Numerical and Computational Methods in Sciences & Engineering
An International Journal

http://dx.doi.org/10.18576/ncmse/040103

Properties of D-metric Spaces and weighted composition Operators Between Hyperbolic Function Spaces

Samy A. Abd-Elhafeez[1], *A. Kamal*[1,2] *and M. Hamza Eissa*[1,*]

[1]Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said 42521, Egypt
[2]Department of Mathematics, College of Science and Arts, AlMithnab, Qassim University, Buridah 51931, Saudi Arabia

Received: 22 June 2021, Revised: 19 Dec. 2021, Accepted: 23 Dec. 2021
Published online: 1 Mar. 2022

Abstract: The two weighted hyperbolic classes ${\mathscr{B}^*}_{\alpha,\,log^\beta}$ are introduced and studied. Moreever, D-metric space in ${\mathscr{B}^*}_{\alpha,\,log^\beta}$ and $Q^*_K(p,q)$ general spaces is presented. We show that the two classes ${\mathscr{B}^*}_{\alpha,\,log^\beta}$ and $Q^*_K(p,q)$ are complete metric space. Finally, we establish the conditions needed for the weighted $uC_\mho$ operator to be bounded and compact.

Keywords: D-metric Spaces, weighted composition operator, Hyperbolic Function Spaces.

1 Introduction

The open unit disk $\Upsilon = \{\gamma \in \mathbb{C} : |\gamma| < 1\}$, where $\mathbb{C}$ is the complex plane and $\partial\Upsilon$ is the boundary. Assuming that $\vartheta(\Upsilon)$ be the space of all analytic functions in Υ. Assuming also that $B(\Upsilon)$ is a concerned subset of $\vartheta(\Upsilon)$ consisting of those $\mathscr{L} \in \vartheta(\Upsilon)$ for which $|\mathscr{L}(\gamma)| < 1$ for all $\gamma \in \Upsilon$.

The concerned Green's function of Υ is given by $\Lambda(\gamma,a) = \log\frac{1}{|\mho(\gamma)|}$, with $\mho(\gamma) = \frac{a-\gamma}{1-\bar{a}\gamma}$, where the points $\gamma, a \in \Upsilon$ give the Möbius transformation by the singular point $\gamma \in \Upsilon$.

The definition of the linear composition operator $C_\mho$ is defined as $C_\mho(\mathscr{L}) = (\mathscr{L}\circ\mho)$ (see [1–3]).

The definition of the weighted composition operator $uc_\mho$ given in [4] as follow

$$(uC_\mho\mathscr{L})(\gamma) = u(\gamma)\mathscr{L}(\mho(\gamma)), \quad \gamma \in \Upsilon \;\; and \;\; u \in \vartheta(\Upsilon)$$

A function $\mathscr{L} \in B(\Upsilon)$ belongs to α-Bloch space $\mathscr{B}_\alpha, 0 < \alpha < \infty$, if

$$||f||_{\mathscr{B}_\alpha} = \sup_{\gamma\in\Upsilon}(1-|\gamma|)^\alpha|\mathscr{L}'(\gamma)| < \infty.$$

The little α-Bloch space $\mathscr{B}_{\alpha,\,0}$ consisting of all $\mathscr{L} \in \mathscr{B}_\alpha$ so that

$$\lim_{|\gamma|\to 1^-}(1-|\gamma|^2)|\mathscr{L}'(\gamma)| = 0.$$

The author in ([5]) introduced the definition of logarithmic Bloch-type space as follows

Definition 1.*Let $\alpha > 0, \beta \geq 0$ and $\mathscr{L}$ be an analytic function in Υ the logarithmic Bloch-type space $\mathscr{B}^\alpha_{log^\beta}$ is defined by*

$$||\mathscr{L}||_{\mathscr{B}^\alpha_{log^\beta}} = \left\{\mathscr{L} \in \vartheta(\Upsilon) : ||\mathscr{L}||_{\mathscr{B}^\alpha_{log^\beta}} = \sup_{\gamma\in\Upsilon}(1-|\gamma|)^\alpha(\ln\frac{e^{\beta/\alpha}}{(1-|\gamma|)})|\mathscr{L}'(\gamma)| < \infty\right\}.$$

Case 1: $\beta = 0$ then $\mathscr{B}^\alpha_{log^\beta}$ becomes the α-Bloch space $\mathscr{B}^\alpha$
Case 2: $\alpha = \beta = 1$ then $\mathscr{B}^\alpha_{log^\beta}$ becomes the logarithmic Bloch space.

* Corresponding author e-mail: mohammed_essa2001@yahoo.com

The authors in ([6]) introduced the definition of $Q_{K(p,q)}$ which has attracted a lot of attention in recent years. It defined as follows

Definition 2. *Let $K:[0,\infty)\to[0,\infty)$ is right-continuous and non decreasing functions. If $0<p<\infty,\ -2<q<\infty$, then an analytic function $\mathscr{L}$ in Υ is said to belong to the space $Q_K(p,q)$ if*

$$||\mathscr{L}||_{Q_K(p,q)} := \sup_{a\in\Upsilon}\int_{\Upsilon}|\mathscr{L}'(\gamma)|^p(1-|\gamma|^2)^q\,K(\Lambda(\gamma,a))dA(\gamma)<\infty.$$

2 Preliminaries

Definition 3. *(see [7]) $\mathscr{B}^*_\alpha$ is define the concerned hyperbolic Bloch space as follow*

$$\mathscr{B}^*_\alpha=\{\mathscr{L}:\mathscr{L}\in B(\Upsilon)\ and\ \sup_{\gamma\in\Upsilon}(1-|\gamma|^2)^\alpha\mathscr{L}^*(\gamma)<\infty\},$$

where $\mathscr{L}^*(\gamma)=\frac{|\mathscr{L}'(\gamma)|}{1-|\mathscr{L}(\gamma)|^2}$, is concerned the hyperbolic derivative of $\mathscr{L}\in B(\Upsilon)$. See([8])

$\mathscr{B}^*_{\alpha,0}$ is define the little concerned hyperbolic Bloch-type class $\mathscr{B}^*_{\alpha,0}$ consisting of all $\mathscr{L}\in\mathscr{B}^*_\alpha$ where

$$\lim_{|\gamma|\to1^-}(1-|\gamma|^2)^\alpha\mathscr{L}^*(\gamma)=0.$$

The norm of $\mathscr{B}^*_\alpha$ is defined by

$$||\mathscr{L}||_{\mathscr{B}^*_\alpha}=|\mathscr{L}(0)|+\sup_{\gamma\in\Upsilon}(1-|\gamma|)^\alpha|\mathscr{L}^*(\gamma)|.$$

Now, we will introduce the definition of weighted hyperbolic Bloch.

Definition 4. *Let $\alpha>0$, for $\mathscr{L}\ \in B(\Upsilon)$ if*

$$||\mathscr{L}||_{\mathscr{B}^*{}_{\alpha,\,lo\Lambda\beta}}=\sup_{\gamma\in\Upsilon}=\sup_{\gamma\in\Upsilon}(1-|\gamma|)^\alpha(\ln\frac{e^{\beta/\alpha}}{(1-|\gamma|)})\mathscr{L}^*(\gamma)<\infty\,,$$

thus $\mathscr{L}$ belongs to the $\mathscr{B}^*{}_{\alpha,\,log^\beta}$.

Definition 5. *(see [1]) Let $K:[0,\infty)\to[0,\infty)$ is right-continuous and non decreasing functions. If $p>0,\ q>-2$, then $\mathscr{L}\in\Upsilon$ is belongs to the space $Q^*_K(p,q)$ if*

$$||\mathscr{L}||_{Q^*_K(p,q)} := \sup_{a\in\Upsilon}\int_{\Upsilon}\mathscr{L}^*(\gamma)^p(1-|\gamma|^2)^q\,K(\Lambda(\gamma,a))dA(\gamma)<\infty.$$

Definition 6. *A weighted composition operator $uC_\mho:\mathscr{B}^*{}_{\alpha,\,log^\beta}\to Q^*_k(p,q)$ is said to be bounded if there is a positive constant C so that $||C_\mho\mathscr{L}||_{\mathscr{L}^*_{\log}}(p,q,s)\le C||\mathscr{L}||_{\mathscr{B}^*{}_{\alpha,\,log^\beta}}$ for all $\mathscr{L}\in\mathscr{B}^*_{p,\,\alpha}$.*

Definition 7. *A weighted composition operator $C_\mho:\mathscr{B}^*{}_{\alpha,\,log^\beta}\to Q^*_K(p,q)$ is said to be compact if it maps any ball in $\mathscr{B}^*_{p,\,\alpha}$ onto a precompact set in $Q^*_K(p,q)$.*

Definition 8. *(see [9]) Let $p,s>0,\alpha<\infty,q>-2$ and $\mho$ is a holomorphic mapping from $\Upsilon\to\Upsilon$. then $C_\mho:\mathscr{B}^*{}_{\alpha,\,log^\beta}\to Q^*_K(p,q)$ is compact if and only if for any bounded sequence $\{\mathscr{L}_n\}_{n\in N}\in\mathscr{B}^*{}_{\alpha,\,log^\beta}$ which converges to zero uniformly on compact subsets of Υ as $n\to\infty$ we have $\lim_{n\to\infty}||C_\mho\mathscr{L}_n||_{Q^*_K(p,q)}=0$.*

3 D-metric space

The properties Topological of D- metric space in general from has been introduced in [10–12].

Definition 9. *[13] If X is a nonempty set and $\mathbb{R}$ a set of real numbers. If the function $D : X \times X \times X \to \mathbb{R}$ satisfies the following conditions:*

(i) $D(l,s,\gamma) \geq 0$ for all $l,s,\gamma \in X$ and equality iff $l = s = \gamma$ (non negativity),

(ii) $D(l,s,\gamma) = D(l,\gamma,s) = \cdots$

(iii) $D(l,s,\gamma) \leq D(l,s,a) + D(l,a,\gamma) + D(a,s,\gamma)$ for all $l,s,\gamma,a \in X$.

Then D is called a $D-$ metric on X.

(X,D) represents D-metric space of a set X with D-metric D. The generalization of D-metric space has been introduced in [11].

Example1.1: [13] Assume that (l,d) is ordinary metric space and function D_1 on X^3 can be define in the following form

$$D_1(l,s,\gamma) = max\{d(l,s), d(s,\gamma), d(\gamma,l)\},$$

for all $l,s,\gamma \in X$. Thus, D_1 is a D-metric on X and (X,D_1) is a D-metric space.

Example1.2: [13] Assume that (X,d) is ordinary metric space and function D_2 on X^3 can be define in the following form

$$D_2(l,s,\gamma) = d(l,s) + d(s,\gamma) + d(\gamma,l)$$

for $l,s,\gamma \in X$. Thus, D_2 is a D-metric on X and (X,D_2) is a D-metric space.

*Remark.*From Example1.1, we can deduce that the D-metric D_1 is the diameter of a set containing of the points l,s,γ in X. From Example1.2, we can deduce that the D-metric D_2 is the sum of the lengths of the sides of a triangle with vertices l,s,γ in X (the perimeter of a triangle).

Definition 10.*(Cauchy sequence , completeness) [14] A sequence (l_n) in a metric space $X = (l,d)$ is called a Cauchy sequence if For every $m,n > N$ and $\varepsilon > 0$ there exist an $N = N(\varepsilon)$ where*

$$d(l_m,l_n) < \varepsilon.$$

If every Cauchy sequence in X converges then the space X is complete.

Theorem 1. *[13] Let (X,d) is ordinary metric on X and $(X,D_1),(X,D_1)$ be corresponding D-metrics on X. Then, (X,D_1) and (X,D_2) are complete if and only if (X,d) is complete.*

4 D-metrics in $\mathscr{B}^*{}_{\alpha,\,lo\Lambda^\beta}$ and $Q^*_K(p,q)$

Now, we present a D-metric space on $\mathscr{B}^*{}_{\alpha,\,log^\beta}$ and $Q^*_K(p,q)$.
Assume that $0 < p, -2 < q < \infty$, and $0 < \alpha < 1$. First, we can get a D-metric in $\mathscr{B}^*{}_{\alpha,\,log^\beta}$, for $\mathscr{L},\Lambda,\vartheta \in \mathscr{B}^*{}_{\alpha,\,log^\beta}$ by

$$\begin{aligned} D(\mathscr{L},\Lambda,\vartheta;\mathscr{B}^*{}_{\alpha,\,log^\beta}) := D_{\mathscr{B}^*{}_{\alpha,\,log^\beta}}(\mathscr{L},\Lambda,\vartheta) + ||\mathscr{L}-\Lambda||_{\mathscr{B}^*{}_{\alpha,\,log^\beta}} + ||\Lambda - h||_{\mathscr{B}^*{}_{\alpha,\,log^\beta}} \\ + ||\vartheta-\mathscr{L}||_{\mathscr{B}^*{}_{\alpha,\,log^\beta}} + |\mathscr{L}(0)-\Lambda(0)| + |\Lambda(0)-\vartheta(0)| + |\vartheta(0)-\mathscr{L}(0)|, \end{aligned} \tag{1}$$

where

$$D_{\mathscr{B}^*{}_{\alpha,\,log^\beta}}(\mathscr{L},\Lambda,\vartheta) := d_{\mathscr{B}^*{}_{\alpha,\,log^\beta}}(\mathscr{L},\Lambda) + d_{\mathscr{B}^*{}_{\alpha,\,log^\beta}}(\Lambda,\vartheta) + d_{\mathscr{B}^*{}_{\alpha,\,log^\beta}}(\vartheta,\mathscr{L})$$

and

$$D_{\mathscr{B}^*{}_{\alpha,\,log^\beta}}(\mathscr{L},\Lambda,\vartheta) := \left(\sup_{\gamma\in\Upsilon} |\mathscr{L}^*(\gamma)-\Lambda^*(\gamma)| + \sup_{\gamma\in\Upsilon} |\Lambda^*(\gamma)-\vartheta^*(\gamma)| + \sup_{\gamma\in\Upsilon} |\vartheta^*(\gamma)-\mathscr{L}^*(\gamma)| \right)$$

$$\times\left((1-|\gamma|)^{\alpha}(\ln\frac{e^{\beta/\alpha}}{(1-|\gamma|)})\mathscr{L}^{*}(\gamma)\right).$$

Therefor, for $\mathscr{L},\Lambda,\vartheta \in Q_K^*(p,q)$ we present a D-metric on $Q_K^*(p,q)$ by

$$D(\mathscr{L},\Lambda,\vartheta;Q_K^*(p,q)) := D_{Q_K^*(p,q)}(\mathscr{L},\Lambda,\vartheta)+||\mathscr{L}-\Lambda||_{Q_K^*(p,q)}+||\Lambda-h||_{Q_K^*(p,q)}+$$
$$||\vartheta-f||_{Q_K^*(p,q)}+|\mathscr{L}(0)-\Lambda(0)|+|\Lambda(0)-\vartheta(0)|+|\vartheta(0)-\mathscr{L}(0)|,$$

where

$$D_{Q_K^*(p,q)}(\mathscr{L},\Lambda,\vartheta) := d_{Q_K^*(p,q)}(\mathscr{L},\Lambda)+d_{Q_K^*(p,q)}(\Lambda,\vartheta)+d_{Q_K^*(p,q)}(\vartheta,\mathscr{L})$$

and

$$d_{Q_K^*(p,q)}(\mathscr{L},\Lambda) := \left(\sup_{\gamma\in\Upsilon}\int_{\Upsilon}|\mathscr{L}^*(\gamma)-\Lambda^*(\gamma)|^p(1-|\gamma|^2)^q(1-|\varphi(\gamma)|^2)^s dA(\gamma)\right)^{\frac{1}{p}}.$$

Proposition 1 *The ${\mathscr{B}^*}_{\alpha,\,log^{\beta}}$ class with the D-metric $D(.,.;{\mathscr{B}^*}_{\alpha,\,log^{\beta}})$ is a complete metric space. Then, $\mathscr{B}^*_{\alpha,\,\log,\,0}$ is a closed (and therefore complete) subspace of ${\mathscr{B}^*}_{\alpha,\,log^{\beta}}$.*

Proof. Assume that $\mathscr{L},\Lambda,\vartheta,a \in {\mathscr{B}^*}_{\alpha,\,log^{\beta}}$. Thus, clearly

(i) $D(\mathscr{L},\Lambda,\vartheta;{\mathscr{B}^*}_{\alpha,\,log^{\beta}}) \geq 0$, for all $\mathscr{L},\Lambda,\vartheta \in {\mathscr{B}^*}_{\alpha,\,log^{\beta}}$.

(ii) $D(\mathscr{L},\Lambda,\vartheta;{\mathscr{B}^*}_{\alpha,\,log^{\beta}}) = D(\mathscr{L},\vartheta,g;{\mathscr{B}^*}_{\alpha,\,log^{\beta}}) = D(\Lambda,\vartheta,\mathscr{L};{\mathscr{B}^*}_{\alpha,\,log^{\beta}})$.

(iii) $D(\mathscr{L},\Lambda,\vartheta;{\mathscr{B}^*}_{\alpha,\,log^{\beta}}) \leq D(\mathscr{L},\Lambda,a;{\mathscr{B}^*}_{\alpha,\,log^{\beta}})+D(\mathscr{L},a,\vartheta;{\mathscr{B}^*}_{\alpha,\,log^{\beta}})+D(a,\Lambda,\vartheta;{\mathscr{B}^*}_{\alpha,\,log^{\beta}})$

for all $\mathscr{L},\Lambda,\vartheta,a \in {\mathscr{B}^*}_{\alpha,\,log^{\beta}}$.

(iv) $D(\mathscr{L},\Lambda,\vartheta;{\mathscr{B}^*}_{\alpha,\,log^{\beta}}) = 0$ implies $\mathscr{L}=\Lambda=\vartheta$.

Then, D is a D-metric on ${\mathscr{B}^*}_{\alpha,\,log^{\beta}}$, and $({\mathscr{B}^*}_{\alpha,\,log^{\beta}},D)$ is D-metric space.

Now, the Theorem 1 can be used to prove the completeness. Assume that $(\mathscr{L}_n)_{n=1}^{\infty}$ is a Cauchy sequence in metric space $({\mathscr{B}^*}_{\alpha,\,log^{\beta}},d)$, So, for any $\varepsilon>0$ there exist $N=N(\varepsilon)\in\mathbb{N}$ where $d(\mathscr{L}_n,\mathscr{L}_m;{\mathscr{B}^*}_{\alpha,\,log^{\beta}})<\varepsilon$, for all $n,m>N$. Since $(\mathscr{L}_n)\subset B(\Upsilon)$, the family $(\mathscr{L}_n)$ is uniformly bounded and hence normal in Υ. Hence, there is $\mathscr{L}\in B(\Upsilon)$ and a subsequence $(\mathscr{L}_{n_j})_{j=1}^{\infty}$ where $\mathscr{L}_{n_j}$ converges to $\mathscr{L}$ uniformly on compact subsets of Υ. It follows that $\mathscr{L}_n$ also converges to $\mathscr{L}$ uniformly on compact subsets, and from the Cauchy formula, the same also holds for the derivatives. Now let $m>N$. Thus, the uniform convergence yields.

$$\left|\mathscr{L}_n^*(\gamma)-\mathscr{L}_m^*(\gamma)\right|(1-|\gamma|)^{\alpha}\left(\ln\frac{e^{\beta/\alpha}}{(1-|\gamma|)}\right)$$
$$=\lim_{n\to\infty}\left|\mathscr{L}_n^*(\gamma)-\mathscr{L}_m^*(\gamma)\right|(1-|\gamma|)^{\alpha}\left(\ln\frac{e^{\beta/\alpha}}{(1-|\gamma|)})\right)$$
$$\leq\lim_{n\to\infty}d(\mathscr{L}_n,\mathscr{L}_m;{\mathscr{B}^*}_{\alpha,\,log^{\beta}})\leq\varepsilon$$

for all $\gamma\in\Upsilon$, and it follows that $||\mathscr{L}||_{{\mathscr{B}^*}_{\alpha,\,log^{\beta}}}\leq||\mathscr{L}_m||_{{\mathscr{B}^*}_{\alpha,\,log^{\beta}}}+\varepsilon$. Then $\mathscr{L}\in{\mathscr{B}^*}_{\alpha,\,log^{\beta}}$ as desired. Moreover, the above inequality and the compactness of the usual ${\mathscr{B}^*}_{\alpha,\,lo\Lambda^{\beta}}$ space tends to $(\mathscr{L}_n)_{n=1}^{\infty}$ converges to $\mathscr{L}$ with respect to the metric d, and $({\mathscr{B}^*}_{\alpha,\,log^{\beta}},D)$ is complete D-metric space.
Since $\lim_{n\to\infty}d(\mathscr{L}_n,\mathscr{L}_m;{\mathscr{B}^*}_{\alpha,\,log^{\beta}})\leq\varepsilon$.

Now, we introduce the characterization of complete D-metric space $D(.,.;Q_K^*(p,q)$.

Proposition 2 *The $Q_K^*(p,q)$ class with the D-metric $D(.,.;Q_K^*(p,q)$ is a complete metric space. Hence, $Q_{K,\,0}^*(p,q)$ is a closed (and therefore complete) subspace of $Q_K^*(p,q)$.*

*Proof.*Assume that $\mathscr{L},\Lambda,\vartheta,a \in Q_K^*(p,q)$. Then clearly

(i) $D(\mathscr{L},\Lambda,\vartheta;Q_K^*(p,q)) \geq 0$, for all $\mathscr{L},\Lambda,\vartheta \in Q_K^*(p,q)$.

(ii)$D(\mathscr{L},\Lambda,\vartheta;Q_K^*(p,q)) = D(\mathscr{L},\vartheta,g;Q_K^*(p,q)) = D(\Lambda,\vartheta,\mathscr{L};Q_K^*(p,q))$.

(iii)$$D(\mathscr{L},\Lambda,\vartheta;Q_K^*(p,q)) \leq D(\mathscr{L},\Lambda,a;Q_K^*(p,q)) + D(\mathscr{L},a,\vartheta;Q_K^*(p,q)) + D(a,\Lambda,\vartheta;Q_K^*(p,q))$$

for all $\mathscr{L},\Lambda,\vartheta,a \in Q_K^*(p,q)$.

(iv)$D(\mathscr{L},\Lambda,\vartheta;Q_K^*(p,q)) = 0$ tends to $\mathscr{L} = \Lambda = \vartheta$.

Therefore, D is a D-metric on $Q_K^*(p,q)$, and $(Q_K^*(p,q),D)$ is D-metric space.

We use the Theorem 1 to proof the complete , assume that $(\mathscr{L}_n)_{n=1}^{\infty}$ be a Cauchy sequence in the metric space $(Q_K^*(p,q),d)$, that is, for any $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ so that $d(\mathscr{L}_n,\mathscr{L}_m;Q_K^*(p,q)) < \varepsilon$, for all $n,m > N$. Since $(\mathscr{L}_n) \subset B(\Upsilon)$, such that $\mathscr{L}_{n_j}$ converges to $\mathscr{L}$ uniformly on compact subsets of Υ. It follows that $\mathscr{L}_n$ also converges to $\mathscr{L}$ uniformly on compact subsets, now assume that $m > N$, and $0 < r < 1$. Then, the by using Fatou's we got

$$\begin{aligned}
&\int_{\Upsilon(0,r)} \left|\mathscr{L}_n^*(\gamma) - \mathscr{L}_m^*(\gamma)\right|^p (1-|\gamma|^2)^q\, K(\Lambda(\gamma,a))dA(\gamma)\\
&= \int_{\Upsilon(0,r)} \lim_{n\to\infty} \left|\mathscr{L}_n^*(\gamma) - \mathscr{L}_m^*(\gamma)\right|^p (1-|\gamma|^2)^q\, K(\Lambda(\gamma,a))dA(\gamma)\\
&\leq \lim_{n\to\infty} \int_{\Upsilon(0,r)} \left|\mathscr{L}_n^*(\gamma) - \mathscr{L}_m^*(\gamma)\right|^p (1-|\gamma|^2)^q\, K(\Lambda(\gamma,a))dA(\gamma) \leq \varepsilon^p,
\end{aligned}$$

and by take $r \to 1^-$, it follows that,

$$\begin{aligned}
&\int_{\Upsilon} (\mathscr{L}_n^*(\gamma))^p (1-|\gamma|^2)^q\, K(\Lambda(\gamma,a))dA(\gamma)\\
&\leq 2^p\varepsilon^p + 2^p \int_{\Upsilon} (\mathscr{L}_m^*(\gamma))^p (1-|\gamma|^2)^q\, K(\Lambda(\gamma,a))dA(\gamma).
\end{aligned}$$

This yields
$$||\mathscr{L}_n||_{Q_K^*(p,q)}^p \leq 2^p ||\mathscr{L}_m||_{Q_K^*(p,q)}^p + 2^p\varepsilon^p.$$

And thus $\mathscr{L} \in Q_K^*(p,q)$. We also find that $\mathscr{L}_n \to \mathscr{L}$ with respect to the metric of $(Q_K^*(p,q),D)$ and $(Q_K^*(p,q),D)$ is complete D-metric space.

5 weighted composition operators of $uC_{\mho} : \mathscr{B}^*{}_{\alpha,\, lo\Lambda^\beta} \to Q_K^*(p,q)$

The boundedness and compactness of weighted composition operators on $\mathscr{B}^*{}_{\alpha,\, log^\beta}$ and $Q_K^*(p,q)$ spaces are studied in this section. We use the following notation in the proof

$$\Phi_{\mho}(\alpha,\beta,p,q;a) = \sup_{a\in\Upsilon} \int_{\Upsilon} \frac{|\mho'(\gamma)|^p (u(\gamma))^p (1-|\gamma|^2)^q\, K(\Lambda(\gamma,a))}{(1-|\mho(\gamma)|)^{\alpha p} \left(\ln \frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)}\right)^p} dA(\gamma)dA(\gamma).$$

For $0 < \alpha < 1$, let we have the two functions $\mathscr{L},\Lambda \in \mathscr{B}^*{}_{\alpha,\, log^\beta}$ with the constant C,

$$(|\mathscr{L}^*(\gamma)| + |\Lambda^*(\gamma)|) \geq \frac{C}{(1-|\mho(\gamma)|)^{\alpha p} \left(\ln \frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)}\right)^p} > 0, \quad for\, each\, z \in \Upsilon.$$

Now, we introduce the following theorem

Theorem 2. *let $\mho$ be a holomorphic mapping from $\Upsilon \to \Upsilon$ and $p,q>0,\quad 0<\alpha\le 1$. Thus the weighted composition operator $uC_{\mho}:\mathscr{B}^*{}_{\alpha,\,log^{\beta}}\to Q_K^*(p,q)$ is bounded if and only if,*

$$\sup_{\gamma\in\Upsilon}\Phi_{\mho}(\alpha,\beta,p,q;a)<\infty. \tag{2}$$

Proof. First direction let $\sup_{\gamma\in\Upsilon}\mho_{\mho}(\alpha,\beta,p,q;a)<\infty$ is achieved, $\mathscr{L}\in\mathscr{B}^*{}_{\alpha,\,log^{\beta}}$ with $||\mathscr{L}||_{\mathscr{B}^*{}_{\alpha,\,log^{\beta}}}\le 1$, we can get

$$\begin{aligned}
&||uC_{\mho}\mathscr{L}||^p_{Q_K^*(p,q)}\\
&=\sup_{a\in\Upsilon}\int_{\Upsilon}(\mathscr{L}^*(\mho(\gamma))^p|\mho'(\gamma)|^p(u(\gamma))^p(1-|\gamma|^2)^q\,K(\Lambda(\gamma,a))dA(\gamma)\\
&\le\sup_{a\in\Upsilon}\int_{\Upsilon}\mathscr{L}^*(\mho(\gamma))^p(1-|\mho(\gamma)|)^{\alpha p}\left(\ln\frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)}\right)^p\\
&\quad\frac{|\mho'(\gamma)|^p(u(\gamma))^p(1-|\gamma|^2)^q\,K(\Lambda(\gamma,a))}{(1-|\mho(\gamma)|)^{\alpha p}\left(\ln\frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)}\right)^p}dA(\gamma)\\
&=||\mathscr{L}||^p_{\mathscr{B}^*{}_{\alpha,\,log^{\beta}}}\sup_{a\in\Upsilon}\int_{\Upsilon}\frac{|\mho'(\gamma)|^p(u(\gamma))^p(1-|\gamma|^2)^q\,K(\Lambda(\gamma,a))}{(1-|\mho(\gamma)|)^{\alpha p}\left(\ln\frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)}\right)^p}dA(\gamma)\\
&=||\mathscr{L}||^p_{\mathscr{B}^*{}_{\alpha,\,log^{\beta}}}\Phi_{\mho}(\alpha,\beta,p,q;a)<\infty.
\end{aligned}$$

Second direction, by using the fact that for each function $\mathscr{L}\in\mathscr{B}^*{}_{\alpha,\,log^{\beta}}$, the analytic function $uC_{\mho}(\mathscr{L})\in Q_K^*(p,q)$. Then, using the functions of lemma 1.2

$$\begin{aligned}
&2^p\left\{||uC_{\mho}\mathscr{L}_1||^p_{Q_K^*(p,q)}+||uC_{\mho}\mathscr{L}_2||^p_{Q_K^*(p,q)}\right\}\\
&=2^p\left\{\sup_{a\in\Upsilon}\int_{\Upsilon}\left((\mathscr{L}_1^*(\mho(\gamma)))^p+(\mathscr{L}_2^*(\mho(\gamma)))^p\right)|\mho'(\gamma)|^p\right.\\
&\quad\left.\times(u(\gamma))^p(1-|\gamma|^2)^q\,K(\Lambda(\gamma,a))dA(\gamma)\right\}\\
&\ge C\left\{\sup_{a\in\Upsilon}\int_{\Upsilon}\frac{|\mho'(\gamma)|^p(u(\gamma))^p(1-|\gamma|^2)^q\,K(\Lambda(\gamma,a))}{(1-|\mho(\gamma)|)^{\alpha p}\left(\ln\frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)}\right)^p}dA(\gamma)\right\}\\
&\ge C\Phi_{\mho}(\alpha,\beta,p,q;a).
\end{aligned}$$

Hence $uC_{\mho}$ is bounded, the proof is completed.

The weighted composition operator $uC_{\mho}:\mathscr{B}^*{}_{\alpha,\,log^{\beta}}\to Q_K^*(p,q)$ is compact if and only if for every sequence $\mathscr{L}_n\in\mathbb{N}\subset Q_K^*(p,q)$ is bounded in $Q_K^*(p,q)$ norm and $\mathscr{L}_n\to 0,\ n\to\infty$, uniformly on compact subset of the unit disk (where $\mathbb{N}$ be the set of all natural numbers), hence,

$$||uC_{\mho}(\mathscr{L}_n)||_{Q_K^*(p,q)}\to 0,n\to\infty.$$

Now, we introduce the compactness in the following theorem:

Theorem 3. *Let $\mho:\Upsilon\to\Upsilon$ and $0<p<\infty,\quad -1<q<\infty,\quad 0<\alpha\le 1$. Thus the following conditions are equivalent:*

(i) $uC_{\mho}:\mathscr{B}^*{}_{\alpha,\,log^{\beta}}\to Q_K^*(p,q)$ is compact.

(ii) $\lim_{r\to 1^-}\sup_{a\in\Upsilon}\Phi_{\mho}(\alpha,\beta,p,q;a)\to 0.$

Proof. The first direction that (ii) achieved. Assume $B:=\bar{B}(\Lambda,\delta)\subset\mathscr{B}^*_{\alpha}$, where $\Lambda\in\mathscr{B}^*_{\alpha}$ and $\delta>0$, is a closed ball, and Assume that $\{\mathscr{L}_n\}_{n=1}^{\infty}\subset B$ is any sequence. We need to arrive that the image has a convergent subsequence in $Q_K^*(p,q)$, which implies that the proof for compactness of $uC_{\mho}$. Again, $\{\mathscr{L}_n\}_{n=1}^{\infty}\subset B(\Upsilon)$ implies that, there is a subsequence

$\{\mathscr{L}_{n_j}\}_{j=1}^{\infty}$ which converges uniformly on the compact subsets of Υ to an analytic function $\mathscr{L}$. By using the derivative of an analytic function from Cauchy formula, the sequence $\{\mathscr{L}'_{n_j}\}_{j=1}^{\infty}$ converges uniformly on compact subsets of Υ to $\mathscr{L}'$. It follows that also the sequences $\{\mathscr{L}_{n_j}\circ\mho\}_{j=1}^{\infty}$ and $\{\mathscr{L}'_{n_j}\circ\mho\}_{j=1}^{\infty}$ converge uniformly on compact subsets of Υ to $\{\mathscr{L}\circ\mho\}$ and $\{\mathscr{L}'\circ\mho\}$, respectively. Therefore, $\mathscr{L}\in B\subset\mathscr{B}_{\alpha}^{*}$ since for any fixed $R, 0<R<1$, the uniform convergence yield $d(\mathscr{L},\Lambda;\mathscr{B}_{\alpha}^{*})\leq\delta$ (see [10] pp.130).
Let $\varepsilon>0$. Since (ii) is achieved, we may fix $r, 0<r<1$, where

$$\sup_{a\in\Upsilon}\int_{|\mho(\gamma)|\geqslant r}\frac{|\mho'(\gamma)|^{p}(u(\gamma))^{p}(1-|\gamma|^{2})^{q}\,K(\Lambda(\gamma,a))}{(1-|\mho(\gamma)|)^{\alpha p}\left(\ln\frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)}\right)^{p}}dA(\gamma)dA(\gamma)\leq\varepsilon. \tag{3}$$

By the uniform convergence, we may fix $N_1\in\mathbb{N}$ where

$$|(\mathscr{L}_{n_j}\circ\mho(0)\,)-(\mathscr{L}\circ\mho(0)\,)|\leq\varepsilon.\quad for\ all\quad j\geqslant N_1. \tag{4}$$

The condition (ii) is known to imply the compactness of $uC_{\mho}:\mathscr{B}^{*}{}_{\alpha,\,log^{\beta}}\to Q_K^{*}(p,q)$, hence, possibly to passing once more to a subsequence and adjusting the notations, we may assume that

$$||(\mathscr{L}_{n_j}\circ\mho\,)-(\mathscr{L}\circ\mho\,)||_{Q_K^{*}(p,q)}\leq\varepsilon.\quad for\ all\quad j\geqslant N_2\quad for\ some\quad N_2\in\mathbb{N}. \tag{5}$$

Now, assume

$$I_1(a,r)=\sup_{a\in\Upsilon}\int_{|\mho(\gamma)|\geqslant r}\Big((\mathscr{L}_{n_j}\circ\mho\,)^{*}(\gamma)-(\Lambda\circ\mho\,)^{*}(\gamma)\Big)^{p}K(\Lambda(\gamma,a))(1-|\gamma|^{2})^{q}dA(\gamma).$$

and

$$I_2(a,r)=\sup_{a\in\Upsilon}\int_{|\mho(\gamma)|\leq r}\Big((\mathscr{L}_{n_j}\circ\mho\,)^{*}(\gamma)-(\Lambda\circ\mho\,)^{*}(\gamma)\Big)^{p}K(\Lambda(\gamma,a))(1-|\gamma|^{2})^{q}dA(\gamma).$$

$\{\mathscr{L}_{n_j}\}_{n=1}^{\infty}\subset B$ and $\mathscr{L}\in B$, it follows from (1) that

$$\begin{aligned}
I_1(a,r)&=\sup_{a\in\Upsilon}\int_{|\mho(\gamma)|\geqslant r}\Big((\mathscr{L}_{n_j}\circ\mho\,)^{*}(\gamma)-(\Lambda\circ\mho\,)^{*}(\gamma)\Big)^{p}K(\Lambda(\gamma,a))(1-|\gamma|^{2})^{q}dA(\gamma)\\
&=\sup_{a\in\Upsilon}\int_{|\mho(\gamma)|\geqslant r}\left|\frac{(\mathscr{L}_{n_j}\circ\mho\,)'(\gamma)}{1-|(\mathscr{L}_{n_j}\circ\mho\,)(\gamma)|^{2}}-\frac{(\Lambda\circ\mho\,)'(\gamma)}{1-|(\Lambda\circ\mho\,)(\gamma)|^{2}}\right|^{p}\\
&K(\Lambda(\gamma,a))(1-|\gamma|^{2})^{q}dA(\gamma)\\
&=\sup_{a\in\Upsilon}\int_{|\mho(\gamma)|\geqslant r}\left|\left(\frac{\mathscr{L}'_{nj}(\mho(\gamma))}{1-|\mathscr{L}_{nj}(\mho(\gamma))|^{2}}-\frac{\Lambda(\mho(\gamma))}{1-|\Lambda(\mho(\gamma))|^{2}}\right)\right.\\
&\left.(1-|\mho(\gamma)|)^{\alpha}\left(\ln\frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)}\right)\right|^{p}\\
&\frac{|\mho'(\gamma)|^{p}(u(\gamma))^{p}(1-|\gamma|^{2})^{q}\,K(\Lambda(\gamma,a))}{(1-|\mho(\gamma)|)^{\alpha p}\left(\ln\frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)}\right)^{p}}dA(\gamma)\\
&\leq D_{\mathscr{B}^{*}{}_{\alpha,\,log^{\beta}}}(\mathscr{L}_{n_j},\Lambda)\sup_{a\in\Upsilon}\int_{|\mho(\gamma)|\geqslant r}\\
&\frac{|\mho'(\gamma)|^{p}(u(\gamma))^{p}(1-|\gamma|^{2})^{q}\,K(\Lambda(\gamma,a))}{(1-|\mho(\gamma)|)^{\alpha p}\left(\ln\frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)}\right)^{p}}dA(\gamma)\\
&\leq\delta\varepsilon.
\end{aligned} \tag{6}$$

On the other hand, for the uniform convergence on compact subsets of Υ, we get an $N_3\in\mathbb{N}$ where for all $j\geqslant N_3$,

$$\left|\left(\frac{\mathscr{L}'_{n_j}(\mho(\gamma))}{1-|\mathscr{L}_{n_j}(\mho(\gamma))|^{2}}-\frac{\Lambda(\mho(\gamma))}{1-|\Lambda(\mho\,(\gamma))|^{2}}\right)(1-|\mho(\gamma)|)^{\alpha}\left(\ln\frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)}\right)\right|\leq\varepsilon.$$

for all z with $|\mho(\gamma)| \leq r$. Thus, for such j,

$$\begin{aligned} I_2(a,r) &= \sup_{a\in\Upsilon} \int_{|\mho(\gamma)|\leq r} \left((\mathscr{L}_{n_j} \circ \mho)^*(\gamma) - (\Lambda \circ \mho)^*(\gamma) \right)^p K(\Lambda(\gamma,a))(1-|\gamma|^2)^q dA(\gamma) \\ &= \sup_{a\in\Upsilon} \int_{|\mho(\gamma)|\leq r} \left| \frac{(\mathscr{L}_{n_j} \circ \mho)'(\gamma)}{1-|(\mathscr{L}_{n_j} \circ \mho)(\gamma)|^2} - \frac{(\Lambda \circ \mho)'(\gamma)}{1-|(\Lambda \circ \mho)(\gamma)|^2} \right|^p \\ & K(\Lambda(\gamma,a))(1-|\gamma|^2)^q dA(\gamma) \\ &= \sup_{a\in\Upsilon} \int_{|\mho(\gamma)|\leq r} \left| \left(\frac{\mathscr{L}'_{nj}(\mho(\gamma))}{1-|\mathscr{L}_{n_j}(\mho(\gamma))|^2} - \frac{\Lambda(\mho(\gamma))}{1-|\Lambda(\mho(\gamma))|^2} \right) \right. \\ & \left. (1-|\mho(\gamma)|)^\alpha \left(\ln \frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)} \right) \right|^p \\ & \frac{|\mho'(\gamma)|^p (u(\gamma))^p (1-|\gamma|^2)^q K(\Lambda(\gamma,a))}{(1-|\mho(\gamma)|)^{\alpha p} \left(\ln \frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)} \right)^p} dA(\gamma). \\ &\leq \varepsilon \sup_{a\in\Upsilon} \int_{|\mho(\gamma)|\leq r} \left(\frac{|\mho'(\gamma)|^p (u(\gamma))^p (1-|\gamma|^2)^q K(\Lambda(\gamma,a))}{(1-|\mho(\gamma)|)^{\alpha p} \left(\ln \frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)|)} \right)^p} \right)^{\frac{1}{p}} dA(\gamma) \\ &\leq C\varepsilon. \end{aligned} \tag{7}$$

where C is the bounded obtained from (2). Combining (4), (5), (6) and (7) we get $\mathscr{L}_{n_j} \to \mathscr{L}$ in $Q^*_K(p,q)$. From the converse direction, Assume that $\mathscr{L}_n(\gamma) := \frac{1}{2} n^{\alpha-1} z^n$ for all $n \in N$, $n \geqslant 2$. Thus $\{\mathscr{L}_n\}_{n=1}^{\infty}$ belongs to the ball $B := \bar{B}(\Lambda,\delta) \subset \mathscr{B}^*_\alpha$ (see [7]). let $C_\mho$ maps the closed ball $\bar{B}(0,3) \subset \mathscr{B}^*_\alpha$ to a compact subset of $Q^*_K(p,q)$, Then, we have an unbounded increasing subsequence $\{\mathscr{L}_{n_j}\}_{j=1}^{\infty}$ where the image of the subsequence $\{C_\mho \mathscr{L}_{n_j}\}_{j=1}^{\infty}$ converges with respect to it's norm. Where, $\{\mathscr{L}_n\}_{n=1}^{\infty}$ and $\{C_\mho \mathscr{L}_{n_j}\}_{j=1}^{\infty}$ converge to 0 function uniformly on compact subsets of Υ, the limit of the last sequence must be 0. Thus,

$$||n_j{}^{\alpha-1} a^{n_j-1}||_{Q^*_K(p,q)} \to 0, j \to \infty. \tag{8}$$

Now assume that $r_j = 1 - \frac{1}{n_j}$. For all numbers $a, rj \leq a < 1$, we get the estimate (see [7]).

$$\frac{n_j{}^{\alpha-1} \mho^{n_j}}{1-a^{n_j}} \geqslant \frac{1}{e(1-a)^\alpha}. \tag{9}$$

Using (9) we obtain

$$\begin{aligned} ||n_j{}^{\alpha-1} a^{n_j-1}||_{Q^*_K(p,q)} &\geqslant \sup_{a\in\Upsilon} \int_{|\mho(\gamma)|\geqslant r} \\ & \frac{|n_j{}^{\alpha} (\mho(\gamma))^{n_j-1} \mho'(\gamma)(u(\gamma))|^p (1-|\gamma|^2)^q K(\Lambda(\gamma,a))}{(1-|\mho(\gamma)^{n_j}|)^2} dA(\gamma). \\ &\geqslant \frac{1}{2e^p} \sup_{a\in\Upsilon} \int_{|\mho(\gamma)|\geqslant r} \frac{|\mho'(\gamma)|^p (u(\gamma))^p (1-|\gamma|^2)^q K(\Lambda(\gamma,a))}{(1-|\mho(\gamma)^{n_j}|)^{\alpha p} \left(\ln \frac{e^{\beta/\alpha}}{(1-|\mho(\gamma)^{n_j}|)} \right)^p} dA(\gamma). \end{aligned} \tag{10}$$

Hence, the condition (ii) follows.

6 Conclusions

Finally, from this work, we obtained the important properties for D-metric spaces. Furthermore, we proved the essential properties for D-metric on $\mathscr{B}^*{}_{\alpha,\, log^\beta}$ and $Q^*_K(p,q)$. in the end, we presented the proof of the boundedness and compactness for the weighted composition operators$uC_\mho$ from $\mathscr{B}^*{}_\alpha$ to $Q^*_K(p,q)$.

Conflict of Interests

There is no conflict of interests by authors regarding the publication of this manuscript.

References

[1] A. Kamal and T. I. Yassen, Some properties of composition operator acting between general hyperbolic type spaces, *International Journal of Mathematical Analysis and Applications* **2**(2) (2015) 17–26.
[2] I. M. Hanafy, A. Kamal and M. H. Eissa, Properties of product T_hC_ϕ operators from $\mathscr{B}^{\alpha}_{log^{\beta}}$ to $Q_{K,\omega}(p,q)$ spaces, *Bulletin of the International Mathematical Virtual Institute* **9** (2019) 159–168.
[3] A. Kamal and M. H. Eissa, Isometries of differentiation and composition operators on zygmund type space, *Bulletin of Pure & Applied Sciences-Mathematics and Statistics* **38**(2) (2019) 592–602.
[4] O. Hu and X. Zhu, Compact generalized weighted composition operators on the bergman space, *Opuscula Mathematica* **37**(2) (2017).
[5] S. Stević, On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces, *Nonlinear Analysis: Theory, Methods & Applications* **71**(12) (2009) 6323–6342.
[6] R. Rashwan, A. El-Sayed Ahmed and A. Kamal, Integral characterizations of weighted Bloch space and $Q_{K,\omega}(p,q)$ spaces, *Mthematica* **74**(1) (2009) 63–76.
[7] F. Pérez-González, J. Rättyä and J. Taskinen, Lipschitz continuous and compact composition operators in hyperbolic classes, *Mediterranean Journal of Mathematics* **8**(1) (2011) 123–135.
[8] T. Hosokawa, Differences of weighted composition operators on the Bloch spaces, *Complex Analysis and Operator Theory* **3**(4) (2009) 847–866.
[9] M. Tjani, Compact composition operators on Besov spaces, *Transactions of the American Mathematical Society* **355**(11) (2003) 4683–4698.
[10] B. Dhage, *A study of some fixed point theorems*, PhD thesis, Marathwada University, Aurangabad, India1984.
[11] B. Dhage, Generalised metric space and mappings with fixed point, *Bulletin of Calcutta Mathematical Society* **84**(4) 329–336.
[12] B. Dhage, On generalized metric spaces and topological structure. II, *Pure Appl. Math. Sci.* **40**(1-2) (1994) 37–41.
[13] B. Dhage, Generalized metric spaces and topological structure. i, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi* **46**(1) (2000) 3–24.
[14] E. Kreyszig, *Introductory Functional Analysis with Applications* (New York: wiley, 1978).