On the diameter of the Kronecker product graph

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Abstract: Let $G_1$ and $G_2$ be two undirected nontrivial graphs. The Kronecker product of $G_1$ and $G_2$ denoted by $G_1 \otimes G_2$ with vertex set $V(G_1) \times V(G_2)$, two vertices $x_1x_2$ and $y_1y_2$ are adjacent if and only if $(x_1, y_1) \in E(G_1)$ and $(x_2, y_2) \in E(G_2)$. This paper presents a formula for computing the diameter of $G_1 \otimes G_2$ by means of the diameters and primitive exponents of factor graphs.

Keywords: graph theory, diameter, Kronecker product, primitive exponent.

1 Introduction

For notation and graph-theoretical terminology not defined here we follow [18]. Specifically, let $G = (V, E)$ be a nontrivial graph with no parallel edges, but loops allowed, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set.

For two graphs $G$ and $H$, Kronecker product $G \otimes H$ is a graph with vertex set $V(G) \times V(H)$ and two vertices $x_1x_2$ and $y_1y_2$ are adjacent when $(x_1, y_1) \in E(G)$ and $(x_2, y_2) \in E(G)$.

As an operation of graphs, Kronecker product $G \otimes H$ was introduced first by Weichsel [15] in 1962. It has been shown that the Kronecker product is a good method to construct larger networks that can generate many good properties of the factor graphs (see [9]), and has received much research attention recently. Some properties and graphic parameters have been investigated [1,2,5,8,11]. The connectivity and diameter are two important parameters to measure reliability and efficiency of a network. Very recently, the connectivity of Kronecker product graph has been deeply studied (see, [3,6,7,11,12,14,16,17]). However, the diameter of Kronecker product graph has not been investigated yet.

In this paper, we determine the diameter of Kronecker product graph by means of primitive exponents and diameters of factor graphs. In particular, we obtain that

$$d(G_1 \otimes G_2) = \begin{cases} 
\gamma_1 & \text{if } \gamma_1 = \gamma_2; \\
\max\{\gamma_2 + 1, d_1\} & \text{if } \gamma_1 > \gamma_2; \\
\max\{\gamma_1 + 1, d_2\} & \text{if } \gamma_1 < \gamma_2,
\end{cases}$$

where $\gamma_i$ and $d_i$ are the primitive exponent and diameter of $G_i$ for $i = 1, 2$, respectively.

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2 Some Lemmas

Let $G$ be a graph. Denote $\gamma(G; x, y)$ to be the minimum integer such that there exists an $(x, y)$-walk of length $k$ for any $k \geq \gamma(G; x, y)$, and $\gamma(G)$ be the minimum integer $\gamma$ for which, for any two vertices $x$ and $y$ in $G$, there exists an $(x, y)$-walk of length $k$ for any integer $k \geq \gamma$. Let $\gamma(G) = \max\{\gamma(G; x, y) : x, y \in V(G)\}$.

If $\gamma(G)$ is well-defined, then $G$ is said to be primitive, and $\gamma(G)$ is called the primitive exponent, exponent for short, of $G$. If $\gamma(G)$ does not exist, then denote $\gamma(G) = \infty$.

Let $K_n^+$ be a graph obtained from a complete graph $K_n$ by appending a loop on each vertex. It is clear that for a graph $G$ without parallel edges of order $n$, $\gamma(G) = 1$ if and only if $G \cong K_n^+$.

Let $A$ be the adjacency matrix of $G$. Equivalently, the exponent of $G$ is the minimum integer $\gamma$ for which $A^\gamma > 0$ and $A^{k} \neq 0$ for any positive integer $k < \gamma$. Let $A_i$ be the adjacent matrix of $G_i$ for $i = 1, 2$. Since for any positive integer $k$, $(A_1 \otimes A_2)^k = A_1^k \otimes A_2^k$, by definition, we have the following result immediately.

**Proposition 2.1** Let $G_i$ be a primitive graph with exponent $\gamma_i$ for $i = 1, 2$, and $G = G_1 \otimes G_2$. Then $\gamma(G) = \max\{\gamma_1, \gamma_2\}$.

The following lemmas will be used in proofs of our main results.

**Lemma 2.1** (Liu et al. [10]) A graph $G$ is primitive if and only if $G$ is connected and contains odd cycles.

**Lemma 2.2** (Liu et al. [10]) Let $G$ be a primitive graph, and let $x$ and $y$ be any pair of vertices in $V(G)$. If there are two $(x, y)$-walks $P_1$ and $P_2$ with lengths $k_1$ and $k_2$, respectively, where $k_1$ and $k_2$ have different parity, then $\gamma(x, y) \leq \max\{k_1, k_2\} - 1$.

**Lemma 2.3** (Delorme and Solé [4]) If $G$ is a primitive graph with diameter $d$, then $\gamma(G) \leq 2d$.

**Lemma 2.4** (Weichesel [15]) Let $G_1$ and $G_2$ be two connected graphs and $G = G_1 \otimes G_2$. Then $G$ is connected if and only if either $G_1$ or $G_2$ contains an odd cycle.

**Lemma 2.5** Let $G = G_1 \otimes G_2$, $x_i$ and $y_i$ be any two vertices in $G_i$, $P_i$ be an $(x_i, y_i)$-walk of length $\ell_i$ in $G_i$ for $i = 1, 2$. If $\ell_1$ and $\ell_2$ have same parity, then there is an $(x_1 x_2, y_1 y_2)$-walk of length $\max\{\ell_1, \ell_2\}$ in $G$.

**Proof.** Without loss of generality, suppose $\ell_1 \geq \ell_2$. Let $k = \ell_1 - \ell_2$. Then $k$ is even. Let $P_1 = (x_1, z_1, \ldots, z_{\ell_1-1}, y_1)$ and $P_2 = (x_2, u_1, \ldots, u_{\ell_2-1}, y_2)$ be an $(x_2, y_2)$-walk of length $\ell_2$ in $G_2$ obtained from $P_2$ by repeating $k$ times of some edge in $P_2$. Then $(x_1 x_2, z_1 u_1, \ldots, z_{\ell_1-1} u_{\ell_2-1}, y_1 y_2)$ is an $(x_1 x_2, y_1 y_2)$-walk of length $\ell_1$ in $G$.

**Lemma 2.6** Let $G$ be a primitive graph with exponent $\gamma$ and order $n > 2$. We have (i) if $\gamma$ is odd, then there exist two vertices $x$ and $y$, and two different vertices $u$ and $v$, such that the shortest odd $(x, y)$-walk and the shortest even $(u, v)$-walk are of length $\gamma$ and $\gamma + 1$, respectively; (ii) if $\gamma$ is even, then there exist two different vertices $p$ and $q$, and two vertices $w$ and $s$, such that the shortest even $(p, q)$-walk and the shortest odd $(w, s)$-walk are of length $\gamma$ and $\gamma + 1$, respectively.

**Proof.** (i) Assume that $\gamma$ is odd. If $\gamma = 1$, then $G \cong K_n^+$. Let $u$ and $v$ be two different vertices in $G$. Then the shortest odd $(u, v)$-walk and the shortest even $(u, v)$-walk are of length $1$ and $\gamma$, respectively. Suppose now $\gamma \geq 2$.

Let $A$ be the adjacency matrix of $G$. By definition of $\gamma$, $A^{\gamma-1} \neq 0$ and $A^{\gamma-2} \neq 0$. These imply that there exist four vertices $x$, $y$, $u$ and $v$ such that there are no odd $(x, y)$-walk and even $(u, v)$-walk with length $\gamma - 2$ and $\gamma - 1$, respectively. Hence there are no odd $(x, y)$-walk and even $(u, v)$-walk with length
no more than $\gamma - 2$ and $\gamma - 1$, respectively. Therefore, the shortest odd $(x, y)$-walk and the shortest even $(u, v)$-walk are of length $\gamma$ and $\gamma + 1$, respectively.

We now show $u \neq v$. If $u = v$, then $(u, w, u)$ is an even $(u, v)$-walk of length 2 for any vertex $w$ adjacent to $u$ in $G$, a contradiction with $\gamma + 1 \geq 3$.

(ii) Assume $\gamma$ is even. If $\gamma = 2$, then $d = d(G) = 1$ or 2 since $d \leq \gamma$. If $d = 1$, then $iG$ is isomorphic to a complete graph $K_n$ with $m$ vertices having loops and $m < n$. Let $p$ be a vertex with no loop and $q \neq p$ be another vertex in $G$. Then the shortest even $(p, q)$-walk and odd $(p, p)$-walk are of length 2 and 3, respectively. If $d = 2$, then there exist two different vertices $p$ and $q$ such that $d_G(p, q) = 2$, and hence the shortest even $(p, q)$-walk and odd $(p, q)$-walk are of length 2 and 3, respectively.

The case when $\gamma > 2$ can be proved by applying the similar discussion as in (i).

Lemma 2.7 Let $G_i$ be a primitive graph with exponent $\gamma_i$ for $i = 1, 2$. $G = G_1 \otimes G_2$, and $x = x_1 x_2 y$ and $y = y_1 y_2$ be two different vertices in $G$. If the shortest odd (resp. even) $(x_1, y_1)$-walk in $G_1$ and the shortest even (resp. odd) $(x_2, y_2)$-walk in $G_2$ are of length $m$ and $n$, respectively, then $d_G(x, y) \geq \min\{m, n\}$.

Proof. Without loss of generality, assume that $m$ is odd and $n$ is even. Let $P = (x_1, x_2, \ldots, u_1 u_2, \ldots, y_1 y_2)$ be a minimum $(x, y)$-path with length $s$ in $G$. Then $(x_1, \ldots, u_1, \ldots, y_1)$ and $(x_2, \ldots, u_2, \ldots, y_2)$ be an $(x_1, y_1)$-walk in $G_1$ and an $(x_2, y_2)$-walk in $G_2$, respectively, and both of them are of length $s$.

If $s$ is odd, then $s \geq m$ since the shortest odd $(x_1, y_1)$-walk in $G_1$ is of length $m$; If $s$ is even, then $s \geq n$ since the shortest even $(x_2, y_2)$-walk in $G_2$ is of length $n$. Therefore $d_G(x, y) = s \geq \min\{m, n\}$. 

3 Main results

Let $G$ be a connected graph with odd cycles and $C^o(G)$ be the set of all odd cycles in $G$. For $C \in C^o(G)$ and $x \in V(G)$, let

$$d_G(x, C) = \min\{d_G(x, y) : y \in V(C)\},$$

and let

$$d^o_G(C) = \max\{d_G(x, C) : x \in V(G - C)\} \quad \text{for} \quad C \in C^o(G),$$

$$l^o(G) = \min\{2d^o_G(C) + |V(C)| - 1 : C \in C^o(G)\}.$$ We define $l^o(G) = \infty$ if $G$ is bipartite.

Theorem 3.1 $\gamma(G) \leq l^o(G)$ for any connected graph $G$.

Proof. If $G$ contains no odd cycles, then $l^o(G) = \infty$, and so the conclusion holds. Suppose that $G$ contains odd cycles. By Lemma 2.1, $G$ is primitive. We only need to prove that for any two vertices $x$ and $y$ in $G$, $\gamma(G; x, y) \leq l^o(G)$.

By definition, there exists an odd cycle $C$ such that $l^o(G) = 2d^o_G(C) + |V(C)| - 1$. Let $d_1 = d_G(x, C)$ and $d_2 = d_G(y, C)$. Then $d_1 \leq d^o_G(C)$ and $d_2 \leq d^o_G(C)$. Let $P_x = (x, x_1, \ldots, x_{d_1})$ and $P_y = (y, y_1, \ldots, y_{d_2})$ be two shortest paths from $x$ and $y$ to $C$, respectively, where $x_{d_1}, y_{d_2} \in V(C)$ (maybe $x_{d_1} = y_{d_2}$). Two vertices $x_{d_1}$ and $y_{d_2}$ partition $C$ into two paths $P_1$ and $P_2$ with lengths $p_1$ and $p_2$, respectively. Then $p_1$ and $p_2$ have different parity, say $p_1 > p_2$. Thus, $P_x \cup P_2 \cup P_y$ and $P_x \cup P_1 \cup P_y$ are two $(x, y)$-walks of length of different parity and at most

$$d_1 + d_2 + p_1 \leq 2d^o_G(C) + |V(C)| = l^o(G) + 1.$$ By Lemma 2.2, $\gamma(G; x, y) \leq l^o(G)$. 

Corollary 3.1 If $G$ is a connected graph with loops and diameter $d$, then $\gamma(G) \leq 2d$.

Let $H_{n,p}$ and $F_{n,p}$ ($p \geq 1$) be two graphs, which are obtained by joining a complete graph $K_p$ and a cycle $C_p$ to the end-vertex $x_{n-p}$ of a path $P_{n-p} = (x_1, x_2, \ldots, x_{n-p})$ with an edge, respectively.

The following result can be deduced by Theorem 3.1.
Corollary 3.2 (Wang and Wang [13]) Let $G$ be a primitive graph with order $\gamma$ and odd girth $p > 3$. Then $\gamma(G) \leq 2n - p - 1$ with equality if and only if $G$ is isomorphic to $F_{n,p}$.

Proof. By Lemma 2.1, $G$ is connected and contains an odd cycle $C$ with $l^p(G) = 2d_{\gamma}^p(C) + |V(C)| - 1$. Since $d_{\gamma}^p(C) \leq n - |V(C)|$ and $|V(C)| \geq p$, by Theorem 3.1, we have that

\[
\gamma(G) \leq l^p(G) = 2d_{\gamma}^p(C) + |V(C)| - 1 \\
\leq 2(n - |V(C)|) + |V(C)| - 1 \\
\leq 2n - p - 1.
\]

The equality implies that all equalities in (1) hold, in particular, $d_{\gamma}^p(C) = n - |V(C)|$ and $|V(C)| = p$. Thus, there is a vertex $x_1$ such that $d_G(x_1, C) = n - p$ in $G$. Suppose $P = (x_1, x_2, \ldots, x_{n-p}, x_{n-p+1})$ is a shortest path from $x_1$ to $C$, where $x_{n-p+1}$ is in $C$. By the minimality of $P$ and primitivity of $G$, it is easy to see that $G$ is isomorphic to $F_{n,p}$. Also, if $G$ is isomorphic to $F_{n,p}$, then the shortest odd closed $(x_1, x_1)$-walk is of length $2(n - p) + p = 2n - p$. This implies there is no closed $(x_1, x_1)$-walk of length $2n - p - 2$. Hence, $\gamma(F_{n,p}) \geq 2n - p - 1$.

Corollary 3.3 If $p \geq 3$, then $\gamma(H_{n,p}) = 2n - 2p + 2$.

Proof. Let $G = H_{n,p}$. Since $G$ contains $K_p$ and $p \geq 3$, $G$ is primitive by Lemma 2.1, and so $d_{\gamma}^p(C) = d_G(x_1, C) = n - p$ for any $C \in C^0(G)$; let $C$ be a cycle of length 3 in $G$. By Theorem 3.1, $\gamma(G) \leq l^p(G) = 2d_{\gamma}^p(C) + |V(C)| - 1 = 2(n - p) + |V(C)| - 1 \leq 2n - 2p + 2$.

It is clear that the shortest odd closed $(x_1, x_1)$-walk is of length $2(n - p) + 3$. This implies there is no closed $(x_1, x_1)$-walk of length $2(n - p) + 1$. Hence, $\gamma(G) \geq 2n - 2p + 2$. The conclusion follows.

Theorem 3.2 Let $G_i$ be a connected graph with diameter $d_i \geq 1$ and exponent $\gamma_i = \gamma(G_i)$ for $i = 1, 2$. $G_1$ contains odd cycles, and $G = G_1 \otimes G_2$. Then the diameter $d(G)$ of $G$ satisfies the following properties.

1. $d(G) \geq \max\{d_1, d_2\}$.
2. If $G_2$ contains odd cycles, then

\[
d(G) \geq \begin{cases} \gamma_1 & \text{if } \gamma_1 = \gamma_2, \\ \min\{\gamma_1, \gamma_2\} + 1 & \text{if } \gamma_1 \neq \gamma_2. \end{cases}
\]

3. $d(G) \leq \max\{\gamma_1, \gamma_2\}$.
4. $d(G) \leq \min\{\max\{\gamma_1 + 1, d_2\}, \max\{\gamma_2 + 1, d_1\}\}$ with equality if $G_2$ is bipartite.

Proof. Since both $G_1$ and $G_2$ are connected and $G_1$ contains odd cycles, by Lemma 2.1 and Lemma 2.5, $\gamma_1$ is well-defined and $G$ is connected. Since $d_1 \geq 1$ and $d_2 \geq 1$, the order of $G_1$ and $G_2$ are no less than 2.

1. For $i = 1, 2$, let $x_i$ and $y_i$ be two vertices in $G_i$ with $d_G(x_i, y_i) = d_i$ and let $P = (x_1x_2, \ldots, u_1u_2, \ldots, y_1y_2)$ be a shortest $(x_1x_2, y_1y_2)$-path in $G_i$. Then $(x_1, \ldots, u_1, \ldots, y_1)$ and $(x_2, \ldots, u_2, \ldots, y_2)$ are two walks in $G_1$ and $G_2$, respectively. Thus $d(G) \geq d(P) \geq \max\{d_1, d_2\}$.

2. Since $G_2$ contains odd cycles, $\gamma_2$ is well-defined by Lemma 2.1. Without loss of generality, assume $\gamma_2 \geq \gamma_1$ and $\gamma_1$ is odd. By Lemma 2.6, there exist two different vertices $x_1$ and $y_1$ such that the shortest even $(x_1, y_1)$-walk is of length $\gamma_1 + 1$ in $G_1$; also there exist two vertices $x_2$ and $y_2$ such that the shortest odd $(x_2, y_2)$-walk is of length $\gamma_2$ or $\gamma_2 + 1$ in $G_2$. By Lemma 2.7, $d_G(x_1x_2, y_1y_2) \geq \min\{\gamma_1 + 1, \gamma_2\}$, and so

\[
d(G) \geq \begin{cases} \gamma_1 & \text{if } \gamma_1 = \gamma_2, \\ \min\{\gamma_1, \gamma_2\} + 1 & \text{if } \gamma_1 \neq \gamma_2. \end{cases}
\]

3. Without loss of generality, suppose that $\gamma_2$ is well-defined and $\gamma_2 \leq \gamma_1$. Let $x = x_1x_2$ and $y = y_1y_2$ be any two different vertices in $G$. By definition of $\gamma$, there exist an $(x_1, y_1)$-walk and an $(x_2, y_2)$-walk of length $\gamma_1$ in $G_1$ and $G_2$, respectively. By Lemma 2.5, there exists an $(x, y)$-walk of length $\gamma_1$, and hence $d(G; x, y) \leq \gamma_1$. By the arbitrariness of $x$ and $y$, we have $d(G) \leq \gamma_1$.

4. Without loss of generality, suppose that $\gamma_2$ is well-defined, and only need to prove $d(G) \leq \max\{\gamma_1 + 1, d_2\}$. Let $x = x_1x_2$ and $y = y_1y_2$ be any two different vertices in $G$ and...
$d_\gamma = d_{G_0}(x_2, y_2)$ (maybe $x_2 = y_2$). If $d_\gamma \geq \gamma_1$, then there exists an $(x_1, y_1)$-walk of length $d_\gamma$ in $G_1$ by definition of $\gamma$. By Lemma 2.5, there exists an $(x, y)$-walk of length $d_\gamma$ in $G$. If $d_\gamma < \gamma_1$, then one of $d_\gamma + \gamma_1$ and $d_\gamma + \gamma_1 + 1$ is even. By definition of $\gamma$, there exist two $(x_1, y_1)$-walks of lengths $\gamma_1$ and $\gamma_1 + 1$ in $G_1$, respectively. By Lemma 2.5, there exists an $(x, y)$-walk of length no more than $\gamma_1 + 1$ in $G$. Thus $d_G(x, y) \leq \max\{\gamma_1 + 1, d_\gamma\}$, and hence $d(G) \leq \max\{\gamma_1 + 1, d_\gamma\}$ by arbitrariness of $x$ and $y$.

Now assume that $G_0$ is bipartite. Let $x_2$ and $y_2$ be two vertices in different parts in $G_0$. Then any $(x_2, y_2)$-walk and closed $(x_2, x_2)$-walk are of odd and even length in $G_0$, respectively. If $\gamma_1 = 1$, then $d(G) \geq d_G(x_2, y_2) \geq 2 = \gamma_1 + 1$ for any two different vertices $x_1$ and $y_1$ in $G_1$ since $|V(G_1)| \geq 2$. Next, assume $\gamma_1 \geq 2$.

By using the Lemma 2.6, we have the following conclusions. If $\gamma_1$ is odd, then there exist two different vertices $x_1$ and $y_1$ such that the shortest even $(x_1, y_1)$-walk is of length $\gamma_1 + 1$ in $G_1$, and hence $d(G) \geq d_G(x_1, y_1) \geq \gamma_1 + 1$. If $\gamma_1$ is even, then there exist two vertices $x_1$ and $y_1$ such that the shortest odd $(x_1, y_1)$-walk is of length $\gamma_1 + 1$ in $G_1$, and hence $d(G) \geq d_G(x_1, y_1) \geq \gamma_1 + 1$. By the conclusion (1), $d(G) \geq d_2$ and hence $d(G) = \max\{\gamma_1 + 1, d_\gamma\}$.

The theorem follows.

Corollary 3.4 Let $G_i$ be a connected graph with diameter $d_i > 1$ and $l_i = l^0(G_i)$ for $i = 1, 2$, $G = G_1 \otimes G_2$. Then

$$d(G) \leq \min\{\max\{l_i + 1, d_\gamma\}, \max\{l_j + 1, d_\gamma\}\}.$$  

Proof. Without loss of generality, we can suppose that both $G_1$ and $G_2$ contain odd cycles. By Theorem 3.1, $\gamma(G_1) \leq l_1$ and $\gamma(G_2) \leq l_2$. The conclusion follows by the conclusion (4) in Theorem 3.2.

Corollary 3.5 Let $G_i$ be a connected graph with diameter $d_i > 1$ for $i = 1, 2$ and $G = G_1 \otimes G_2$. If $G_1$ contains odd cycles, then $d(G) \leq \max\{2d_1, d_2\}$.

Proof. By Lemma 2.3, $\gamma(G_1) \leq 2d_1$. The Theorem follows by the conclusion (4) in Theorem 3.2.

The following result, obtained by Leskovec et al. [9], can be deduced by Theorem 3.2 immediately.

Corollary 3.6 (Leskovec et al. [9]) Let $G_i$ be a connected graph with diameter $d_i > 1$ and there is a loop on every vertex of $G_i$, for $i = 1, 2$. Then $d(G_1 \otimes G_2) = \max\{d_1, d_2\}$.

Proof. It is clear that $\gamma(G_1) = d_1$ and $\gamma(G_2) = d_2$ since each of $G_1$ and $G_2$ has a loop on every vertex. The conclusion follows by the conclusions (1) and (3) in Theorem 3.2.

Corollary 3.7 Let $G$ be a primitive graph with order $n \geq 2$. Then $\gamma(G) = d(G \otimes K_2) - 1$.

By Theorem 3.2, we immediately obtain our main results in this paper.

Theorem 3.3 Let $G_i$ be a connected graph with diameter $d_i \geq 1$ and exponent $\gamma_i = \gamma(G_i)$ for $i = 1, 2$. If $G_1$ contains odd cycles, then

$$d(G_1 \otimes G_2) = \left\{ \begin{array}{ll} \gamma_1 & \text{if } \gamma_1 = \gamma_2; \\
\max\{\gamma_1 + 1, d_1\} & \text{if } \gamma_1 > \gamma_2; \\
\max\{\gamma_1 + 1, d_2\} & \text{if } \gamma_1 < \gamma_2. \end{array} \right.$$  

In Theorem 3.3, we consider the diameter of the Kronecker product of two graphs $G_1$ and $G_2$ with order no less than 2. Next, we consider the case that at least one of $G_1$ and $G_2$ with order 1. Let $G$ be a connected graph with order $n$ and no parallel edges. We have noted in Section 2, $\gamma(G) = 1$ if and only if $G \cong K_n^+$. For a graph $H$ with order 1, if $G \otimes H$ is connected, then $H \cong K_1^+$ since $G \otimes K_1$ is empty. It is easy to see that $K_1^+ \otimes G \cong G$ and then $d(K_1^+ \otimes G) = d(G)$.

In the following, we show the diameters for some special Kronecker product of two graphs only by using the diameters of factor graphs.
Theorem 3.4 Let $G_i$ be a connected graph with order $n_i ≥ 2$ for $i = 1, 2$. Then $d(G_1 ⊗ G_2) = 1$ if and only if $G_1 \cong K^+_{n_1}$ and $G_2 \cong K^+_{n_2}$.

Proof. The sufficiency is obviously.
Now we show the necessity. By contradiction. Without loss of generality, assume $G_1 \not\cong K^+_{n_1}$. Then either there exists a vertex $x$ such that it does not contain a loop or $d(G_1) ≥ 2$. Then $d(G) ≥ d_G(x, y, z) ≥ 2$ for any two different vertices $y, z \in V(G_2)$ or $d(G) ≥ (d(G_1) ≥ 2$ by the conclusion (1) in Theorem 3.2. ∎

Theorem 3.5 Let $G \not\cong K^+_{n}$ be a connected graph with order $n ≥ 2$ and $m ≥ 2$. Then

$$d(K^+_{m} ⊗ G) = \begin{cases} 2, & d(G) = 1; \\ d(G), & d(G) ≥ 2. \end{cases}$$

Proof. The Theorem follows by Theorem 3.3 since $\gamma(K^+_{m}) = 1$ and $\gamma(G) ≥ 2$. ∎

Theorem 3.6 Let $G$ be a connected graph with diameter $d > 1$ and $H$ be a complete $t$ partite graph with $t > 3$. Then

$$d(G ⊗ H) = \begin{cases} d, & d ≥ 3; \\ 2, & d ≤ 2 \text{ and } \gamma(G) ≤ 2; \\ 3, & d ≤ 2 \text{ and } \gamma(G) > 2. \end{cases}$$

Proof. It is clear that $d(H) ≥ 2$, $H$ is primitive and $\gamma(H) = 2$. The Theorem follows by Theorem 3.3. ∎

Corollary 3.8 Let $G$ be $H_{n,b}$ or $F_{n,b}$ with odd cycles and diameter $d_1 > 1$, and $H$ be any connected graph with diameter $d_2 ≥ 1$.

1. If $H$ is bipartite, then $d(G ⊗ H) = \max\{2d_1 + 1, d_2\}$.
2. If $H = G_{n_2,d_2}$ is non-bipartite, then

$$d(G ⊗ H) = \begin{cases} 2d_1 & \text{if } d_1 = d_2; \\ \max\{d_1, 2d_2 + 1\} & \text{if } d_1 > d_2; \\ \max\{d_2, 2d_1 + 1\} & \text{if } d_1 < d_2. \end{cases}$$

Proof. By Lemma 2.1, $G$ is primitive since $G$ contains odd cycles. By Lemma 2.5, $G ⊗ H$ is connected. By Corollary 3.1 and 3.2, $\gamma(G) = 2d_1$. If $H$ is bipartite, then $H$ is not primitive by Lemma 2.1. Thus $\gamma(H) > \gamma(G)$, and hence, $d(G ⊗ H) = \max\{2d_1 + 1, d_2\}$ by Theorem 3.3. If $H = G_{n_2,d_2}$ is non-bipartite, then $\gamma(H) = 2d_2$. The conclusion follows by Theorem 3.3 immediately. ∎

Corollary 3.9 Let $C_m$ be an odd cycle and $H$ be a connected graph with order $n$ and diameter $d ≥ 1$.

1. If $H$ is bipartite, then $d(C_m ⊗ H) = \max\{m, d\}$. Hence $d(C_m ⊗ P_n) = \max\{m, n - 1\}$, and $d(C_m ⊗ C_n) = \max\{m, \frac{n}{2}\}$ if $n$ is even.
2. If $H = C_n$ and $n$ is odd, then

$$d(C_m ⊗ C_n) = \begin{cases} m - 1 & \text{if } m = n, \\ \max\{m, \frac{m}{2}\} & \text{if } m > n, \\ \max\{m, \frac{n}{2}\} & \text{if } m < n. \end{cases}$$

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References


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