

Generalized Schwarz Algorithm For A Class Of Variational Inequalities

Bouzoualegh Ikram* and Saadi Samira

Department of Mathematics, University Badji Mokhtar Annaba, Annaba, Algeria

Received: 22 Sep. 2020, Revised: 21 Nov. 2020, Accepted: 4 Dec. 2020

Published online: 1 Jan. 2021

Abstract: We consider a decomposition to m -subdomains of the obstacle problem, which is modeled by a variational inequality of first species, using the auxiliary sequences, and we have proved a alternating relation between the solutions on each subdomain. We also proved a geometrical convergence between the n th iteration and the solution of the initial problem, we obtained a result on the error estimate that contains a logarithmic factor with an extra power of $|\log(h)|$.

Keywords: L^∞ error estimate. Finite element method. Variational inequalities. Schwarz algorithm.

1 Introduction

The Schwarz alternating method of decomposing the domain has lately been found to be effective means for solving elliptic partial differential equations on a multi processing computing system. Pierre-Louis Lions represented the starting point of an intense research activity to develop this tool of calculation, see [1]-[3]. In this paper, we are interested in the Schwarz alternating method which is used to solve a class of elliptic variational inequality in the context of overlapping nonmatching grids, precisely in the error analysis in the maximum norm of obstacle type problems. The maximum error analysis of overlapping nonmatching grids for the obstacle problem which Ω is the union of two subdomains has been investigated in [4]. The same error analysis of a nonmatching grids for linear and nonlinear elliptic partial differential equations as well as elliptic quasi-variational inequalities has been addressed in [5]-[9]. In this paper we consider a domain Ω which is the union of m overlapping sub-domains where each sub-domain has its own triangulation. To prove the main result, we introduce the m continuous and discrete Schwarz sequences as well as prove a main result concerning the error estimate of solution in L^∞ -norm, taking into account the combination of geometrical convergence and uniform convergence of finite element approximation.

This paper consists of two parts: In the first, we formulate

the problem of continuous and discrete elliptic variational inequality we show the monotonicity and stability of discrete solution, and define the Schwarz algorithm for m subdomains with overlapping nonmatching grids. In the second part, we establish m auxiliary Schwarz sequences, and prove the main result of this work.

2 The generalized Schwarz alternating method

2.1 Elliptic obstacle problem

Let Ω be a convex domain in $\mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega$.

We consider the bilinear form

$$a(u, v) = \int_{\Omega} (\nabla u \cdot \nabla v) dx, \quad (1)$$

the linear form

$$(f, v) = \int_{\Omega} f(x) \cdot v(x) dx, \quad (2)$$

the right hande-side

$$f \in L^\infty(\Omega), \quad (3)$$

* Corresponding author e-mail: bouzoualeghikaram2020@gmail.com

the obstacle

$$\Psi \in W^{2,\infty}(\Omega) \quad \text{such that} \quad \Psi \geq 0 \quad \text{on} \quad \partial\Omega, \quad (4)$$

and the nonempty convex set

$$K_g = \{v \in H^1(\Omega) : v = g \quad \text{on} \partial\Omega, v \leq \Psi \text{ in } \Omega\}, \quad (5)$$

where g is a regular function defined on $\partial\Omega$.

We consider the obstacle problem: find $u \in K_g$ such that

$$a(u, v - u) \geq (f, v - u), \quad \forall v \in K_g, \quad (6)$$

Let V_h be the space of finite elements consisting of continuous piecewise linear functions. The discrete counterpart of (6) consists of finding $u_h \in K_{gh}$ such that

$$a(u_h, v - u_h) \geq (f, v - u_h), \quad \forall v \in K_{gh}, \quad (7)$$

where

$$K_{gh} = \{v \in V_h : v = \pi_h g \quad \text{on} \partial\Omega, v \leq r_h \Psi \quad \text{in} \Omega\}, \quad (8)$$

π_h is an interpolation operator on $\partial\Omega$, and r_h is the usual finite element restriction operator on Ω . The lemma below establishes a monotonicity property of the solution of (6) with respect to the obstacle and the boundary condition.

Lemma 2.1 Let $(\Psi, g); (\tilde{\Psi}, \tilde{g})$ be a pair of data, and $u = \sigma(\Psi, g); \tilde{u} = \sigma(\tilde{\Psi}, \tilde{g})$ the corresponding solutions of (6). If $\Psi \geq \tilde{\Psi}$ and $g \geq \tilde{g}$, then $\sigma(\Psi, g) \geq \sigma(\tilde{\Psi}, \tilde{g})$.

Proof. let $v = \min(0, u - \tilde{u})$. In the region where v is negative ($v < 0$), we have

$$u < \tilde{u} \leq \psi \leq \tilde{\psi} \quad (9)$$

which means that the obstacle is inactive for u .

Thus, for v , we have

$$a(u, v) = (f, v) \quad (10)$$

$$\tilde{u} + v \leq \tilde{\psi} \quad (11)$$

so

$$a(\tilde{u}, v) = (f, v) \quad (12)$$

Subtracting (10) and (12) from each other, we obtain

$$a(\tilde{u} - u, v) \geq 0 \quad (13)$$

but,

$$a(v, v) = a(u - \tilde{u}, v) = -a(\tilde{u} - u, v) \leq 0 \quad (14)$$

so

$$v = 0 \quad (15)$$

Then,

$$u \geq \tilde{u} \quad (16)$$

which completes the proof.

The proof for the discrete case is similar.

Proposition 2.2 Under the notations and conditions of the preceding lemma, we have

$$\|u - \tilde{u}\|_{L^\infty(\Omega)} \leq \|\Psi - \tilde{\Psi}\|_{L^\infty(\Omega)} + \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}, \quad (17)$$

Proof. Setting

$$\phi \leq \|\Psi - \tilde{\Psi}\|_{L^\infty(\Omega)} + \|g - \tilde{g}\|_{L^\infty(\partial\Omega)} \quad (18)$$

we have

$$\begin{aligned} \psi &\leq \tilde{\psi} + \psi - \tilde{\psi} \leq \tilde{\psi} + |\psi - \tilde{\psi}| \leq \tilde{\psi} + \|\psi - \tilde{\psi}\|_{L^\infty(\Omega)} \\ &\leq \psi + \|\psi - \tilde{\psi}\|_{L^\infty(\Omega)} + \|g - \tilde{g}\|_{L^\infty(\partial\Omega)} \end{aligned} \quad (19)$$

Hence,

$$\psi \leq \tilde{\psi} + \phi \quad (20)$$

On the other hand, we have

$$\begin{aligned} g &\leq \tilde{g} + g - \tilde{g} \leq \tilde{g} + |g - \tilde{g}| \leq \tilde{g} + \|g - \tilde{g}\|_{L^\infty(\partial\Omega)} \\ &\leq \tilde{g} + \|g - \tilde{g}\|_{L^\infty(\partial\Omega)} + \|\psi - \tilde{\psi}\|_{L^\infty(\Omega)} \end{aligned} \quad (21)$$

so

$$g \leq \tilde{g} + \phi \quad (22)$$

Now, making use of **Lemma 2.1**, we obtain

$$\sigma(\psi, g) \leq \sigma(\psi + \phi, g + \phi) = \sigma(\tilde{\psi}, \tilde{g}) + \phi \quad (23)$$

or

$$\sigma(\psi, g) - \sigma(\tilde{\psi}, \tilde{g}) \leq \phi \quad (24)$$

Similarly, interchanging the roles of the couples (ψ, g) and $(\tilde{\psi}, \tilde{g})$, we obtain

$$\sigma(\tilde{\psi}, \tilde{g}) - \sigma(\psi, g) \leq \phi \quad (25)$$

The proof for the discrete case is similar.

Remark 2.3 if $\psi = \tilde{\psi}$, then (17) becomes

$$\|u - \tilde{u}\|_{L^\infty(\Omega)} \leq \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}, \quad (26)$$

Theorem 2.4 [10] Under conditions (3) and (4), there exists a constant C independent of h such that

$$\|u - u_h\|_{L^\infty(\Omega)} \leq ch^2 \log|h|^2, \quad (27)$$

2.2 The continuous Schwarz sequences

Consider the model obstacle problem: find $u \in K_0 (g = 0)$ such that

$$a(u, v) \geq f(v - u) \quad \forall v \in K_0, \quad (28)$$

We decompose Ω into m overlapping subdomains such that

$$\Omega = \bigcup_{i=1}^m \Omega_i, \quad \Omega_i \cap \Omega_j \neq \emptyset, \quad i = \overline{1, m}, \quad j = \overline{1, m}, \quad i \neq j \quad (29)$$

and u satisfies the local regularity condition

$$u|_{\Omega_i} \in W^{2,p}(\Omega_i); \quad 2 < p < \infty, \quad (30)$$

We denote by $\partial\Omega_i$ the boundary of Ω_i , and $\Gamma_{ij} = \partial\Omega_i \cap \Omega_j, i \neq j$. The intersection of $\bar{\Gamma}_i$ and $\bar{\Gamma}_j; i \neq j$ is assumed to be empty. Choosing $u_0 = \Psi$, we define the alternating Schwarz sequences (u_i^{n+1}) on Ω_i such that $u_i^{n+1} \in K$ solves

$$a_i(u_i^{n+1}, v - u_i^{n+1}) \geq (f_i, v - u_i^{n+1}) \quad \text{in } \Omega_i \quad (31)$$

$$u_i^{n+1} = u_j^{n+1} \quad \text{on } \Gamma_{ij}$$

where $i = \overline{1, m}, j = \overline{1, m}, i \neq j$ and

$$1_{ij} = \begin{cases} 1 & \text{if } i > j \\ 0 & \text{if } i < j \end{cases}$$

2.3 Geometrical convergence

Theorem 2.5 *The sequences $(u_1^{n+1}), (u_2^{n+1}), \dots, (u_m^{n+1}); n \geq 0$ produced by the Schwarz alternating method converge geometrically to the solution u of the obstacle problem (28). More precisely, there exist m constants $k_1, k_2, \dots, k_m \in (0, 1), \forall i = \overline{1, m-1}, j = \overline{2, m}, i < j$ such that for all $n \geq 0$*

$$\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} \leq k_i^n k_j^n \|u - u^0\|_{L^\infty(\Gamma_{ij})} \quad (32)$$

$$\|u_j - u_j^{n+1}\|_{L^\infty(\Omega_j)} \leq k_i^{n+1} k_j^n \|u - u^0\|_{L^\infty(\Gamma_{ji})}$$

we consider a function $w_l \in L^\infty(\Omega_l)$ continuous in $\bar{\Omega}_l \setminus (\bar{\Gamma}_l \cap \partial\Omega)$

such that

$$\begin{cases} \Delta w_l = 0 & \text{in } \Omega_l, l = \overline{1, m} \\ w_l = \begin{cases} 0 & \text{on } \partial\Omega_l \setminus \bar{\Gamma}_l \\ 1 & \text{on } \Gamma_l \end{cases} \end{cases}$$

and

$$k_t = \sup\{w_s(x) \mid x \in \partial\Omega_t \cap \Omega, t \neq s\} \in (0, 1) \quad (33)$$

$$\forall t, s = \overline{1, m}$$

Proof. From the principle of the maximum

$$\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} \leq \|u_i - u_i^{n+1}\|_{L^\infty(\Gamma_{ij})}$$

and

$$\begin{aligned} \|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} &\leq \|u_j - u_j^n\|_{L^\infty(\Gamma_{ij})} \leq \|w_i u_j - w_i u_j^n\|_{L^\infty(\Gamma_{ij})} \\ &\leq \|w_i u_j - w_i u_j^n\|_{L^\infty(\Omega_j)} \\ &\leq \|w_i u_j - w_i u_j^n\|_{L^\infty(\Gamma_{ji})} \\ &\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|u_j - u_j^n\|_{L^\infty(\Gamma_{ji})} \\ &\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_j - w_j u_j^n\|_{L^\infty(\Gamma_{ji})} \\ &\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_i - w_j u_i^n\|_{L^\infty(\Gamma_{ji})} \\ &\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_i - w_i u_i^n\|_{L^\infty(\Omega_i)} \\ &\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_i - w_j u_i^n\|_{L^\infty(\Gamma_{ij})} \\ &\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j\|_{L^\infty(\Gamma_{ij})} \|u_i - u_i^n\|_{L^\infty(\Gamma_{ij})} \end{aligned}$$

using (33), so

$$\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} \leq k_i k_j \|u_i - u_i^n\|_{L^\infty(\Gamma_{ij})}$$

By induction, we obtain

$$\begin{aligned} \|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} &\leq k_i^n k_j^n \|u_i - u_i^1\|_{L^\infty(\Gamma_{ij})} \\ &\leq k_i^n k_j^n \|u - u^0\|_{L^\infty(\Gamma_{ij})} \end{aligned}$$

where $u_i^1 = u^0$ on $\Gamma_{ij}, u = 0$ on $\partial\Omega_i \cap \partial\Omega$. Similarly, we have

$$\begin{aligned}
\|u_j - u_j^{n+1}\|_{L^\infty(\Omega_j)} &\leq \|u_j - u_j^{n+1}\|_{L^\infty(\Gamma_{ji})} \\
&\leq \|u_i - u_i^{n+1}\|_{L^\infty(\Gamma_{ji})} \\
&\leq \|w_j u_i - w_j u_i^{n+1}\|_{L^\infty(\Gamma_{ji})} \\
&\leq \|w_j u_i - w_j u_i^{n+1}\|_{L^\infty(\Omega_i)} \\
&\leq \|w_j u_i - w_j u_i^{n+1}\|_{L^\infty(\Gamma_{ij})} \\
&\leq \|w_j\|_{L^\infty(\Gamma_{ij})} \|u_i - u_i^{n+1}\|_{L^\infty(\Gamma_{ij})} \\
&\leq \|w_j\|_{L^\infty(\Gamma_{ij})} \|w_i u_j - w_i u_j^n\|_{L^\infty(\Gamma_{ij})} \\
&\leq \|w_j\|_{L^\infty(\Gamma_{ij})} \|w_i u_j - w_i u_j^n\|_{L^\infty(\Omega_j)} \\
&\leq \|w_j\|_{L^\infty(\Gamma_{ij})} \|w_i u_j - w_i u_j^n\|_{L^\infty(\Gamma_{ji})} \\
&\leq \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j\|_{L^\infty(\Gamma_{ij})} \|u_j - u_j^n\|_{L^\infty(\Gamma_{ji})} \\
&\leq k_i k_j \|u_i - u_j^n\|_{L^\infty(\Gamma_{ji})}
\end{aligned}$$

By induction, we obtain

$$\begin{aligned}
\|u_j - u_j^{n+1}\|_{L^\infty(\Omega_j)} &\leq k_i^n k_j^n \|u_j - u_i^1\|_{L^\infty(\Gamma_{ji})} \\
&\leq k_i^n k_j^n \|w_j u_i - w_j u_i^1\|_{L^\infty(\Gamma_{ji})} \\
&\leq k_i^n k_j^n \|u_i - u_i^1\|_{L^\infty(\Omega_i)} \\
&\leq k_i^n k_j^n \|u_i - u_i^1\|_{L^\infty(\Gamma_{ij})} \\
&\leq k_i^{n+1} k_j^n \|u_i - u_i^1\|_{L^\infty(\Gamma_{ij})} \\
&\leq k_i^{n+1} k_j^n \|u_j - u_j^0\|_{L^\infty(\Gamma_{ij})} \\
&\leq k_i^{n+1} k_j^n \|u_j - u_j^0\|_{L^\infty(\Omega_j)} \\
&\leq k_i^{n+1} k_j^n \|u - u^0\|_{L^\infty(\Gamma_{ji})}
\end{aligned}$$

2.4 The discretization

let τ^{h_i} be a standard regular and quasi-uniform finite element triangulation in Ω_i , h_i is the meshsizes. We assume that every two triangulations are mutually independent on $\Omega_i \cap \Omega_j$, in the sense that a triangle belonging to one triangulation does not necessarily belong to the other, $i = \overline{1, m}$, $j = \overline{1, m}$, $i \neq j$

Let $V_{h_{ij}} = V_{h_{ij}}(\Omega_i)$ be the space of continuous piecewise linear functions on, τ^{h_i} which vanish on $\partial\Omega \cap \Omega_i$.

For $w \in C(\bar{\tau}_i)$ we define

$$V_{h_{ij}}^{(w)} = \{v \in V_{h_{ij}} : v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega; v = \pi_{h_{ij}}(w) \text{ on } \Gamma_{ij}\}$$

where $\pi_{h_{ij}}$ denotes a suitable interpolation operator on Γ_{ij}
We define the discrete Schwarz sequence:

$$u_{ih}^{n+1} \in V_{h_{ij}}^{(u_{jh}^{n+1})}$$

solves

$$\begin{aligned}
a_i(u_{ih}^{n+1}, v - u_{ih}^{n+1}) &\geq (f_i, v - u_{ih}^{n+1}) \quad \forall v \in V_{h_{ij}}^{(u_{jh}^{n+1})} \\
u_{ih}^{n+1} &\leq r_h \cdot \psi, \quad v \leq r_h \cdot \psi
\end{aligned} \tag{34}$$

3 L^∞ -error analysis

3.1 Definition of m auxiliary sequences

For $\omega_0^{ih} = u_0^{ih} = r_h \psi$; $i = \overline{1, m}$, we define the sequences $\omega_{ih}^{n+1} \in V_{h_{ij}}^{(u_j^{n+1})}$ such that

$$\begin{aligned}
a_i(\omega_{ih}^{n+1}, v - \omega_{ih}^{n+1}) &\geq (f_i, v - \omega_{ih}^{n+1}) \quad \forall v \in V_{h_{ij}}^{(u_j^{n+1})} \\
\omega_{ih}^{n+1} &\leq r_h \cdot \psi, \quad v \leq r_h \cdot \psi
\end{aligned} \tag{35}$$

To simplify the notation, we take

$$\begin{aligned}
|\cdot|_{ij} &= \|\cdot\|_{L^\infty(\Gamma_{ij})} \\
\|\cdot\|_i &= \|\cdot\|_{L^\infty(\Omega_i)} \quad h_{ij} = h \quad \pi_{h_{ij}} = \pi_h
\end{aligned} \tag{36}$$

Lemma 3.1 For $i = \overline{1, m-1}$, $j = \overline{2, m}$, $i < j$

$$\|u_i^{n+1} - u_{ih}^{n+1}\|_i \leq \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^n \|u_j^p - \omega_{jh}^p\|_j \tag{37}$$

$$\|u_j^{n+1} - u_{jh}^{n+1}\|_j \leq \sum_{p=0}^{n+1} \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i$$

Proof. By induction

for $n = 0$, using the discrete version of **Remark 2.3**, we get

$$\begin{aligned}
\|u_i^1 - u_{ih}^1\|_i &\leq \|u_i^1 - \omega_{ih}^1\|_i + \|\omega_{ih}^1 - u_{ih}^1\|_i \\
&\leq \|u_i^1 - \omega_{ih}^1\|_i + |\pi_h u_j^0 - \pi_h u_{jh}^0|_{ij} \\
&\leq \|u_i^1 - \omega_{ih}^1\|_i + |u_j^0 - u_{jh}^0|_{ij} \\
&\leq \|u_i^1 - \omega_{ih}^1\|_i + \|u_j^0 - u_{jh}^0\|_j
\end{aligned}$$

$$\begin{aligned}\|u_j^1 - u_{jh}^1\|_j &\leq \|u_j^1 - \omega_{jh}^1\|_j + \|\omega_{jh}^1 - u_{jh}^1\|_j \\ &\leq \|u_j^1 - \omega_{jh}^1\|_j + |\pi_h u_i^1 - \pi_h u_{ih}^1|_{ji} \\ &\leq \|u_j^1 - \omega_{jh}^1\|_j + |u_i^1 - u_{ih}^1|_{ji} \\ &\leq \|u_j^1 - \omega_{jh}^1\|_j + \|u_i^1 - u_{ih}^1\|_i \\ &\leq \|u_j^1 - \omega_{jh}^1\|_j + \|u_i^1 - \omega_{ih}^1\|_i \\ &\quad + \|u_j^0 - u_{jh}^0\|_j\end{aligned}$$

So

$$\begin{aligned}\|u_i^1 - u_{ih}^1\|_i &\leq \sum_{p=1}^1 \|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^0 \|u_j^p - \omega_{jh}^p\|_j \\ \|u_j^1 - u_{jh}^1\|_j &\leq \sum_{p=0}^1 \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^1 \|u_i^p - \omega_{ih}^p\|_i\end{aligned}$$

For $n = 1$, using the discrete version of **Remark 2.3**:

$$\begin{aligned}\|u_i^2 - u_{ih}^2\|_i &\leq \|u_i^2 - \omega_{ih}^2\|_i + \|\omega_{ih}^2 - u_{ih}^2\|_i \\ &\leq \|u_i^2 - \omega_{ih}^2\|_i + |\pi_h u_j^1 - \pi_h u_{jh}^1|_{ji} \\ &\leq \|u_i^2 - \omega_{ih}^2\|_i + \|u_j^1 - u_{jh}^1\|_j \\ &\leq \|u_i^2 - \omega_{ih}^2\|_i + \|u_j^1 - u_{jh}^1\|_j \\ &\leq \|u_i^2 - \omega_{ih}^2\|_i + \|u_j^1 - \omega_{jh}^1\|_j \\ &\quad + \|u_i^1 - \omega_{ih}^1\|_i + \|u_j^0 - u_{jh}^0\|_j\end{aligned}$$

$$\begin{aligned}\|u_j^2 - u_{jh}^2\|_j &\leq \|u_j^2 - \omega_{jh}^2\|_j + \|\omega_{jh}^2 - u_{jh}^2\|_j \\ &\leq \|u_j^2 - \omega_{jh}^2\|_j + |\pi_h u_i^2 - \pi_h u_{ih}^2|_{ji} \\ &\leq \|u_j^2 - \omega_{jh}^2\|_j + |u_i^2 - u_{ih}^2|_{ji} \\ &\leq \|u_j^2 - \omega_{jh}^2\|_j + \|u_i^2 - u_{ih}^2\|_i \\ &\leq \|u_j^2 - \omega_{jh}^2\|_j + \|u_i^2 - \omega_{ih}^2\|_i \\ &\quad + \|u_j^1 - \omega_{jh}^1\|_j + \|u_i^1 - \omega_{ih}^1\|_i \\ &\quad + \|u_j^0 - u_{jh}^0\|_j\end{aligned}$$

So

$$\begin{aligned}\|u_i^2 - u_{ih}^2\|_i &\leq \sum_{p=1}^2 \|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^1 \|u_j^p - \omega_{jh}^p\|_j \\ \|u_j^2 - u_{jh}^2\|_j &\leq \sum_{p=0}^2 \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^2 \|u_i^p - \omega_{ih}^p\|_i\end{aligned}$$

We suppose that

$$\|u_j^n - u_{jh}^n\|_j \leq \sum_{p=0}^n \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^n \|u_i^p - \omega_{ih}^p\|_i$$

Then, using the discrete version of **Remark 2.3** again:

$$\begin{aligned}\|u_i^{n+1} - u_{ih}^{n+1}\|_i &\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + \|\omega_{ih}^{n+1} - u_{ih}^{n+1}\|_i \\ &\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + |\pi_h u_j^n - \pi_h u_{jh}^n|_{ji} \\ &\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + \|u_j^n - u_{jh}^n\|_j \\ &\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + \|u_j^n - u_{jh}^n\|_j \\ &\leq \|u_i^{n+1} - u_{ih}^{n+1}\|_i + \sum_{p=0}^n \|u_j^p - \omega_{jh}^p\|_j \\ &\quad + \sum_{p=1}^n \|u_i^p - \omega_{ih}^p\|_i\end{aligned}$$

Then

$$\|u_i^{n+1} - u_{ih}^{n+1}\|_i \leq \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^n \|u_j^p - \omega_{jh}^p\|_j$$

$$\begin{aligned}\|u_j^{n+1} - u_{jh}^{n+1}\|_j &\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + \|\omega_{jh}^{n+1} - u_{jh}^{n+1}\|_j \\ &\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + |\pi_h u_i^n - \pi_h u_{ih}^n|_{ji} \\ &\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + |u_i^n - u_{ih}^n|_{ji} \\ &\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + \|u_i^n - u_{ih}^n\|_i \\ &\leq \|u_j^{n+1} - u_{jh}^{n+1}\|_j + \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i \\ &\quad + \sum_{p=0}^n \|u_j^p - \omega_{jh}^p\|_j\end{aligned}$$

Then

$$\|u_j^{n+1} - u_{jh}^{n+1}\|_j \leq \sum_{p=0}^{n+1} \|u_j^p - \omega_{jh}^p\|_j + \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i$$

Lemma 3.2 $\forall i = \overline{1, m-1}, j = \overline{2, m}, i < j$. Then there exists a constant c independent of h and n such that

$$\|u_i^{n+1} - u_{ih}^{n+1}\|_i \leq 2(n+1)Ch^2 |\log h|^3 \quad (38)$$

$$\|u_j^{n+1} - u_{jh}^{n+1}\|_j \leq (2n+3)Ch^2|\log h|^3$$

Proof. By induction
for $n = 0$, using **Theorem 2.4**

$$\begin{aligned}\|u_i^1 - u_{ih}^1\|_i &\leq \|u_i^1 - \omega_{ih}^1\|_i + \|\omega_{ih}^1 - u_{ih}^1\|_i \\ &\leq \|u_i^1 - \omega_{ih}^1\|_i + \|u_j^0 - u_{jh}^0\|_j \\ &\leq ch^2 \log|h|^2 + ch^2 |\log h|^2 \\ &\leq 2ch^2 |\log h|^2\end{aligned}$$

$$\begin{aligned}\|u_j^1 - u_{jh}^1\|_j &\leq \|u_j^1 - \omega_{jh}^1\|_j + \|\omega_{jh}^1 - u_{jh}^1\|_j \\ &\leq \|u_j^1 - \omega_{jh}^1\|_j + \|u_i^1 - u_{ih}^1\|_i \\ &\leq ch^2 |\log h|^2 + 2ch^2 \log|h|^2 \\ &\leq 3ch^2 |\log h|^2\end{aligned}$$

Now we suppose that

$$\|u_j^n - u_{jh}^n\|_j \leq (2n+1)Ch^2|\log h|^2$$

$$\begin{aligned}\|u_i^{n+1} - u_{ih}^{n+1}\|_i &\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + \|\omega_{ih}^{n+1} - u_{ih}^{n+1}\|_i \\ &\leq \|u_i^{n+1} - \omega_{ih}^{n+1}\|_i + \|u_j^n - u_{jh}^n\|_j \\ &\leq ch^2 |\log h|^2 + (2n+1)ch^2 |\log h|^2 \\ &\leq 2(n+1)ch^2 |\log h|^2\end{aligned}$$

$$\begin{aligned}\|u_j^{n+1} - u_{jh}^{n+1}\|_j &\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + \|\omega_{jh}^{n+1} - u_{jh}^{n+1}\|_j \\ &\leq \|u_j^{n+1} - \omega_{jh}^{n+1}\|_j + \|u_i^{n+1} - u_{ih}^{n+1}\|_i \\ &\leq ch^2 |\log h|^2 + 2(n+1)ch^2 |\log h|^2 \\ &\leq (2n+3)ch^2 |\log h|^2\end{aligned}$$

3.2 L^∞ error estimate

Theorem 3.3 Let $h = \max(h_i, h_j)$, $i = \overline{1, m-1}$; $j = \overline{2, m}$ and $i < j$. Then, there exists a constant C independent of both h and n such that

$$\|u_M - u_{Mh}^{n+1}\|_{L^\infty(\Omega_M)} \leq Ch^2 |\log h|^3; \quad M = \overline{i, j} \quad (39)$$

Proof. Let us give the proof for $M = i$. The case $M = j$ is similar.

For $N = i$, let $k = \max(k_i, k_j)$

Using **Theorem 2.5**, **lemma 3.2**, we obtain

$$\begin{aligned}\|u_i - u_{ih}^{n+1}\|_i &\leq \|u_i - u_i^{n+1}\|_i + \|u_i^{n+1} - u_{ih}^{n+1}\|_i \\ &\leq k^{2n} |u - u^0|_{ij} + \|u_i^{n+1} - u_{ih}^{n+1}\|_i \\ &\leq k^{2n} |u - u^0|_{ij} + \sum_{p=1}^{n+1} \|u_i^p - \omega_{ih}^p\|_i + \sum_{p=0}^n \|u_j^p - \omega_{jh}^p\|_j \\ &\leq k^{2n} |u - u^0|_{ij} + 2(n+1)Ch^2 |\log h|^2\end{aligned}$$

We suppose that

$$k^{2n} \leq h^2$$

we obtain

$$\|u_i - u_{ih}^{n+1}\|_i \leq Ch^2 |\log h|^3$$

4 Conclusion

In this work, we have established an approach of the alternating Schwarz algorithm for m overlapping subdomains with nonmatching grids, for the class of elliptic variational inequality. This type of estimation which we have obtained relies on the geometrical convergence and the error estimate between the continuous and discrete Schwarz iterates. We contend that this result plays an important role in the study of the numerical analysis for the class of elliptic variational inequality in the context overlapping nonmatching grids using the parallel Schwarz method.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

References

- [1] P.L. Lion, On the Schwarz alternating method.I, First International Symposiumon Domain Decomposition Methods for Partial Differential Equations, SIAM, Philadelphia (1988) 1-42.
- [2] P.L. Lion, On the Schwarz alternating method.II, stochastic interpretation and order proprieties, domain decomposition methods, Second International Symposiumon Domain Decomposition Methods for Partial Differential Equations, SIAM, Philadelphia (1989) 47-70.

- [3] P.L. Lion, On the Schwarz alternating method.III, A Variant for Nonoverlapping Subdomains, Proc. 3rd Conference on Domain Decomposition Methods, *SIAM*, Philadelphia (1990) 202-223.
- [4] M. Boulbrachene and S. Saadi, Maximum norma nalysis of an overlapping nonmatching grids method for the obstacle problem, *Adv. Differ. Equ* (2006) 1-10.
- [5] M. Boulbrachene and Q.A. Farei, Maximum norm error analysis of nonmatching grids finite element method for linear elliptic PDEs, *Appl. Math. Comput.* **238** (2014) 21-29.
- [6] A. Harbi and M. Boulbrachene, Maximum norm analysis of nonmatching grids method for nonlinear elliptic pdes, *J. Appl. Math* (2011).
- [7] M. Haiour and S. Boulaares, Overlapping Domain Decomposition Methods for Elliptic Quasi-Variational Inequalities Related to Impulse Control Problem With Mixed Boundary Conditions, *Indian Acad. Sci. (Math. Sci.)*, **121** (2011) 481-493.
- [8] S. Saadi and A. Mehri, A new error estimate on uniform norm of Schwarz algorithm for elliptic quasi-variational inequalities with nonlinear source, *J. Inequal. Appl* (2018) 1-13.
- [9] S. Saadi and A. Mehri, L^∞ -error of Schwarz algorithm for elliptic quasi-variational inequalities related in impulse control problem, *Aust. J. Math. Anal. Appl*(2014) 1-13.
- [10] P. Cortey-Dumont, On finite element approximation in the L^∞ -norm of variational inequalities, *Numer. Math*, **47** (1985) 45-57.



Ikram Bouzoualeghis is a doctoral, She is preparing a PhD thesis on Numerical Analysis at University of Badji Mokhtar, Annaba, Algeria. She received her master degree in Mathematics in 2018 from the University of 20 Août 1955 Skikda, Algeria.



Saadi Samira is currently a Professor at Department of Mathematics Faculty of Science at University of Badji Mokhtar, Annaba, Algeria. She received her PhD degree in mathematics on 2006 a specializing Numerical analysis and his M.S. degree in 1997 both from University of Badji Mokhtar, Annaba, Algeria. She research interests include: Numerical Analysis of Variational and Quasi-Variational Inequalities, Hamilton Jacobi Bellman-equations, Numerical Methods for PDEs. She has published more than 09 papers in international refereed journals. She has also supervised 10 Master Theses, 6 PhD Theses, and served as external examiner in more than 40 Master and PhD Theses.