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Expansion Formulae Involving the Product of q-Analogue of Hypergeometric Function and Fox's H-Function

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Abstract: The main objective of present article is to drive expansion formulas for the product of basic analogue of hypergeometric function and Fox's H-function by the application of the q-Leibniz rule for Weyl type and Reimann-Liouville type fractional q-derivatives. Some special cases involving product of q-analogue of hypergeometric function, G-function and MacRobert's E-function have been discussed.

Keywords: q-Leibniz rule, Weyl type fractional q-derivative, Reimann-Liouville type fractional q-derivative, q-analogue of hypergeometric function, Fox's H-function, Meijer's G-function, MacRobert's E-function.

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1 Introduction and Preliminaries

In last few years, many researchers have attention to the q-calculus and fractional q-differential equations due to numerous applications of the q-calculus in physics, statistics and mathematics. It is also called the quantum calculus can be dated back to 1908, Jackson's work [1] and fractional q-calculus is the q-analogous of the ordinary fractional calculus.

For $0 <| q |< 1$ the q-shifted factorial is defined as [2]:

$$(b;q)_k \equiv (q^b;q)_k = \begin{cases} 1 & (k=0) \\ \prod_{s=0}^{k-1}(1-bq^s) & (k \in \mathbb{N}), \end{cases} \tag{1}$$

here, $b,q \in \mathbb{C}$ and $b \neq q^{-l}(l \in N_0)$.

The q-derivative of a function $h(t)$ is defined as [2]:

$$D_q\{h(t)\} = \frac{d_q}{d_qt}\{h(t)\} = \frac{h(qt)-h(t)}{qt-t}. \tag{2}$$

From above, we observe and notice that

$$\lim_{q\to 1} D_q\{h(t)\} = \frac{d}{dt}\{h(t)\}, \tag{3}$$

if, given function $h(t)$ is differentiable.

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The q-integral of a function $h(t)$ is defined as [2]:

$$\int_0^s h(t)d_qt = s(1-q)\sum_{l=0}^{\infty} q^l h(tq^l), \tag{4}$$

$$\int_s^{\infty} h(t)d_qt = s(1-q)\sum_{l=0}^{\infty} q^{-l} h(tq^{-l}), \tag{5}$$

$$\int_0^{\infty} h(t)d_qt = s(1-q)\sum_{l=-\infty}^{\infty} q^l h(q^l). \tag{6}$$

Then, (1) can be expressed in terms of q-gamma function as follows:

$$(b;q)_k = \frac{\Gamma_q(b+k)(1-q)^k}{\Gamma_q(b)} \tag{7}$$

where the q-gamma function defined as [2] :

$$\Gamma_q(b) = \frac{(q;q)_\infty}{(q^b;q)_\infty(1-q)^{b-1}}. \tag{8}$$

For $k \in N_0$, q-shifted factorial with negative subscript is defined as follow:

$$(b;q)_{-k} = \frac{1}{(1-bq^{-1})(1-bq^{-2})(1-bq^{-3})...(1-bq^{-k})}. \tag{9}$$

which yields

$$(b;q)_{-k} = \frac{1}{(bq^{-k};q)_k} = \frac{(-q/b)^k q^{\binom{k}{2}}}{(q/b;q)_k} \tag{10}$$

We also write that

$$(b;q)_\infty = \prod_{s=0}^{\infty}(1-bq^s), \tag{11}$$

here, $b,q \in \mathbb{C}$.

From (1), (9) and (10), we can stated that:

$$(b;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}, \quad (k \in \mathbb{Z}) \tag{12}$$

The following identities defined in [2] are important to prove our main results:

$$(bq^{-n};q)_n = \left(\frac{q}{b};q\right)_n \left(\frac{-b}{q}\right)^n q^{-\binom{n}{2}}, \quad (n \in \mathbb{Z}) \tag{13}$$

$$(b;q)_{n-k} = \frac{(b;q)_n}{(b^{-1}q^{1-n};q)_k}(-qb^{-1})^k q^{\binom{k}{2}-nk}, \quad (n,k \in \mathbb{Z}) \tag{14}$$

$$(bq^{-n};q)_k = \frac{(b;q)_k(qb^{-1};q)_n}{(b^{-1}q^{1-k};q)_n} q^{-nk}, \quad (n \in \mathbb{Z}) \tag{15}$$

$$(bq^k;q)_{n-k} = \frac{(b;q)_n}{(b;q)_k} \quad (n \in \mathbb{Z}) \tag{16}$$

$$(bq^n;q)_k = \frac{(b;q)_k(bq^k;q)_n}{(b;q)_n} \tag{17}$$

Agarwal [3], introduced the Reimann-Liouville type fractional q-derivative as follow:

$$D^u_{z,q}\{f(z)\} = \frac{1}{\Gamma_q(-u)}\int_0^z (z-tq)_{-u-1} f(t) d_q t, \tag{18}$$

The above equation (18), exist $\forall$ values of u.

From the equation (4), the above operator (18) can be stated as:

$$D^u_{z,q}\{f(z)\} = \frac{z^{-u}(1-q)}{\Gamma_q(-u)}\sum_{k=0}^{\infty} q^k (1-q^{k+1})_{-u-1} f(zq^k). \tag{19}$$

For $f(z) = z^{\eta-1}$, the above equation yields to

$$D^u_{z,q}\{z^{\eta-1}\} = \frac{\Gamma_q(\eta)}{\Gamma_q(\eta-u)} z^{\eta-u-1} \tag{20}$$

where, $\Re(\eta) > 0$, and exist $\forall$ values of u.

The q-extension of the Leibniz rule involving Reimann-Liouville type fractional q-derivative defined as follows [3]:

$$D^u_{z,q}\{U_1(z)V_1(z)\} = \sum_{r=0}^{\infty} \frac{(-1)^r q^{r(r+1)/2}(q^{-u};q)_r}{(q;q)_r} D^{u-r}_{z,q}\{U_1(zq^r)\} D^r_{z,q}\{V_1(z)\} \tag{21}$$

where, $U_1(z)$ and $V_1(z)$ are two regular functions with power series representation $U_1(z) = \sum_{s=0}^{\infty} a_s z^s, \ \mid z \mid < S_1$; $V_1(z) = \sum_{s=0}^{\infty} b_s z^s \ , \mid z \mid < S_2$, then for above result (21), $\mid z \mid < S = \min(S_1, S_2)$.

The q- Weyl fractional derivative operator introduce by Al-Salam [4] as follows:

$${}_xD^u_{\infty,q}\{h(x)\} = \frac{q^{-u(1+u)/2}}{\Gamma_q(-u)}\int_x^{\infty} (t-x)_{-u-1} h(tq^{1+u}) d_q t. \tag{22}$$

where, $\Re(u) < 0$ and from the equation (5), the above operator (22), can be stated as follows:

$${}_xD^u_{\infty,q}\{h(x)\} = \frac{q^{u(1-u)/2} x^{-u}(1-q)}{\Gamma_q(-u)}\sum_{k=0}^{\infty} q^{uk}(1-q^{k+1})_{-u-1} h(xq^{u-k}) \tag{23}$$

where, $\Re(u) < 0$.

and

$$(a-b)_u = a^u \prod_{n=0}^{\infty}\left[\frac{1-(b/a)q^n}{1-(b/a)q^{n+u}}\right] \tag{24}$$

For, $h(x) = x^{-a}$, the above equation reduces to

$${}_xD^u_{\infty,q}\{x^{-a}\} = \frac{\Gamma_q(a+u)}{\Gamma_q(a)} q^{-ua+u(1-u)/2} x^{-a-u} \tag{25}$$

The q-extension of the Leibniz rule involving Weyl type fractional q-derivatives defined as follow [5]:

$${}_zD^u_{\infty,q}\{U_1(z)V_1(z)\} = \sum_{r=0}^{u} \frac{(-1)^r q^{r(r+1)/2}(q^{-u};q)_r}{(q;q)_r} {}_zD^{u-r}_{\infty,q}\{U_1(z)\} {}_zD^r_{\infty,q}\{V_1(zq^{u-r})\} \tag{26}$$

To, drive our main results, we need following q-analogue of Special functions :

Generalised q-hypergeometric abnormal type series ${}_r\Phi_s$ is defined as follow [6]:

$${}_r\Phi_s\left[\begin{matrix} b_1,\ldots,b_r;q;z \\ c_1,\ldots c_s;q^{\sigma}\end{matrix}\right] = \sum_{l=0}^{\infty} \frac{(b_1,\ldots\ldots,b_r;q)_l}{(q,c_1,\ldots c_s;q)_l} z^l q^{\sigma l(l+1)/2}, \tag{27}$$

where, $\sigma > 0$.

For $\sigma = 0$, above series (27), becomes generalised q-hypergeometric series ${}_r\Phi_s$ as [6]:

$$ {}_r\Phi_s \left[\begin{matrix} b_1, ..., b_r; \\ c_1, ..., c_s; \end{matrix} \; q, z \right] = \sum_{l=0}^{\infty} \frac{(b_1, ..., b_r; q)_l}{(q, c_1, ..., c_s; q)_l} z^l, \tag{28} $$

where, convergence conditions for of the series (28), are as follows, for $|\, q \,| < 1$, $\begin{cases} \forall z, & if \;\; r \le s \\ |\, z \,| < 1, & if \;\; r = s+1 \end{cases}$

The basic analogue of the H-function defined as [7]:

$$ H_{C,D}^{m,n} \left[z; q \left| \begin{matrix} (a,\gamma) \\ (b,\delta) \end{matrix} \right. \right] = \frac{1}{2\pi\imath} \int_{\mathscr{C}} \frac{\prod_{j=1}^{m} G(q^{b_j - \delta_j s}) \prod_{j=1}^{n} G(q^{1-a_j-\gamma_j s}) \pi z^s}{\prod_{j=m+1}^{D} G(q^{1-b_j+\delta_j s}) \prod_{j=n+1}^{C} G(q^{a_j - \delta_j s}) G(q^{1-s}) sin\pi s} ds, \tag{29} $$

where,

$$ G(q^{\alpha}) = \frac{1}{\prod_{n=0}^{\infty}(1 - q^{\alpha+n})} = \frac{1}{(q^{\alpha}; q)_{\infty}}, \tag{30} $$

and $0 \le m \le D$, $0 \le n \le C$; γ_j and δ_j are all positive integers. For more details about Fox's H function, we refer see, [7].

Further notice that for $\gamma_j = \delta_i = 1$, $j = 1, ..., C$, $i = 1, ...D$, then Fox's H-function (29), becomes the q-analogue of the Meijer's G-function studied by Saxena et. al. [7] as follow:

$$ \begin{aligned} H_{C,D}^{m,n} \left[z; q \left| \begin{matrix} (a,1) \\ (b,1) \end{matrix} \right. \right] &\equiv G_{C,D}^{m,n} \left[z; q \left| \begin{matrix} a_1, ..., a_A \\ b_1, ..., b_B \end{matrix} \right. \right] \\ &= \frac{1}{2\pi\imath} \int_{\mathscr{C}} \frac{\prod_{j=1}^{m} G(q^{b_j - s}) \prod_{j=1}^{n} G(q^{1-a_j-s}) \pi z^s}{\prod_{j=m+1}^{D} G(q^{1-b_j+s}) \prod_{j=n+1}^{C} G(q^{a_j - s}) G(q^{1-s}) sin\pi s} ds, \end{aligned} \tag{31} $$

where, $0 \le m \le D$, $0 \le n \le C$ and $\Re[slog(z) - logsin\pi s] < 0$.

If we substitute $n = 0$ and $m = D$ in (31), then we have the basic analogue of MacRobert's E-function studied by Agarwal [8] as follows:

$$ G_{C,D}^{D,0} \left[z; q \left| \begin{matrix} a_1, ..., a_C \\ b_1, ..., b_D \end{matrix} \right. \right] \equiv E_q[D; b_j : C; a_j : z] = \frac{1}{2\pi\imath} \int_{\mathscr{C}} \frac{\prod_{j=1}^{D} G(q^{b_j - s}) \pi z^s}{\prod_{j=1}^{C} G(q^{a_j - s}) G(q^{1-s}) sin\pi s} ds, \tag{32} $$

where, $\Re[slog(z) - logsin\pi s] < 0$.

2 Main Results

Here, we drive some expansion formulae involving the product of q- analogue of hypergeometric function and Fox's H-function by substituting appropriate values to the functions $U_1(z)$ and $V_1(z)$ in the q-Leibniz rule via Weyl type and Reimann-Liouville type fractional q-derivative.

Theorem 1. *The following summation result hold true:*

$$ \begin{aligned} & {}_r\Phi_s \left[\begin{matrix} a_1, ..., a_r; q; \rho(zq^u) \\ b_1, ...b_s; q^{\sigma} \end{matrix} \right] H_{C+1,D+1}^{m+1,n} \left[\rho(zq^u)^k; q \left| \begin{matrix} (a,\gamma), (1-\lambda-t, k) \\ (1-\lambda-t+u, k), (b,\delta) \end{matrix} \right. \right] \\ = & \sum_{r=0}^{\infty} \frac{(q^{-u}; q)_r (q^{1-\lambda-r}; q)_u q^{(\lambda+u)r}}{(q;q)_r (q^{\lambda}; q)_r} {}_{r+1}\Phi_{s+1} \left[\begin{matrix} a_1, ..., a_r, \lambda; q; \rho(zq^r) \\ b_1, ...b_s, \lambda + r - u; q^{\sigma} \end{matrix} \right] H_{C+1,D+1}^{m+1,n} \left[\rho(zq^r)^k; q \left| \begin{matrix} (a,\gamma), (0,k) \\ (r,k), (b,\delta) \end{matrix} \right. \right] \end{aligned} \tag{33} $$

Proof. From the equation (21), we have:

$$ D_{z,q}^{u}\{U_1(z)V_1(z)\} = \sum_{r=0}^{\infty} \frac{(-1)^r q^{r(r+1)/2(q^{-u};q)_r}}{(q;q)_r} D_{z,q}^{u-r}\{U_1(zq^r)\} D_{z,q}^{r}\{V_1(z)\} \tag{34} $$

To, prove our result, we begin with $U_1(z) = z^{\lambda-1}{}_r\Phi_s\left[\begin{matrix} a_1,...,a_r;q;\rho z \\ b_1,...b_s;q^\sigma \end{matrix}\right]$ and $V_1(z) = H^{m,n}_{C,D}\left[\rho z^k;q\left|\begin{matrix}(a,\gamma)\\(b,\delta)\end{matrix}\right.\right]$ in the above equation to obtain

$$D^u_{z,q}\left\{z^{\lambda-1}{}_r\Phi_s\left[\begin{matrix} a_1,...,a_r;q;\rho z \\ b_1,...b_s;q^\sigma \end{matrix}\right] H^{m,n}_{C,D}\left[\rho z^k;q\left|\begin{matrix}(a,\gamma)\\(b,\delta)\end{matrix}\right.\right]\right\}$$
$$= \sum_{r=0}^{\infty}\frac{(-1)^r q^{r(r+1)/2(q^{-u};q)_r}}{(q;q)_r} D^{u-r}_{z,q}\left\{(zq^r)^{\lambda-1}{}_r\Phi_s\left[\begin{matrix} a_1,...,a_r;q;\rho(zq^r) \\ b_1,...b_s;q^\sigma \end{matrix}\right]\right\} D^r_{z,q}\left\{H^{m,n}_{C,D}\left[\rho z^k;q\left|\begin{matrix}(a,\gamma)\\(b,\delta)\end{matrix}\right.\right]\right\} \tag{35}$$

Denote the left-hand of the above equation by L

$$L \equiv D^u_{z,q}\left\{z^{\lambda-1}{}_r\Phi_s\left[\begin{matrix} a_1,...,a_r;q;\rho z \\ b_1,...b_s;q^\sigma \end{matrix}\right] H^{m,n}_{C,D}\left[\rho z^k;q\left|\begin{matrix}(a,\gamma)\\(b,\delta)\end{matrix}\right.\right]\right\} \tag{36}$$

Than view of definitions (27), (29), the above equation becomes

$$L = \sum_{t=0}^{\infty}\frac{(a_1,.....,a_r;q)_t}{(q,b_1,...b_s;q)_t}\rho^t q^{\sigma t(t+1)/2}$$
$$\frac{1}{2\pi\iota}\int_{\mathscr{C}}\frac{\prod_{j=1}^{m}G(q^{b_j-\delta_j s})\prod_{j=1}^{n}G(q^{1-a_j-\gamma_j s})\pi\rho^s}{\prod_{j=m+1}^{D}G(q^{1-b_j+\delta_j s})\prod_{j=n+1}^{C}G(q^{a_j-\gamma_j s})G(q^{1-s})\sin\pi s} D^u_{z,q}\{z^{\lambda+t+ks-1}\}ds \tag{37}$$

On using the equation (20),

$$D^u_{z,q}\{z^{\lambda+t+ks-1}\} = \frac{\Gamma_q(\lambda+t+ks)}{\Gamma_q(\lambda+t+ks-u)} z^{\lambda+t+ks-u-1} \tag{38}$$

Then using the identities (7),(10), (12) and (30), we have

$$D^u_{z,q}\{z^{\lambda+t+ks-1}\} = \frac{G(q^{1-\lambda-t+u-ks})}{(-1)^u q^{(1-\lambda-t-ks)u}q^{u(u-1)/2}(1-q)^u G(q^{1-\lambda-t-ks})} z^{\lambda+t+ks-u-1} \tag{39}$$

Then substitute value of (39) in the equation (37), we have

$$L = \sum_{t=0}^{\infty}\frac{(a_1,.....,a_r;q)_t}{(q,b_1,...b_s;q)_t}\rho^t q^{\sigma t(t+1)/2}$$
$$\frac{1}{2\pi\iota}\int_{\mathscr{C}}\frac{\prod_{j=1}^{m}G(q^{b_j-\delta_j s})\prod_{j=1}^{n}G(q^{1-a_j-\gamma_j s})\pi\rho^s}{\prod_{j=m+1}^{D}G(q^{1-b_j+\delta_j s})\prod_{j=n+1}^{C}G(q^{a_j-\gamma_j s})G(q^{1-s})\sin\pi s}$$
$$\times\frac{G(q^{1-\lambda-t+u-ks})}{(-1)^u q^{(1-\lambda-t-ks)u}q^{u(u-1)/2}(1-q)^u G(q^{1-\lambda-t-ks})} z^{\lambda+t+ks-u-1}ds \tag{40}$$

On re-arranging the terms and using the definitions, (27), (29), we get

$$L = \frac{(-1)^{-u}z^{\lambda-u-1}q^{(\lambda-1)u}q^{u(1-u)/2}}{(1-q)^u}{}_r\Phi_s\left[\begin{matrix} a_1,...,a_r;q;\rho(zq^u) \\ b_1,...b_s;q^\sigma \end{matrix}\right] H^{m+1,n}_{C+1,D+1}\left[\rho(zq^u)^k;q\left|\begin{matrix}(a,\gamma),(1-\lambda-t,k)\\(1-\lambda-t+u,k),(b,\delta)\end{matrix}\right.\right] \tag{41}$$

Now denote right-hand side of (35), by R

$$R \equiv \sum_{r=0}^{\infty}\frac{(-1)^r q^{r(r+1)/2(q^{-u};q)_r}}{(q;q)_r} D^{u-r}_{z,q}\left\{(zq^r)^{\lambda-1}{}_r\Phi_s\left[\begin{matrix} a_1,...,a_r;q;\rho(zq^r) \\ b_1,...b_s;q^\sigma \end{matrix}\right]\right\} D^r_{z,q}\left\{H^{m,n}_{C,D}\left[\rho z^k;q\left|\begin{matrix}(a,\gamma)\\(b,\delta)\end{matrix}\right.\right]\right\} \tag{42}$$

Now, from definition of (27) and applying (20) and some identities (8), (14) and (15), we get

$$D^{u-r}_{z,q}\left\{(zq^r)^{\lambda-1}{}_r\Phi_s\left[\begin{matrix} a_1,...,a_r;q;\rho(zq^r) \\ b_1,...b_s;q^\sigma \end{matrix}\right]\right\} = \sum_{t=0}^{\infty}\frac{(a_1,.....,a_r;q)_t}{(q,b_1,...b_s;q)_t}\rho^t q^{\sigma t(t+1)/2}q^{rt}q^{(\lambda-1)r}$$
$$\frac{q^{-ut}(1-q)^{r-u}(q^\lambda;q)_t(q^{1-\lambda-r};q)_u}{(-1)^u(q^\lambda;q)_r(q^{\lambda+r-u};q)_t q^{u-\lambda u-ut}q^{u(u-1)/2-ur}} \tag{43}$$

Then from definition (29) and applying (20) and some identities (8), (10), (12) and (30), we get

$$D_{z,q}^{r}\left\{H_{A,B}^{m,n}\left[\rho z^{k};q\left|\begin{matrix}(a,\gamma)\\(b,\delta)\end{matrix}\right.\right]\right\}=\frac{1}{2\pi\iota}\int_{\mathscr{C}}\frac{\prod_{j=1}^{m}G(q^{b_j-\delta_j s})\prod_{j=1}^{n}G(q^{1-a_j-\gamma_j s})\pi\rho^{s}}{\prod_{j=m+1}^{D}G(q^{1-b_j+\delta_j s})\prod_{j=n+1}^{C}G(q^{a_j-\gamma_j s})G(q^{1-s})\sin\pi s}\frac{G(q^{r-ks})(1-q)^{-r}z^{ks-r}}{(-1)^{r}q^{-ksr}q^{r(r-1)/2}G(q^{-ks})}ds \tag{44}$$

Now, using (44),(43) in (42) and re-arranging some terms, we get

$$R=\frac{z^{\lambda-u-1}q^{(\lambda-1)u}(-1)^{u}q^{u(1-u)/2}}{(1-q)^{u}}\sum_{r=0}^{\infty}\frac{(q^{-u};q)_r(q^{1-\lambda-r};q)_u q^{(\lambda+u)r}}{(q;q)_r(q^{\lambda};q)_r}\times {}_{r+1}\Phi_{s+1}\left[\begin{matrix}a_1,\ldots,a_r,\lambda;q;\rho(zq^{r})\\b_1,\ldots b_s,\lambda+r-u;q^{\sigma}\end{matrix}\right]H_{C+1,D+1}^{m+1,n}\left[\rho(zq^{r})^{k};q\left|\begin{matrix}(a,\gamma),(0,k)\\(r,k),(b,\delta)\end{matrix}\right.\right] \tag{45}$$

So, from (41) and (45), we have our desired result.

$${}_{r}\Phi_{s}\left[\begin{matrix}a_1,\ldots,a_r;q;\rho(zq^{u})\\b_1,\ldots b_s;q^{\sigma}\end{matrix}\right]H_{C+1,D+1}^{m+1,n}\left[\rho(zq^{u})^{k};q\left|\begin{matrix}(a,\gamma),(1-\lambda-t,k)\\(1-\lambda-t+u,k),(b,\delta)\end{matrix}\right.\right]=\sum_{r=0}^{\infty}\frac{(q^{-u};q)_r(q^{1-\lambda-r};q)_u q^{(\lambda+u)r}}{(q;q)_r(q^{\lambda};q)_r}{}_{r+1}\Phi_{s+1}\left[\begin{matrix}a_1,\ldots,a_r,\lambda;q;\rho(zq^{r})\\b_1,\ldots b_s,\lambda+r-u;q^{\sigma}\end{matrix}\right]H_{C+1,D+1}^{m+1,n}\left[\rho(zq^{r})^{k};q\left|\begin{matrix}(a,\gamma),(0,k)\\(r,k),(b,\delta)\end{matrix}\right.\right] \tag{46}$$

Corollary 1. *The following result hold true:*

$${}_{r}\Phi_{s}\left[\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots b_s\end{matrix}q;\rho(zq^{u})\right]H_{C+1,D+1}^{m+1,n}\left[\rho(zq^{u})^{k};q\left|\begin{matrix}(a,\gamma),(1-\lambda-t,k)\\(1-\lambda-t+u,k),(b,\delta)\end{matrix}\right.\right]=\sum_{r=0}^{\infty}\frac{(q^{-u};q)_r(q^{1-\lambda-r};q)_u q^{(\lambda+u)r}}{(q;q)_r(q^{\lambda};q)_r}{}_{r+1}\Phi_{s+1}\left[\begin{matrix}a_1,\ldots,a_r,\lambda\\b_1,\ldots b_s,\lambda+r-u\end{matrix}q;\rho(zq^{r})\right]H_{C+1,D+1}^{m+1,n}\left[\rho(zq^{r})^{k};q\left|\begin{matrix}(a,\gamma),(0,k)\\(r,k),(b,\delta)\end{matrix}\right.\right] \tag{47}$$

Proof. On substitute $\sigma=0$ in (1) and using (28), we get our desired result.

Corollary 2. *The following result hold true:*

$${}_{r}\Phi_{s}\left[\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots b_s\end{matrix}q;\rho(zq^{u})\right]G_{C+1,D+1}^{m+1,n}\left[\rho(zq^{u});q\left|\begin{matrix}a_1\ldots a_C,(1-\lambda-t)\\(1-\lambda-t+u),b_1\ldots b_D\end{matrix}\right.\right]=\sum_{r=0}^{\infty}\frac{(q^{-u};q)_r(q^{1-\lambda-r};q)_u q^{(\lambda+u)r}}{(q;q)_r(q^{\lambda};q)_r}{}_{r+1}\Phi_{s+1}\left[\begin{matrix}a_1,\ldots,a_r,\lambda\\b_1,\ldots b_s,\lambda+r-u\end{matrix}q;\rho(zq^{r})\right]G_{C+1,D+1}^{m+1,n}\left[\rho(zq^{r});q\left|\begin{matrix}a_1\ldots a_C,0\\r,b_1\ldots b_D\end{matrix}\right.\right] \tag{48}$$

Proof. On putting $\sigma=0$, $k=1$ and $\gamma_j=\delta_i=1$, $j=1,\ldots,C$ and $i=1,\ldots,D$ in (1) and using (31), we get our desired result.

Corollary 3. *The following result hold true:*

$${}_{r}\Phi_{s}\left[\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots b_s\end{matrix}q;\rho(zq^{u})\right]E_q[D+1,b_j;(1-\lambda-t+u):C+1,a_j;(1-\lambda-t);\rho(zq^{u})]=\sum_{r=0}^{\infty}\frac{(q^{-u};q)_r(q^{1-\lambda-r};q)_u q^{(\lambda+u)r}}{(q;q)_r(q^{\lambda};q)_r}{}_{r+1}\Phi_{s+1}\left[\begin{matrix}a_1,\ldots,a_r,\lambda\\b_1,\ldots b_s,\lambda+r-u\end{matrix}q;\rho(zq^{r})\right]E_q[D+1,b_j;r:C+1,a_j;0;\rho(zq^{r})] \tag{49}$$

Proof. On considering $\sigma=0$, $k=1$, $m=D$, $n=0$ and $\gamma_j=\delta_i=1$, $j=1,\ldots,C$ and $i=1,\ldots,D$ in (1) and using (32), we get our desired result.

Theorem 2. *The following summation result hold true:*

$${}_{r}\Phi_{s}\left[\begin{matrix}a_1,\ldots,a_r;q;\frac{\rho}{(zq^{u})}\\b_1,\ldots b_s;q^{\sigma}\end{matrix}\right]H_{C+1,D+1}^{m+1,n}\left[\rho(zq^{u})^{k};q\left|\begin{matrix}(a,\gamma),(\lambda+t,k)\\(\lambda+t+u,k),(b,\delta)\end{matrix}\right.\right]=\sum_{r=0}^{u}\frac{(q^{-u};q)_r(q^{\lambda};q)_u q^{r(r+1)/2}}{(q;q)_r(q^{1-\lambda-u};q)_r}{}_{r+1}\Phi_{s+1}\left[\begin{matrix}a_1,\ldots,a_r,\lambda+u-r;q;\frac{\rho}{(zq^{u-r})}\\b_1,\ldots b_s,\lambda+r-u;q^{\sigma}\end{matrix}\right]\times H_{C+1,D+1}^{m+1,n}\left[\rho(zq^{u-r})^{k};q\left|\begin{matrix}(a,\gamma),(0,k)\\(r,k),(b,\delta)\end{matrix}\right.\right] \tag{50}$$

Proof. From the equation (26), we have:

$$ {}_zD^u_{\infty,q}\{U_1(z)V_1(z)\} = \sum_{r=0}^{u} \frac{(-1)^r q^{r(r+1)/2(q^{-u};q)_r}}{(q;q)_r} {}_zD^{u-r}_{\infty,q}\{U_1(z)\}{}_zD^r_{\infty,q}\{V_1(zq^{u-r})\} \tag{51} $$

Then consider $U_1(z) = z^{-\lambda}{}_r\Phi_s\left[\begin{matrix} a_1,...,a_r;q;\frac{\rho}{z} \\ b_1,...b_s;q^{\sigma^z} \end{matrix}\right]$ and $V_1(z) = H^{m,n}_{C,D}\left[\rho z^k;q\left|\begin{matrix}(a,\gamma)\\(b,\delta)\end{matrix}\right.\right]$ and with same parallel line of proof as (1) and using some identities (8), (10), (12), with results (25), (30) we get our desired result.

Corollary 4. *The following result hold true:*

$$ \begin{aligned} &{}_r\Phi_s\left[\begin{matrix} a_1,...,a_r \\ b_1,...b_s \end{matrix} q;\frac{\rho}{zq^u}\right] H^{m+1,n}_{C+1,D+1}\left[\rho(zq^u)^k;q\left|\begin{matrix}(a,\gamma),(\lambda+t,k)\\(\lambda+t+u,k),(b,\delta)\end{matrix}\right.\right] \\ &= \sum_{r=0}^{u} \frac{(q^{-u};q)_r(q^\lambda;q)_u q^{r(r+1)/2}}{(q;q)_r(q^{1-\lambda-u};q)_r} {}_{r+1}\Phi_{s+1}\left[\begin{matrix} a_1,...,a_r,\lambda+u-r \\ b_1,...b_s,\lambda \end{matrix} q;\frac{\rho}{zq^{u-r}}\right] H^{m+1,n}_{C+1,D+1}\left[\rho(zq^{u-r})^k;q\left|\begin{matrix}(a,\gamma),(0,k)\\(r,k),(b,\delta)\end{matrix}\right.\right] \end{aligned} \tag{52} $$

Proof. On substitute $\sigma = 0$ in (2) and using (28), we get our desired result.

Corollary 5. *The following result hold true:*

$$ \begin{aligned} &{}_r\Phi_s\left[\begin{matrix} a_1,...,a_r \\ b_1,...b_s \end{matrix} q;\frac{\rho}{zq^u}\right] G^{m+1,n}_{C+1,D+1}\left[\rho(zq^u);q\left|\begin{matrix}a_1...a_C,(\lambda+t)\\(\lambda+t+u),b_1...b_D\end{matrix}\right.\right] \\ &= \sum_{r=0}^{u} \frac{(q^{-u};q)_r(q^\lambda;q)_u q^{r(r+1)/2}}{(q;q)_r(q^{1-\lambda-u};q)_r} {}_{r+1}\Phi_{s+1}\left[\begin{matrix} a_1,...,a_r,\lambda+u-r \\ b_1,...b_s,\lambda \end{matrix} q;\frac{\rho}{zq^{u-r}}\right] G^{m+1,n}_{C+1,D+1}\left[\rho(zq^{u-r});q\left|\begin{matrix}a_1...a_C,0\\r,b_1...b_D\end{matrix}\right.\right] \end{aligned} \tag{53} $$

Proof. On putting $\sigma = 0$, $k = 1$ and $\gamma_j = \delta_i = 1$, $j = 1,...,C$ and $i = 1,...,D$ in (2) and using (31), we get our desired result.

Corollary 6. *The following result hold true:*

$$ \begin{aligned} &{}_r\Phi_s\left[\begin{matrix} a_1,...,a_r \\ b_1,...b_s \end{matrix} q;\frac{\rho}{zq^u}\right] E_q[D+1,b_j;(\lambda+t+u):C+1,a_j;(\lambda+t);\rho(zq^u)] \\ &= \sum_{r=0}^{u} \frac{(q^{-u};q)_r(q^\lambda;q)_u q^{r(r+1)/2}}{(q;q)_r(q^{1\lambda-u};q)_r} {}_{r+1}\Phi_{s+1}\left[\begin{matrix} a_1,...,a_r,\lambda+u-r \\ b_1,...b_s,\lambda \end{matrix} q;\frac{\rho}{zq^{u-r}}\right] E_q[D+1,b_j;r:C+1,a_j;0;\rho(zq^{u-r})] \end{aligned} \tag{54} $$

Proof. On substitute $\sigma = 0$, $k = 1$, $m = D$, $n = 0$ and $\gamma_j = \delta_i = 1$, $j = 1,...,C$ and $i = 1,...,D$ in (2) and using (32), we get our desired result.

3 Some Examples

Example 1. By putting $r = 2$ and $s = 1$ in the Theorem (1) with the help of generalised q- hypergeometric function (28), it becomes:

$$ \begin{aligned} &{}_2\Phi_1\left[\begin{matrix} a_1,a_2 \\ b_1 \end{matrix} q;\rho(zq^u)\right] H^{m+1,n}_{C+1,D+1}\left[\rho(zq^u)^k;q\left|\begin{matrix}(a,\gamma),(1-\lambda-t,k)\\(1-\lambda-t+u,k),(b,\delta)\end{matrix}\right.\right] \\ &= \sum_{r=0}^{\infty} \frac{(q^{-u};q)_r(q^{1-\lambda-r};q)_u q^{(\lambda+u)r}}{(q;q)_r(q^\lambda;q)_r} {}_3\Phi_2\left[\begin{matrix} a_1,a_2,\lambda \\ b_1,\lambda+r-u \end{matrix} q;\rho(zq^r)\right] H^{m+1,n}_{C+1,D+1}\left[\rho(zq^r)^k;q\left|\begin{matrix}(a,\gamma),(0,k)\\(r,k),(b,\delta)\end{matrix}\right.\right] \end{aligned} \tag{55} $$

Example 2. If we set $r = 1$ and $s = 1$ in the Theorem (1) then by using the generalised q- hypergeometric function (28), we have:

$$ \begin{aligned} &{}_1\Phi_1\left[\begin{matrix} a \\ b \end{matrix} q;\rho(zq^u)\right] H^{m+1,n}_{C+1,D+1}\left[\rho(zq^u)^k;q\left|\begin{matrix}(a,\gamma),(1-\lambda-t,k)\\(1-\lambda-t+u,k),(b,\delta)\end{matrix}\right.\right] \\ &= \sum_{r=0}^{\infty} \frac{(q^{-u};q)_r(q^{1-\lambda-r};q)_u q^{(\lambda+u)r}}{(q;q)_r(q^\lambda;q)_r} {}_2\Phi_2\left[\begin{matrix} a,\lambda \\ b,\lambda+r-u \end{matrix} q;\rho(zq^r)\right] H^{m+1,n}_{C+1,D+1}\left[\rho(zq^r)^k;q\left|\begin{matrix}(a,\gamma),(0,k)\\(r,k),(b,\delta)\end{matrix}\right.\right] \end{aligned} \tag{56} $$

*Example 3.*If we set $r = 2$ and $s = 1$ in the Theorem (2), it becomes in the following form by using (28):

$$
\begin{aligned}
&{}_2\Phi_1\left[\begin{matrix}a_1,a_2\\ b_1\end{matrix}\quad q;\frac{\rho}{zq^u}\right]H^{m+1,n}_{C+1,D+1}\left[\rho(zq^u)^k;q\left|\begin{matrix}(a,\gamma),(\lambda+t,k)\\(\lambda+t+u,k),(b,\delta)\end{matrix}\right.\right]\\
&=\sum_{r=0}^{u}\frac{(q^{-u};q)_r(q^\lambda;q)_u q^{r(r+1)/2}}{(q;q)_r(q^{1-\lambda-u};q)_r}\,{}_3\Phi_2\left[\begin{matrix}a_1,a_2,\lambda+u-r\\ b_1,\lambda\end{matrix}\quad q;\frac{\rho}{zq^{u-r}}\right]H^{m+1,n}_{C+1,D+1}\left[\rho(zq^{u-r})^k;q\left|\begin{matrix}(a,\gamma),(0,k)\\(r,k),(b,\delta)\end{matrix}\right.\right]
\end{aligned}
\tag{57}
$$

*Example 4.*If $r = 1$ and $s = 1$ in the Theorem (2) and using (28), we get:

$$
\begin{aligned}
&{}_1\Phi_1\left[\begin{matrix}a\\ b\end{matrix}\quad q;\frac{\rho}{zq^u}\right]H^{m+1,n}_{C+1,D+1}\left[\rho(zq^u)^k;q\left|\begin{matrix}(a,\gamma),(\lambda+t,k)\\(\lambda+t+u,k),(b,\delta)\end{matrix}\right.\right]\\
&=\sum_{r=0}^{u}\frac{(q^{-u};q)_r(q^\lambda;q)_u q^{r(r+1)/2}}{(q;q)_r(q^{1-\lambda-u};q)_r}\,{}_2\Phi_2\left[\begin{matrix}a,\lambda+u-r\\ b,\lambda\end{matrix}\quad q;\frac{\rho}{zq^{u-r}}\right]H^{m+1,n}_{C+1,D+1}\left[\rho(zq^{u-r})^k;q\left|\begin{matrix}(a,\gamma),(0,k)\\(r,k),(b,\delta)\end{matrix}\right.\right]
\end{aligned}
\tag{58}
$$

4 Concluding Remark

In a series of papers by many researchers [5,9,10,11,12,13], investigated many applications of q-Leibniz rule given by (21) and (26) and deduced many fascinating transformations and summation formulas involving many q- hypergeometric functions of one and more variables including the basic q-analogue of Fox's H- function. Here, we have introduced summation formulas for the product of basic hypergeometric function and q-analogue of Fox's H-function by the application of the q-Leibniz rule for Weyl type q-derivative and Reimann-Liouville type q-derivative. Then, we have drive some particular cases of the results presented in this article. Our results are new in theory of quantum calculus.

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