ-left ideal, Γ-right ideal) of S.

Proof. We prove this result for Γ-right ideals. Let C be a Γ- right ideal of S and $x,y \in S$. If $x \in C$ then $x\alpha y \in C$ implies that $\overline{\mu}_{C_a^b}(x) = a = \overline{\mu}_{C_a^b}(x\alpha y)$ and $\overline{\gamma}_{C_a^b}(x) = b = \overline{\gamma}_{C_a^b}(x\alpha y)$. If $x \notin C$ then $\overline{\mu}_{C_a^b}(x) = b \geq \overline{\mu}_{C_a^b}(x\alpha y)$ and $\overline{\gamma}_{C_a^b}(x) = a \leq \overline{\gamma}_{C_a^b}(x\alpha y)$. Hence $C_a^b = (\overline{\mu}_{C_a^b}, \overline{\gamma}_{C_a^b})$ is an intuitionistic N-fuzzy Γ-right ideal of S.

Conversely, we suppose that $C_a^b = (\overline{\mu}_{C_a^b}, \overline{\gamma}_{C_a^b})$ is an intuitionistic N-fuzzy Γ-right ideal of S. Let $x \in C$ then $\overline{\mu}_{C_a^b}(x) = a$ and $\overline{\gamma}_{C_a^b}(x) = b$. For $y \in S$ and $\alpha \in \Gamma$, $\overline{\mu}_{C_a^b}(x\alpha y) \leq \overline{\mu}_{C_a^b}(x) = a$ but $\overline{\mu}_{C_a^b}(x\alpha y) \geq a$ implies that $\overline{\mu}_{C_a^b}(x\alpha y) = a$ implies that $x\alpha y \in C \Rightarrow C\Gamma S \subseteq C$. Hence C is a Γ-right ideal of S. The result for other cases is similar.

Definition 20. Let $Q = (\overline{\mu}_Q, \overline{\gamma}_Q)$ be an INFS in S. Then Q is called an intuitionistic N-fuzzy Γ-quasi-ideal of S if for all $x \in S$,

$$\overline{\mu}_Q(x) \leq \max\{(\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S)(x), (\overline{\mu}_S \circ_\Gamma \overline{\mu}_Q)(x)\} \text{ and}$$
$$\overline{\gamma}_Q(x) \geq \min\{(\overline{\gamma}_Q \circ_\Gamma \overline{\gamma}_S)(x), (\overline{\gamma}_S \circ_\Gamma \overline{\gamma}_Q)(x)\},$$

Alternatively, above conditions can be written as

$$\overline{\mu}_Q \leq \overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \vee \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q \text{ and}$$

$$\overline{\gamma}_Q \geq \overline{\gamma}_Q \circ_\Gamma \overline{\gamma}_S \wedge \overline{\gamma}_S \circ_\Gamma \overline{\gamma}_Q.$$

Proposition 3. Every intuitionistic N-fuzzy Γ-quasi-ideal of S is an intuitionistic N-fuzzy Γ-subsemigroup of S.

Proof. Let $Q = (\overline{\mu}_Q, \overline{\gamma}_Q)$ be an intuitionistic N-fuzzy Γ-quasi-ideal of S. As, $\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \leq \overline{\mu}_Q \circ_\Gamma \overline{\mu}_Q$ and $\overline{\mu}_S \circ_\Gamma \overline{\mu}_Q \leq \overline{\mu}_Q \circ_\Gamma \overline{\mu}_Q$ implies that

$$\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \vee \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q \leq \overline{\mu}_Q \circ_\Gamma \overline{\mu}_Q.$$

Since Q is a an intuitionistic N-fuzzy Γ-quasi-ideal of S, so

$$\overline{\mu}_Q \leq \overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \vee \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q.$$

This implies that $\overline{\mu}_Q \leq \overline{\mu}_Q \circ_\Gamma \overline{\mu}_Q$. Similarly, $\overline{\gamma}_Q \circ_\Gamma \overline{\gamma}_Q \leq \overline{\gamma}_Q$. Hence by Proposition 2, $Q = (\overline{\mu}_Q, \overline{\gamma}_Q)$ is an intuitionistic N-fuzzy Γ-subsemigroup of S.

Lemma 4. Let $\{Q_i, i \in I\}$ be a collection of intuitionistic N-fuzzy Γ-quasi-ideals of S then $\bigcap_{i \in I} Q_i$ is also an intuitionistic N-fuzzy Γ-quasi-ideal of S.

Proof. Straightforward.

Lemma 5. Every intuitionistic N-fuzzy Γ-left (right) ideal of S is an intuitionistic N-fuzzy Γ-quasi-ideal of S.

Proof. Let $L = (\overline{\mu}_L, \overline{\gamma}_L)$ be an intuitionistic N-fuzzy Γ-left ideal of S then $\overline{\mu}_L \leq \mu_S \circ_\Gamma \overline{\mu}_L$ and $\overline{\gamma}_L \geq \overline{\gamma}_S \circ_\Gamma \overline{\gamma}_L$.
Now,
$\overline{\mu}_L \leq \overline{\mu}_S \circ_\Gamma \overline{\mu}_L$ implies that $\overline{\mu}_L \leq \overline{\mu}_S \circ_\Gamma \overline{\mu}_L \vee \overline{\mu}_L \circ_\Gamma \overline{\mu}_S$.
Similarly,
$\overline{\gamma}_L \geq \overline{\gamma}_S \circ_\Gamma \overline{\gamma}_L$ implies that $\overline{\gamma}_L \geq \overline{\gamma}_S \circ_\Gamma \overline{\gamma}_L \wedge \overline{\gamma}_L \circ_\Gamma \overline{\gamma}_S$.
Hence $L = (\overline{\mu}_L, \overline{\gamma}_L)$ is an intuitionistic N-fuzzy Γ-quasi-ideal of S.

Theorem 5. Let $Q = (\overline{\mu}_Q, \overline{\gamma}_Q)$ be an INFS in S. Then Q is an intuitionistic N-fuzzy Γ-quasi-ideal of S if and only if $N_Q(t,s)$ is a Γ-quasi-ideal of S, for all $t, s \in [-1, 0]$ with $t + s \geq -1$.

Proof. We suppose that $Q = (\overline{\mu}_Q, \overline{\gamma}_Q)$ is an intuitionistic N-fuzzy Γ-quasi-ideal of S then

$$\overline{\mu}_Q \leq \overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \vee \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q.$$

Let $m \in N_Q(t,s)\Gamma S$ then $m = n\alpha y$ for $n \in N_Q(t,s)$, $y \in S$ and $\alpha \in \Gamma$. Since $n \in N_Q(t,s)$ implies that $\overline{\mu}_Q(n) \leq t$ and $\overline{\gamma}_Q(n) \geq s$. Now,

$$\begin{aligned}(\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S)(m) &= (\overline{\mu}_{Q \circ_\Gamma S})(m) \\ &= \min_{m = n\alpha y} \{\max(\overline{\mu}_Q(n), \overline{\mu}_S(y))\} \\ &\leq \max(t, -1) \leq t.\end{aligned}$$

Similarly, $(\overline{\mu}_S \circ_\Gamma \overline{\mu}_Q)(m) \leq t$ implies that

$$\max\{(\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S)(m), (\overline{\mu}_S \circ_\Gamma \overline{\mu}_Q)(m)\} \leq t$$
$$i.e.\ (\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \vee \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q)(m) \leq t.$$

By supposition, $\overline{\mu}_Q(m) \leq (\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \vee \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q)(m)$

implies that $\overline{\mu}_Q(m) \leq t$. Similarly, we can prove that $\overline{\gamma}_Q(m) \geq s$. This implies that $m = n\alpha y \in N_Q(t,s)$ implies that $N_Q(t,s)\Gamma S \subseteq N_Q(t,s)$. This gives that $N_Q(t,s)\Gamma S \cap S\Gamma N_Q(t,s) \subseteq N_Q(t,s)$. Hence $N_Q(t,s)$ is a Γ-quasi-ideal of S.

Conversely, we suppose that $N_Q(t,s)$ is a Γ-quasi-ideal of S, for all $t, s \in [-1, 0]$ with $t + s \geq -1$. We have to prove that $Q = (\overline{\mu}_Q, \overline{\gamma}_Q)$ is an intuitionistic N-fuzzy Γ-quasi-ideal of S. On contrary, we suppose that there exist some $m \in S$ such that

$$(\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \vee \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q)(m) < \overline{\mu}_Q(m)$$

$$\Rightarrow (\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S)(m) \vee (\overline{\mu}_S \circ_\Gamma \overline{\mu}_Q)(m) < \overline{\mu}_Q(m).$$

Now, choose a $t \in [-1, 0]$ such that $(\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S)(m) \vee (\overline{\mu}_S \circ_\Gamma \overline{\mu}_Q)(m) \leq t < \overline{\mu}_Q(m)$. Now, $(\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S)(m) \leq t$ implies that $m \in N_Q(t,s)\Gamma S$ and $(\overline{\mu}_S \circ_\Gamma \overline{\mu}_Q)(m) \leq t$ implies that $m \in S\Gamma N_Q(t,s)$. This implies that $m \in N_Q(t,s)\Gamma S \cap S\Gamma N_Q(t,s) \subseteq N_Q(t,s) \Rightarrow m \in N_Q(t,s) \Rightarrow \overline{\mu}_Q(m) \leq t$, which is a contradiction.

$$\text{So } (\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \vee \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q)(m) \geq \overline{\mu}_Q(m),\ \forall\, m \in S.$$

This implies that, $(\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \vee \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q) \geq \overline{\mu}_Q$. Similarly, $\overline{\gamma}_Q \circ_\Gamma \overline{\gamma}_S \wedge \overline{\gamma}_S \circ_\Gamma \overline{\gamma}_Q \leq \overline{\gamma}_Q$. Hence, $Q = (\overline{\mu}_Q, \overline{\gamma}_Q)$ is an intuitionistic N-fuzzy Γ-quasi-ideal of S.

Example 11. Let $S = \{a, b, c\}$ and $\Gamma = \{\alpha\}$. Then S is a Γ-semigroup under the operation defined in the bellow table.

α	a	b	c
a	a	a	a
b	a	b	b
c	a	c	c

Here $\{a, b\}$ is a Γ-quasi-ideal of S. Now, define $\overline{\mu}_A : S \to [-1, 0]$ and $\overline{\gamma}_A : S \to [-1, 0]$ as $\overline{\mu}_A(a) = -0.4, \overline{\mu}_A(b) = -0.5, \overline{\mu}_A(c) = -0.3$ and $\overline{\gamma}_A(a) = \overline{\gamma}_A(c) = -0.1, \overline{\gamma}_Q(b) = -0.2$. Then

$$N_A(t,s) = \begin{bmatrix} S & \text{if } t, s \in [-0.3, 0] \\ \{a, b\} & \text{if } t, s \in [-0.5, -0.3) \\ \Phi & \text{if } t, s \in [-1, -0.5) \end{bmatrix}.$$

Obviously, $N_A(t,s)$ is a Γ-quasi-ideal of S, for all $t, s \in [-1, 0]$ with $t + s \geq -1$. Then by Theorem 5, $A = (\overline{\mu}_A, \overline{\gamma}_A)$ is an intuitionistic N-fuzzy Γ-quasi-ideal of S.

Theorem 6. A nonempty subset C of S is a Γ-quasi-ideal of S if and only if $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C}, \overline{\gamma}_{\overline{\chi}_C})$ is an intuitionistic N-fuzzy Γ-quasi-ideal of S.

Proof. Let C be a Γ-quasi-ideal of S. For $m \in S$ either $m \in C$ or $m \notin C$.
If $m \in C$ then $\overline{\mu}_{\overline{\chi}_C}(m) = -1$ and $(\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S)(m) \geq -1$ implies that $(\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S)(m) \geq -1 = \overline{\mu}_{\overline{\chi}_C}(m)$ implies that $(\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S)(m) \vee (\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C})(m)$
$= ((\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S) \vee (\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C}))(m) \geq \overline{\mu}_{\overline{\chi}_C}(m)$.
If $m \notin C$ then either $m = a\alpha b$ or $m \neq a\alpha b$, for $a, b \in S$ and $\alpha \in \Gamma$. When $m \neq a\alpha b$, for $a, b \in S$ then $(\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S)(m) = 0$. Also $\overline{\mu}_{\overline{\chi}_C}(m) = 0$ implies that $(\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S)(m) = \overline{\mu}_{\overline{\chi}_C}(m)$. When $m = a\alpha b$, for $a, b \in S$ then maximum one of a, b may contained in C otherwise if $a, b \in C$ then $m = a\alpha b \in C\Gamma S \cap S\Gamma C \subseteq C$ (Since C is a Γ-quasi-ideal of S). $\Rightarrow m \in C$, which is a contradiction. We have following cases:
(i) $m = a\alpha b$ and $a \notin C, b \notin C$. In this case

$$\begin{aligned}(\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S)(m) &= (\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S)(a\alpha b)\\ &= \min\{\max(\overline{\mu}_{\overline{\chi}_C}(a), \overline{\mu}_S(b))\}\\ &= \max\{(0, -1)\}\\ &= 0.\end{aligned}$$

This implies that $(\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S)(m) = 0$. Similarly, $(\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C})(m) = 0$. Since $\overline{\mu}_{\overline{\chi}_C}(m) = 0$, which implies that

$$((\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S) \vee (\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C}))(m) = \overline{\mu}_{\overline{\chi}_C}(m).$$

(ii) $m = a\alpha b$ and exactly one of a, b contained in C. Let $a \in C$ and $b \notin C$ then $\overline{\mu}_{\overline{\chi}_C}(m) = 0$ and $\overline{\mu}_{\overline{\chi}_C}(a) = -1$. Also

$$\begin{aligned}(\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S)(m) &= (\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S)(a\alpha b)\\ &= \min\{\max(\overline{\mu}_{\overline{\chi}_C}(a), \overline{\mu}_S(b))\}\\ &= \max\{(-1, -1)\}\\ &= -1,\\ \overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C})(m) &= (\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C})(a\alpha b)\\ &= \min\{\max(\overline{\mu}_S(a), \overline{\mu}_{\overline{\chi}_S}(b))\}\\ &= \max\{(-1, 0)\}\\ &= 0.\end{aligned}$$

$$\begin{aligned}\Rightarrow (\overline{\mu}_{\overline{\chi}_C} &\circ_\Gamma \overline{\mu}_S)(m) \vee (\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C})(m)\\ &= \max((\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S)(m), (\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C})(m))\\ &= \max(-1, 0)\\ &= 0.\end{aligned}$$

This implies that $((\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S) \vee (\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C}))(m) = \overline{\mu}_{\overline{\chi}_C}(m)$. Hence, for all $m \in S$,

$$\begin{aligned}&((\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S) \vee (\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C}))(m) \geq \overline{\mu}_{\overline{\chi}_C}(m)\\ &\quad\Rightarrow\ (\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S) \vee (\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C}) \geq \overline{\mu}_{\overline{\chi}_C}.\end{aligned}$$

Similarly, we can verify that $\overline{\gamma}_{\overline{\chi}_C} \circ_\Gamma \overline{\gamma}_S \wedge \overline{\gamma}_S \circ_\Gamma \overline{\gamma}_{\overline{\chi}_C} \leq \overline{\gamma}_{\overline{\chi}_C}$. Hence, $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C}, \overline{\gamma}_{\overline{\chi}_C})$ is an intuitionistic N-fuzzy Γ-quasi-ideal of S.

Conversely, we suppose that $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C}, \overline{\gamma}_{\overline{\chi}_C})$ is an intuitionistic N-fuzzy Γ-quasi-ideal of S. Let $x \in C\Gamma S \cap S\Gamma C$ implies that $x \in C\Gamma S$ and $x \in S\Gamma C$ implies that $x = a_1\alpha_1 b_1 = a_2\alpha_2 b_2$, where $a_1, b_2 \in C, b_1, a_2 \in S$ and $\alpha_1, \alpha_2 \in \Gamma$. Then

$$(\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S)(x) = (\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S)(a_1\alpha_1 b_1) = -1$$
$$(\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C})(x) = (\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C})(a_2\alpha_2 b_2) = -1.$$
$$\Rightarrow ((\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S) \vee (\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C}))(x) = -1.$$

Since, $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C}, \overline{\gamma}_{\overline{\chi}_C})$ is an intuitionistic N-fuzzy Γ-quasi-ideal of S then

$$((\overline{\mu}_{\overline{\chi}_C} \circ_\Gamma \overline{\mu}_S) \vee (\overline{\mu}_S \circ_\Gamma \overline{\mu}_{\overline{\chi}_C})(x) \geq \overline{\mu}_{\overline{\chi}_C}(x)$$
$$\Rightarrow -1 \geq \overline{\mu}_{\overline{\chi}_C}(x) \text{ but } \overline{\mu}_{\overline{\chi}_C}(x) \geq -1.$$

This gives $\overline{\mu}_{\overline{\chi}_C}(x) = -1$. Similarly, $\overline{\gamma}_{\overline{\chi}_C}(x) = 0$ implies that $x \in C$ that is $C\Gamma S \cap S\Gamma C \subseteq C$. Hence, C is a Γ-quasi-ideal of S.

Lemma 6. A nonempty subset C of S is a Γ-quasi-ideal of S if and only if C_a^b is an intuitionistic N-fuzzy Γ-quasi-ideal of S.

Proof. Straightforward.

Definition 21. Let $B = (\overline{\mu}_B, \overline{\gamma}_B)$ be an INFS in S. Then B is called an intuitionistic N-fuzzy Γ-bi-ideal of S if,

i) B is an intuitionistic N-fuzzy Γ-subsemigroup of S.
ii) For all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$,

$$\overline{\mu}_B(x\alpha z\beta y) \leq \max(\overline{\mu}_B(x), \overline{\mu}_B(y)) \text{ and}$$
$$\overline{\gamma}_B(x\alpha z\beta y) \geq \min(\overline{\gamma}_B(x), \overline{\gamma}_B(y)).$$

Example 12. Let S be the Γ-semigroup as given in Example 9. Now define, $\overline{\mu}_B : S \to [-1, 0]$ and $\overline{\gamma}_B : S \to [-1, 0]$ such that $B = (\overline{\mu}_B, \overline{\gamma}_B) = \{< a, -0.3, -0.1 >, < b, -0.7, -0.2 >, < c, -0.5, -0.5 >\}$. Then by simple calculations we can verify that $B = (\overline{\mu}_B, \overline{\gamma}_B)$ is an intuitionistic N-fuzzy Γ-bi-ideal of S.

Proposition 4. Let $B = (\overline{\mu}_B, \overline{\gamma}_B)$ be an INFS in S. Then B is an intuitionistic N-fuzzy Γ-bi-ideal of S if and only if

(1) $\overline{\mu}_B \circ_\Gamma \overline{\mu}_B \geq \overline{\mu}_B$ and $\overline{\gamma}_B \circ_\Gamma \overline{\gamma}_B \leq \overline{\gamma}_B$
(2) $\overline{\mu}_B \circ_\Gamma \overline{\mu}_S \circ_\Gamma \overline{\mu}_B \geq \overline{\mu}_B$ and $\overline{\gamma}_B \circ_\Gamma \overline{\gamma}_S \circ_\Gamma \overline{\gamma}_B \leq \overline{\gamma}_B$

Proof. We suppose that $B = (\overline{\mu}_B, \overline{\gamma}_B)$ is an intuitionistic N-fuzzy Γ-bi-ideal of S then it is Γ-subsemigroup of S and by *Proposition* 2, (1) holds. Now for (2), let $m \in S$. If $m \neq x\alpha y$, for $x, y \in S$ and $\alpha \in \Gamma$ then

$$(\overline{\mu}_B \circ_\Gamma \overline{\mu}_S \circ_\Gamma \overline{\mu}_B)(m) = 0 \geq \overline{\mu}_B(m) \text{ and}$$
$$(\overline{\gamma}_B \circ_\Gamma \gamma_S \circ_\Gamma \overline{\gamma}_B)(m) = -1 \leq \overline{\gamma}_B(m).$$

If $m = x\alpha y$ and $x = u\delta v$, for $u, v \in S$ and $\delta \in \Gamma$ then

$$\begin{aligned}
(\overline{\mu}_B \circ_\Gamma \overline{\mu}_S \circ_\Gamma \overline{\mu}_B)(m) \\
&= \min_{m=x\alpha y}\{\max\{(\overline{\mu}_B \circ_\Gamma \overline{\mu}_S)(x), \overline{\mu}_B(y)\}\} \\
&= \min_{m=x\alpha y}\{\max\{\min_{x=u\delta v}\{\max(\overline{\mu}_B(u), \overline{\mu}_S(v)), \overline{\mu}_B(y)\}\} \\
&= \min_{m=x\alpha y}\{\max\{\min_{x=u\delta v}\{\max(\overline{\mu}_B(u), -1), \overline{\mu}_B(y)\}\} \\
&= \min_{m=x\alpha y}\{\min_{x=u\delta v}\{\max\{\overline{\mu}_B(u), \overline{\mu}_B(y)\}\} \\
&= \min_{m=u\delta v\alpha y}(\max(\overline{\mu}_B(u), \overline{\mu}_B(y)) \\
&\geq \min_{m=u\delta v\alpha y}\overline{\mu}_B(u\delta v\alpha y), \\
&= \overline{\mu}_B(m)
\end{aligned}$$

$\Rightarrow$ $(\overline{\mu}_B \circ_\Gamma \overline{\mu}_S \circ_\Gamma \overline{\mu}_B)(m) \geq \overline{\mu}_B(m)$, for all $m \in S$. This implies that $\overline{\mu}_B \circ_\Gamma \overline{\mu}_S \circ_\Gamma \overline{\mu}_B \geq \overline{\mu}_B$. Similarly, $\overline{\gamma}_B \circ_\Gamma \overline{\gamma}_S \circ_\Gamma \overline{\gamma}_B \leq \overline{\gamma}_B$.

Conversely, we suppose that (1) and (2) holds for any intuitionistic N-fuzzy subset $B = (\overline{\mu}_B, \overline{\gamma}_B)$ of S. Let $m = x\alpha z\beta y$ for $x, y, z \in S$, $\alpha, \beta \in \Gamma$ then

$$\begin{aligned}
\overline{\mu}_B(x\alpha z\beta y) = \overline{\mu}_B(m) \\
&\leq (\overline{\mu}_B \circ_\Gamma \overline{\mu}_S \circ_\Gamma \overline{\mu}_B)(m) \\
&= (\overline{\mu}_B \circ_\Gamma \overline{\mu}_S \circ_\Gamma \overline{\mu}_B)(x\alpha z\beta y) \\
&= \min_{m=(x\alpha z)\beta y}\{\max\{(\overline{\mu}_B \circ_\Gamma \overline{\mu}_S)(x\alpha z), \overline{\mu}_B(y)\}\} \\
&\leq \max\{\min_{n=x\alpha z}\{\max(\overline{\mu}_B(x), \overline{\mu}_S(z))\}, \overline{\mu}_B(y)\} \\
&= \min_{n=x\alpha z}\{\max\{\max(\overline{\mu}_B(x), -1), \overline{\mu}_B(y)\}\} \\
&= \min_{n=x\alpha z}\{\max(\overline{\mu}_B(x), \overline{\mu}_B(y)\} \\
&\leq \max(\overline{\mu}_B(x), \overline{\mu}_B(y)).
\end{aligned}$$

Similarly, we can show that $\overline{\gamma}_B(x\alpha z\beta y) \geq \min(\overline{\gamma}_B(x), \overline{\gamma}_B(y))$. Hence $B = (\overline{\mu}_B, \overline{\gamma}_B)$ is an intuitionistic N-fuzzy Γ-bi ideal of S.

Lemma 7. Every intuitionistic N-fuzzy Γ-quasi-ideal of S is an intuitionistic N-fuzzy Γ-bi-ideal of S.

Proof. Let $Q = (\overline{\mu}_Q, \overline{\gamma}_Q)$ be an intuitionistic N-fuzzy Γ-quasi-ideal of S . By *Proposition* 3, Q is intuitionistic N-fuzzy Γ-ternary subsemigroup of S then

$$\overline{\mu}_Q \circ_\Gamma \overline{\mu}_Q \geq \overline{\mu}_Q \text{ and } \overline{\gamma}_Q \circ_\Gamma \overline{\gamma}_Q \leq \overline{\gamma}_Q \text{ holds.}$$

$$\begin{aligned}
\text{Also } \overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q &\geq \overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \circ_\Gamma \overline{\mu}_S, (\text{since } \overline{\mu}_Q \geq \overline{\mu}_S) \\
&\geq \overline{\mu}_Q \circ_\Gamma \overline{\mu}_S, (\text{since } \overline{\mu}_S \circ_\Gamma \overline{\mu}_S \geq \overline{\mu}_S) \\
\text{and } \overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q &\geq \overline{\mu}_S \circ_\Gamma \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q \geq \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q.
\end{aligned}$$

This implies that

$$\overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q \geq \overline{\mu}_Q \circ_\Gamma \overline{\mu}_S \vee \overline{\mu}_S \circ_\Gamma \overline{\mu}_Q \geq \overline{\mu}_Q.$$

Similarly, we can show that $\overline{\gamma}_Q \circ_\Gamma \overline{\gamma}_S \circ_\Gamma \overline{\gamma}_Q \leq \overline{\gamma}_Q$. Hence, $Q = (\overline{\mu}_Q, \overline{\gamma}_Q)$ is an intuitionistic N-fuzzy Γ-bi-ideal of S.

Lemma 8. Every intuitionistic N-fuzzy Γ-left (right)-ideal of S is an intuitionistic N-fuzzy Γ-bi-ideal of S.

Proof. Straightforward.

Lemma 9. Let $\{B_i, i \in I\}$ be a collection of intuitionistic N-fuzzy Γ-bi-ideals of S then $\bigcap_{i\in I} B_i$ is also an intuitionistic N-fuzzy Γ-bi-ideal of S.

Proof. Straightforward.

Theorem 7. Let $B = (\overline{\mu}_B, \overline{\gamma}_B)$ be an INFS in S. Then B is an intuitionistic N-fuzzy Γ-bi-ideal of S if and only if $N_B(t,s)$ is a Γ-bi-ideal of S, for all $t, s \in [-1, 0]$ with $t + s \geq -1$.

Proof. We suppose that $B = (\overline{\mu}_B, \overline{\gamma}_B)$ is an intuitionistic N-fuzzy Γ-bi-ideal of S and $m \in N_B(t,s)\Gamma S\Gamma N_B(t,s)$. Then $m = n\alpha x\beta o$ for $n, o \in N_B(t,s)$, $x \in S$ and $\alpha, \beta \in \Gamma$. Since $n, o \in N_B(t,s)$ implies that $\overline{\mu}_B(n), \overline{\mu}_B(o) \leq t$ and $\overline{\gamma}_B(n), \overline{\gamma}_B(o) \geq s$. Now, since B is intuitionistic N-fuzzy Γ-bi-ideal of S so

$$\overline{\mu}_B(m) = \overline{\mu}_B(n\alpha x\beta o) \leq \max(\overline{\mu}_B(n), \overline{\mu}_B(o))\} = \max(t,t) = t$$

and

$$\overline{\gamma}_B(m) = \overline{\gamma}_B(n\alpha x\beta o) \geq \min(\overline{\gamma}_B(n), \overline{\gamma}_B(o))\} = \min(s,s) = s.$$

This implies that $m \in N_B(t,s)$ implies that $N_B(t,s)\Gamma S\Gamma N_B(t,s) \subseteq N_B(t,s)$. Hence, $N_B(t,s)$ is a Γ-bi-ideal of S.

Conversely, we suppose that $N_B(t,s)$ is a Γ-bi-ideal of S, for all $t, s \in [-1, 0]$ with $t + s \geq -1$. Let $x, y \in S$ such that $\overline{\mu}_B(x) = t_x, \overline{\gamma}_B(x) = s_x$ with $-1 \leq t_x + s_x \leq 0$ and $\overline{\mu}_B(y) = t_y, \overline{\gamma}_B(y) = s_y$ with $-1 \leq t_y + s_y \leq 0$. Then $x \in N_B(t_x, s_x)$ and $y \in N_B(t_y, s_y)$. We may assume that $t_x \leq t_y$ and $s_x \geq s_y$ then $N_B(t_x, s_x) \subseteq N_B(t_y, s_y)$. This implies that $x, y \in N_B(t_y, s_y)$. Since $N_B(t_y, s_y)$ is a Γ-bi-ideal of S then for $z \in S$, $\alpha, \beta \in \Gamma, x\alpha z\beta y \in N_B(t_y, s_y)$, we have

$$\overline{\mu}_B(x\alpha z\beta y) \leq t_y = \max(t_x, t_y) = \max(\overline{\mu}_B(x), \overline{\mu}_B(y)) \text{ and}$$

$$\overline{\gamma}_B(x\alpha z\beta y) \geq s_y = \min(s_x, s_y) = \min(\overline{\gamma}_B(x), \overline{\gamma}_B(y)).$$

This holds for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Hence $B = (\overline{\mu}_B, \overline{\gamma}_B)$ is an intuitionistic N-fuzzy Γ-bi-ideal of S.

Theorem 8. A nonempty subset C of S is a Γ-bi-ideal of S if and only if $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C}, \overline{\gamma}_{\overline{\chi}_C})$ is an intuitionistic N-fuzzy Γ-bi-ideal of S.

Proof. We suppose that C is a Γ-bi-ideal of S then it is a Γ-subsemigroup of S and by *Theorem* 3, $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C}, \overline{\gamma}_{\overline{\chi}_C})$ is an intuitionistic N-fuzzy Γ-subsemigroup of S. Also $C\Gamma S\Gamma C \subseteq C$. Now for any $x, y, z \in S, \alpha, \beta \in \Gamma$, $x\alpha z\beta y \in S$. We have following cases,

(i) If $x, y \in C$ then $x\alpha z\beta y \in C\Gamma S\Gamma C \subseteq C$ implies that $\overline{\mu}_{\overline{\chi}_C}(x) = \overline{\mu}_{\overline{\chi}_C}(y) = -1 = \overline{\mu}_{\overline{\chi}_C}(x\alpha z\beta y)$ and $\overline{\gamma}_{\overline{\chi}_C}(x) = \overline{\gamma}_{\overline{\chi}_C}(y) = 0 = \overline{\gamma}_{\overline{\chi}_C}(x\alpha z\beta y)$. Hence $\overline{\mu}_{\overline{\chi}_C}(x\alpha z\beta y) = -1 = \max(\overline{\mu}_{\overline{\chi}_C}(x), \overline{\mu}_{\overline{\chi}_C}(y))$. Also $\overline{\gamma}_{\overline{\chi}_C}(x\alpha z\beta y) = 0 = \min(\overline{\gamma}_{\overline{\chi}_C}(x), \overline{\gamma}_{\overline{\chi}_C}(y))$.

(ii) If either $x \notin C$ or $y \notin C$ then either $\overline{\mu}_{\overline{\chi}_C}(x) = 0, \overline{\gamma}_{\overline{\chi}_C}(x) = -1$ or $\overline{\mu}_{\overline{\chi}_C}(y) = 0, \overline{\gamma}_{\overline{\chi}_C}(y) = -1$

implies that, $\max(\overline{\mu}_{\overline{\chi}_C}(x),\overline{\mu}_{\overline{\chi}_C}(y)) = 0$ and $\min(\overline{\gamma}_{\overline{\chi}_C}(x),\overline{\gamma}_{\overline{\chi}_C}(y)) = -1$. But $\overline{\mu}_{\overline{\chi}_C}(x\alpha z\beta y) \leq 0$ and $\overline{\gamma}_{\overline{\chi}_C}(x\alpha z\beta y) \geq -1$. This implies that $\overline{\mu}_{\overline{\chi}_C}(x\alpha z\beta y) \leq \max(\overline{\mu}_{\overline{\chi}_C}(x),\overline{\mu}_{\overline{\chi}_C}(y))$ and $\overline{\gamma}_{\overline{\chi}_C}(x\alpha z\beta y) \geq \min(\overline{\gamma}_{\overline{\chi}_C}(x),\overline{\gamma}_{\overline{\chi}_C}(y))$.

(iii) If $x \notin A$ and $y \notin A$ and $z \notin A$.It same like case (ii). Hence, $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C},\overline{\gamma}_{\overline{\chi}_C})$ is an intuitionistic N-fuzzy Γ-bi-ideal of S.

Conversely, we suppose that $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C},\overline{\gamma}_{\overline{\chi}_C})$ is an intuitionistic N-fuzzy Γ-bi-ideal of S. For any $m \in C\Gamma S\Gamma C$ there exists $x,y \in C, z \in S$ and $\alpha,\beta \in \Gamma$ such that $m = x\alpha z\beta y$. Then $\overline{\mu}_{\overline{\chi}_C}(x) = \overline{\mu}_{\overline{\chi}_C}(y) = -1$ implies that $\max(\overline{\mu}_{\overline{\chi}_C}(x),\overline{\mu}_{\overline{\chi}_C}(y)) = -1$ and $\overline{\gamma}_{\overline{\chi}_C}(x) = \overline{\gamma}_{\overline{\chi}_C}(y) = 0$ implies that $\min(\overline{\gamma}_{\overline{\chi}_C}(x),\overline{\gamma}_{\overline{\chi}_C}(y)) = 0$. Since $\overline{\chi}_C = (\overline{\mu}_{\overline{\chi}_C},\overline{\gamma}_{\overline{\chi}_C})$ is an intuitionistic N-fuzzy Γ-bi-ideal of S implies that $\overline{\mu}_{\overline{\chi}_C}(x\alpha z\beta y) \leq \max(\overline{\mu}_{\overline{\chi}_C}(x),\overline{\mu}_{\overline{\chi}_C}(y)) = -1$ and $\overline{\gamma}_{\overline{\chi}_C}(x\alpha z\beta y) \geq \min(\overline{\gamma}_{\overline{\chi}_C}(x),\overline{\gamma}_{\overline{\chi}_C}(y)) = 0$. But by definition $\overline{\mu}_{\overline{\chi}_C}(x\alpha z\beta y) \geq -1$ and $\overline{\gamma}_{\overline{\chi}_C}(x\alpha z\beta y) \leq 0$. This gives that $\overline{\mu}_{\overline{\chi}_C}(x\alpha z\beta y) = -1$ and $\overline{\gamma}_{\overline{\chi}_C}(x\alpha z\beta y) = 0$ implies that $m = x\alpha z\beta y \in C$. This implies that $C\Gamma S\Gamma C \subseteq C$. Hence C is a Γ-bi-ideal of S.

Lemma 10. A nonempty subset C of S is a Γ-bi-ideal of S if and only if C_a^b is an intuitionistic N-fuzzy Γ-bi-ideal of S.

Proof. Straightforward.

The following examples shows that the converses of *Proposition* 3, *Lemma* 5, *Lemma* 7 and *Lemma* 8 are not true in general.

Example 13. Let $S = \{a,b,c,d,e\}$ and $\Gamma = \{\alpha\}$. Then S is Γ-semigroup along with the operation defined in the bellow table.

α	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	a
c	a	e	c	c	e
d	a	b	d	d	b
e	a	e	a	c	a

Then $\{a,b\}$ and $\{a,c\}$ are Γ-quasi-ideals of S but both are neither Γ-left ideal nor Γ-right ideals of S. Define, $\overline{\mu}_Q : T \to [-1,0]$ and $\overline{\gamma}_Q : T \to [-1,0]$ as, $\overline{\mu}_Q(a) = -0.8, \overline{\mu}_Q(b) = -0.5, \overline{\mu}_Q(c) = -0.8, \overline{\mu}_Q(d) = -0.3, \overline{\mu}_Q(e) = -0.3$ and $\overline{\gamma}_Q(a) = 0, \overline{\gamma}_Q(b) = -0.3, \overline{\gamma}_Q(c) = 0, \overline{\gamma}_Q(d) = -0.1, \overline{\gamma}_Q(e) = 0$. Let,

$$N_Q(t,s) = \begin{bmatrix} S & \text{if } t,s \in [-0.5,0] \\ \{a,c\} & \text{if } t,s \in [-0.8,-0.5) \\ \Phi & \text{if } t,s \in [-1,-0.8) \end{bmatrix}.$$

Obviously, $N_Q(t,s)$ is a Γ-quasi-ideal of S but neither a Γ-left nor a Γ-right ideal. By Theorem 5, $Q = (\overline{\mu}_Q,\overline{\gamma}_Q)$ is an intuitionistic N-fuzzy Γ-quasi-ideal of S but by Theorem 5, Q is neither an intuitionistic N-fuzzy Γ-left nor an intuitionistic N-fuzzy Γ-right ideal of S.

Example 14. Let $S = \{a,b,c,d\}$ and $\Gamma = \{\alpha\}$. Then S is Γ-semigroup along with the operation defined in the table bellow,

α	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

Then $\{a,c\}$ is a Γ-subsemigroup of S but not a Γ-quasi-ideal. Moreover $\{a,c\}$ is a Γ-bi-ideal but neither a Γ-left nor a Γ-right ideal of S. Now define $\overline{\mu}_A : T \to [-1,0]$ and $\overline{\gamma}_A : T \to [-1,0]$ as, $\overline{\mu}_A(a) = -0.6, \overline{\mu}_A(b) = -0.4, \overline{\mu}_A(c) = -0.5, \overline{\mu}_A(d) = -0.4$ and $\overline{\gamma}_A(a) = 0, \overline{\gamma}_A(b) = -0.1, \overline{\gamma}_A(c) = -0.4, \overline{\gamma}_A(d) = -0.3$. Let,

$$N_A(t,s) = \begin{bmatrix} S & \text{if } t,s \in [-0.4,0] \\ \{a,c\} & \text{if } t,s \in [-0.7,-0.4) \\ \Phi & \text{if } t,s \in [-1,-0.7) \end{bmatrix}.$$

Then $N_A(t,s)$ is a Γ-subsemigroup of S and a Γ-bi-ideal of S for all $t,s \in [-1,0]$ with $t+s \geq -1$ but not a Γ-quasi-ideal and not a Γ-left as well as Γ-right ideal of S. By Theorem 1 and Theorem 8, $A = (\overline{\mu}_A,\overline{\gamma}_A)$ is an intuitionistic N-fuzzy Γ-subsemigroup and an intuitionistic N-fuzzy Γ-bi-ideal of S. But by Theorem 2, A is neither an intuitionistic N-fuzzy Γ-left ideal of S nor an intuitionistic N-fuzzy Γ-right ideal of S and by Theorem 5, A is not an intuitionistic N-fuzzy Γ-quasi-ideal of S.

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Muhammad Akram did his M.Sc with distinction from the "University of The Punjab", Lahore and M.Phil from Quaid-i-Azam University, Islamabad, Pakistan. Since 2008 he is a permanent faculty member at the "University of Gujrat", Gujrat, Pakistan. He is a Ph.D research scholar at the "Universiti Tun Hussein Onn Malaysia" Malaysia. His research interests are algebraic structures, fuzzy and intuitionistic fuzzy algebraic structures etc. He published several research papers in his research area.

J. Kavikumar received his M.Sc and Ph.D from Annamalai University, India. Since 2006 he has been at Universiti Tun Hussein Onn Malaysia, Malaysia. He has published many papers in national and international journals. His research interests center on the algebra, Fuzzy algebra, Fuzzy functional analysis and Fuzzy numerical analysis.

Zaffar Iqbal received his MSc and M. Phil in Mathematics from Quaid-i-Azam University, Islamabad, Pakistan and Ph.D in Mathematics from Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan. He is a permanent faculty member at University of Gujrat, Pakistan. He has published many papers on Braid Groups and related topics. His research interests focus on Combinatorial problems in braids groups and knot theory as well as Fuzzy algebraic structures.