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Travelling Wave Solutions for Baroclinic Potential Vorticity Equation

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Abstract: In this work we study the baroclinic potential vorticity (BPV) equation. We apply a variety of ansatze approaches. The analysis leads to a variety of travelling wave solutions of distinct structures.

Keywords: Baroclinic potential vorticity equation; the tanh-coth method; the sin-cos ansatz

1 Introduction

In recent years, there has been an increasing interest in the study of nonlinear evolution models that exhibit significant features of scientific phenomena. It is necessary to determine exact solutions for these nonlinear equations to enable us to get an insight through qualitative and quantitative properties of these equations. Many powerful methods [1,2,3,4,5,6,7] were established to achieve the exact solutions of nonlinear equations.

The aim of this work is to present a reliable treatment for studying the baroclinic potential vorticity (BPV) equation. The (3+1)-dimensional BPV equation reads

$$\begin{aligned} v_t + u_x v_y - u_y v_x + \beta u_x = 0, \\ v = u_{xx} + u_{yy} + u_{zz}, \end{aligned} \tag{1}$$

where β is a constant, and $u \equiv u(x,y,z,t)$ and $v \equiv v(x,y,z,t)$. Eliminating v, the BPV takes the form

$$\begin{aligned} (u_{xx} + u_{yy} + u_{zz})_t + u_x(u_{xx} + u_{yy} + u_{zz})_y \\ -u_y(u_{xx} + u_{yy} + u_{zz})_x + \beta u_x = 0. \end{aligned} \tag{2}$$

In [1], it was indicated that there are difficulties to derive solutions for the BPV equation (2) by some specific methods such as symmetry reductions or the Jacobi elliptic method and other methods as well. However, the reductive perturbation method was used in [1] where three types of generalized (2+1)-dimensional KP equations were derived from the BPV equation.

In this work we will apply a variety of useful methods to obtain travelling wave solutions for the baroclinic potential vorticity (BPV) equation (2). The obtained solutions show distinct physical structures. The constraints that will guarantee the existence of specific solutions will be investigated.

2 The (3+1)-Dimensional BPV Equation

As stated before we will apply a variety of methods to determine travelling wave solutions. We will begin our analysis by introducing the wave variable

$$\xi = kx + ry + sz - \omega t, \tag{3}$$

where k, r, and s are constants, and ω is the dispersion relation.

2.1 The tanh/coth Method

The tanh method [7,8,9,10,11,12,13] is now well-known in the literature, hence we skip details of this method. Using the balance method, between linear and nonlinear terms of the BPV equation (2), we find $M = -1$. In view of this, the tanh method admits the use of the solution in the form

$$u(x,y,z,t) = \frac{1}{a_0 + a_1 \tanh(kx + ry + sz - \omega t)}, \tag{4}$$

where a_0, a_1 are parameters that will be determined. Substituting (4) into (2), collecting the coefficients of

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$\tanh^i(\xi)$, $i=0,1,2$, and equating each coefficient to zero we find

$$\begin{aligned} a_0 &= \pm a_1, \\ \omega &= \tfrac{\beta k}{4(k^2+r^2+s^2)}, \end{aligned} \tag{5}$$

where a_1 will be left as a free parameter. This will give the singular solution

$$\begin{aligned} &u(x,y,z,t) \\ &= \pm\tfrac{1}{a_1(1+\tanh(kx+ry+sz-(\beta k/(4(k^2+r^2+s^2)))t))}. \end{aligned} \tag{6}$$

The obtained solution blows up for $x\to-\infty$.

However, using the coth ansatz, we may also assume the solution takes the form

$$u(x,y,z,t) = \pm\frac{1}{a_0+a_1\coth(kx+ry+sz-wt)}. \tag{7}$$

Proceeding as presented earlier we obtain

$$\begin{aligned} a_0 &= \pm a_1, \\ \omega &= \tfrac{\beta k}{4(k^2+r^2+s^2)}. \end{aligned} \tag{8}$$

This in turn gives the singular solution

$$\begin{aligned} &u(x,y,z,t) \\ &= \pm\tfrac{1}{a_1(1+\coth(kx+ry+sz-(\beta k/(4(k^2+r^2+s^2)))t))}, \end{aligned} \tag{9}$$

which also blows up for $x\to-\infty$.

2.2 The tan /cot Ansatze

The tan ansatz introduces the solution in the form

$$u(x,y,z,t) = \frac{1}{a_0+a_1\tan(kx+ry+sz-\omega t)}, \tag{10}$$

where a_0, a_1 are parameters that will be determined. Substituting (10) into (2), collecting the coefficients of $\tan^i(\xi)$, $i=0,1,2$, and equating each coefficient to zero we find

$$\begin{aligned} a_0 &= \pm i a_1, \quad i-\sqrt{-1}, \\ \omega &= -\tfrac{\beta k}{4(k^2+r^2+s^2)}, \end{aligned} \tag{11}$$

where a_1 will be left as a free parameter. This will give the complex solution

$$\begin{aligned} &u(x,y,z,t) \\ &= \pm\tfrac{1}{a_1(i+\tan(kx+ry+sz+(\beta k/(4(k^2+r^2+s^2)))t))}. \end{aligned} \tag{12}$$

However, using the cot ansatz, we may also assume the solution takes the form

$$u(x,y,z,t) = \pm\frac{1}{a_0+a_1\cot(kx+ry+sz-\omega t)}. \tag{13}$$

Proceeding as before we find

$$\begin{aligned} a_0 &= \pm i a_1, \quad i=\sqrt{-1}, \\ \omega &= -\tfrac{\beta k}{4(k^2+r^2+s^2)}. \end{aligned} \tag{14}$$

This result leads to the complex solution

$$\begin{aligned} &u(x,y,z,t) \\ &= \pm\tfrac{1}{a_1(i+\coth(kx+ry+sz+(\beta k/(4(k^2+r^2+s^2)))t)}, \end{aligned} \tag{15}$$

which also blows up for $x\to-\infty$.

2.3 The sinh/cosh Ansatze

The sinh ansatz admits the use of the solution in the form

$$u(x,y,z,t) = a_0+a_1\sinh(kx+ry+sz-\omega t), \tag{16}$$

where a_0 and a_1 are parameters that will be determined later. Substituting (25) into (2), collecting the coefficients of cosh, and solving the resulting equation we obtain

$$\omega = \frac{\beta k}{k^2+r^2+s^2}, \tag{17}$$

where a_0 and a_1 are left as free parameters. Consequently, we obtain the solitary pattern solution

$$u(x,y,z,t) = a_0+a_1\sinh(kx+ry+sz-\frac{\beta k}{k^2+r^2+s^2}t). \tag{18}$$

In a like manner, we can use the cosh ansatz where we can set the solution in the form

$$u(x,y,z,t) = a_0+a_1\cosh(kx+ry+sz-\omega t), \tag{19}$$

where a_0 and a_1 are parameters that will be determined later. Proceeding as before we find

$$\omega = \frac{\beta k}{k^2+r^2+s^2}, \tag{20}$$

where a_0 and a_1 are left as free parameters. Consequently, we obtain the solitary pattern solution

$$u(x,y,z,t) = a_0+a_1\cosh(kx+ry+sz-\frac{\beta k}{k^2+r^2+s^2}t). \tag{21}$$

2.4 The sin/cos Ansatze

To derive periodic solutions that satisfy the BPV equation, it is normal to use the sin/cos ansatze. The sin ansatz introduces the solution in the form

$$u(x,y,z,t) = a_0 + a_1 \sin(kx+ry+sz-\omega t), \tag{22}$$

where a_0 and a_1 are parameters. Substituting (22) into (2), collecting the coefficients of cos, and solving the resulting equation we obtain

$$\omega = -\frac{\beta k}{k^2+r^2+s^2}, \tag{23}$$

where a_0 and a_1 are left as free parameters. This in turn gives the periodic solution

$$u(x,y,z,t) = a_0 + a_1 \sin(kx+ry+sz+\frac{\beta k}{k^2+r^2+s^2}t). \tag{24}$$

To use the cos ansatz, we assume the solution takes the form

$$u(x,y,z,t) = a_0 + a_1 \cos(kx+ry+sz-\omega t), \tag{25}$$

where a_0 and a_1 are parameters to be determined. Proceeding as before we find

$$\omega = -\frac{\beta k}{k^2+r^2+s^2}, \tag{26}$$

where a_0 and a_1 are left as free parameters. Consequently, we obtain the periodic solution

$$u(x,y,z,t) = a_0 + a_1 \cos(kx+ry+sz+\frac{\beta k}{k^2+r^2+s^2}t). \tag{27}$$

3 Discussion

We examined the baroclinic potential vorticity (BPV) equation. We applied a variety of methods and specific ansatze. We derived a variety of travelling wave solutions with distinct physical structures. The obtained solutions vary from singular solutions, through solitary pattern solutions to periodic solutions.

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