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Some Transcendence Results from a Harmless Irrationality Theorem

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Abstract: The arithmetic nature of values of some functions such as $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, e^z, and $\ln z$, is a relevant topic in number theory. For instance, all those functions return transcendental values for non-zero algebraic values of z ($z \neq 1$ in the case of $\ln z$). On the other hand, not even an irrationality proof is known for some numbers like $\ln \pi$, $\pi + e$ and $\pi\, e$, though it is well-known that at least one of the last two numbers is irrational. In this note, we first generalize the last result, showing that at least one of the sum and product of any two transcendental numbers is transcendental. We then use this to show that, given any non-null complex number $z \neq 1/e$, at least two of the numbers $\ln z$, $z+e$ and $z\,e$ are transcendental. It is also shown that $\cosh z$, $\sinh z$ and $\tanh z$ return transcendental values for all $z = r \ln t$, $r \in \mathbb{Q}$, $r \neq 0$, t being any transcendental number. The analogue for common trigonometric functions is also proved.

Keywords: Irrationality proofs, Transcendental numbers, Trigonometric functions, Hyperbolic functions

1 Introduction

As usual, let $\mathbb{Q}$ denote the set of all rational numbers, i.e. the numbers which can be written as p/q, p and q being integers, $q \neq 0$. Also, let $\mathscr{A}$ denote the set of all algebraic numbers (over $\mathbb{Q}$), i.e. the complex numbers z which are roots of some polynomial equation in $\mathbb{Z}[z]$. All other complex numbers — i.e. $z \notin \mathscr{A}$ — are called *transcendental* numbers.[1] Though the existence of irrational numbers such as $\sqrt{2}$ remounts to the ancient Greeks, no example of a transcendental number was known at the beginning of the 19th century, which reflects the difficulty of showing that a given number is transcendental. The existence of 'non-algebraic' numbers was conjectured by Euler in 1744, in his *Introductio in analysin infinitorum*, where he claims, without a proof, that "the logarithms of (rational) numbers which are not powers of the base are neither rational nor (algebraic) irrational, so they should be called *transcendental*." A such proof appeared only in 1844, when Liouville showed that any number that has a rapidly converging sequence of distinct rational approximations must be transcendental [10]. In particular, he used his approximation theorem to show that the series $\sum_{k=0}^{\infty} 1/(2^{k!})$ converges to a transcendental number. From the work of Cantor on set theory in 1874, one knows that the set $\mathscr{A}$ is countable whereas the set $\mathbb{R}$ of all real numbers is uncountable, so *'almost all' real numbers are transcendental*. However, it remained an important unsolved problem to prove the transcendence of naturally occurring numbers, such as e (the natural logarithm base) and π (the Archimedes' constant). Then, in 1873 Hermite proved that e^r is transcendental for all rational $r \neq 0$ (in particular, e is transcendental) [5]. In 1882, Lindemann proved the following extension of Hermite's result [9].

Lemma 1 (Hermite-Lindemann) *The number e^{α} is transcendental for all algebraic $\alpha \neq 0$.*

This implies the transcendence of π, as follows from Euler's identity $e^{i\pi} = -1$, which is equivalent to the impossibility of squaring the circle with only ruler and compass, a problem that remained open by more than two thousand years. Lemma 1 is equivalent to the transcendence of $\ln\alpha$ for all $\alpha \in \mathscr{A}$, $\alpha \neq 0, 1$.[2] Based upon these first results, in 1885 Weierstrass succeeded in proving a much more general result.

[1] All rational numbers are roots of $qz - p = 0$, thus algebraic. Hence all transcendental numbers are irrational.

[2] We are interpreting the complex function $\ln z$ as its principal value, with the argument lying in the interval $(-\pi, \pi]$.

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Lemma 2 (Lindemann-Weierstrass) *Given an integer $n > 0$, whenever $\alpha_0, \ldots, \alpha_n$ are distinct algebraic numbers, the numbers $e^{\alpha_0}, \ldots, e^{\alpha_n}$ are linearly independent over $\mathscr{A}$. That is, for any $\beta_0, \ldots, \beta_n \in \mathscr{A}$ not all zero,*

$$\sum_{k=0}^{n} \beta_k \, e^{\alpha_k} \neq 0 \, .$$

For a proof, see, e.g., Theorem 1.4 of Ref. [1] or Theorem 1.8 of Ref. [2]. As an immediate consequence, when one takes $\alpha_0 = 0$ and $\beta_0 \neq 0$, one concludes that

Corollary 1 *Given an integer $n > 0$, being $\alpha_1, \ldots, \alpha_n$ distinct non-zero algebraic numbers and $\beta_1, \ldots, \beta_n \in \mathscr{A}$ not all zero, $\sum_{k=1}^{n} \beta_k \, e^{\alpha_k}$ is a transcendental number.*

From Euler's formula $e^{\pm i\,\theta} = \cos\theta \pm i \, \sin\theta$, it follows that $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$. Analogously, the basic hyperbolic functions are defined as $\cosh\theta := (e^{\theta} + e^{-\theta})/2$ and $\sinh\theta := (e^{\theta} - e^{-\theta})/2$. From Corollary 1, it follows that

Corollary 2 *For any algebraic $\alpha \neq 0$, all numbers $\cos\alpha$, $\sin\alpha$, $\cosh\alpha$, and $\sinh\alpha$ are transcendental.*

The relevance of investigating the transcendence of powers and logarithms of algebraic numbers was acknowledged by Hilbert in his famous lecture "Mathematical Problems" in 1900, at the 2nd International Congress of Mathematicians [6], being the content of his 7th problem, in which he questioned the arithmetic nature of $e^{i\pi z}$ for $z \in \mathscr{A}$.[3] Of course, for $z = r$ a rational it was already known that both $\cos(r\pi)$ and $\sin(r\pi)$ are algebraic, so $e^{i\pi r}$ is algebraic,[4] but the case of *irrational* algebraic values of z remained unsolved until 1934, when Gelfond and Schneider, working independently, showed that [4,15]

Lemma 3 (Gelfond-Schneider) *If $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$ are algebraic numbers, then any value of α^{β} is transcendental.*

For a proof, see e.g. Theorem 2.1 of Ref. [2] or Theorem 10.1 (and Sec. 4 of Chap. 10) of Ref. [13]. This lemma promptly implies the transcendence of $2^{\sqrt{2}}$ and $e^{\pi} = i^{-2i}$, two real numbers mentioned by Hilbert [6]. It also follows that $e^{i\pi\beta} = \left(e^{i\pi}\right)^{\beta} = (-1)^{\beta}$ is a transcendental number for every $\beta \in \mathscr{A} \setminus \mathbb{Q}$, which solves Hilbert's 7th problem.[5]

[3] Hilbert himself remarked that he expected this problem to be harder than showing the Riemann hypothesis!

[4] It was proved by Lehmer in 1933 that, for rational $r = k/n$, $n > 2$, the numbers $2\cos(2\pi r)$ and $2\sin(2\pi r)$ are *algebraic integers* (i.e., roots of monic polynomial equations in $\mathbb{Z}[x]$) [7].

[5] Indeed, given $a, b \in \mathscr{A} \bigcap \mathbb{R}$, if either $a = 0$ and $b \notin \mathbb{Q}$ or $a \neq 0$, then $e^{(a+bi)\,\pi} = e^{(b-ia)\,i\pi} = (-1)^{b-ia}$ is transcendental.

2 Further transcendence results

Lemma 3 has a logarithmic version, namely

Lemma 4 (Log version) *$\log_{\beta}\alpha = \ln\alpha/\ln\beta$ is a transcendental number whenever α and β are non-zero algebraic numbers, $\beta \neq 1$, and $\log_{\beta}\alpha \notin \mathbb{Q}$.*

This form appears, e.g., in Theorem 10.2 of Ref. [13]. It has a consequence for tangent arcs, as noted by Margolius in Ref. [11].

Corollary 3 (Margolius) *If x is rational and $x \neq 0, \pm 1$, then the number $\dfrac{\arctan x}{\pi}$ is transcendental.*

Proof. Write $x = \tan\theta$, $x \in \mathbb{Q}$, $x \neq 0, \pm 1$. Then

$$\frac{\arctan x}{\pi} = \frac{\theta}{\pi} = \frac{1/i \cdot \ln(z/|z|)}{1/i \cdot \ln(-1)}$$
$$= \frac{\ln\left(\pm 1/\sqrt{1+x^2} + x\,i/\sqrt{1+x^2}\right)}{\ln(-1)} \, , \tag{1}$$

which follows by taking $z = \pm 1 + x\,i$ in $\ln(z/|z|) = i\,\theta$, which in turn comes from the exponential representation $z = |z|\, e^{i\,\theta}$. Clearly, the last expression in Eq. (1) is a ratio of two logs with algebraic arguments,[6] so Lemma 4 applies and θ/π has to be either rational or transcendental. However, it is irrational because, being $r \in \mathbb{Q}$, $x = \tan\theta = \tan(r\,\pi)$ is rational only when $x = 0, \pm 1$, as proved in Corollary 3.12 of Ref. [13].[7]

□

In particular, it follows that the Plouffe's constant $\arctan\left(\frac{1}{2}\right)/\pi$ is transcendental [11]. Let us extend Margolius' result to all basic trigonometric arcs. Hereafter, the word 'trig' will stand for any of $\{\cos, \sin, \tan, \cot, \sec, \csc\}$.

Theorem 1 (Extension of Margolius' result) *If x is a real algebraic number, then $\dfrac{\mathrm{arctrig}(x)}{\pi}$ is either a rational or transcendental number.*

Proof. The proof is similar to the previous one, being enough to take $x = \mathrm{trig}(\theta)$, $x \in \mathscr{A} \bigcap \mathbb{R}$, and write

$$\frac{\mathrm{arctrig}(x)}{\pi} = \frac{\theta}{\pi} = \frac{\ln(z/|z|)}{\ln(-1)} \, , \tag{2}$$

[6] Since $\mathscr{A}$ is a *field*, then, given any $\alpha, \beta \in \mathscr{A}$, all the numbers $\alpha \pm \beta$, $\alpha\,\beta$, and α/β ($\beta \neq 0$) are also algebraic (see, e.g., Sec. 6.6 and Theorem 6.12 of Ref. [3]). More generally, given $r \in \mathbb{Q}$ and $\alpha \in \mathscr{A}$, $\alpha \neq 0$, if z is any complex algebraic (respectively, transcendental) number then all numbers $z \pm \alpha$, αz, z/α, and z^r are also algebraic (respectively, transcendental), the only exception being $z^0 = 1$ for $z \notin \mathscr{A}$.

[7] The irrationality of θ/π is also nicely proved by Margolius in Theorem 3 of Ref. [11] by exploring the properties of sequences of primitive Pythagorean triples formed on writing $x = a/b$, a and b being distinct non-zero integers.

$z \neq 0$. The choice of z now changes accordingly to the function represented by 'trig'. For $\arccos x$, choose $z = x \pm \sqrt{1-x^2}\, i$. For $\text{arcsec}\, x$, choose $z = \pm 1 \pm \sqrt{x^2-1}\, i$. For $\arcsin x$, choose $z = \pm\sqrt{1-x^2} + x\,i$. For $\text{arccsc}\, x$, choose $z = \pm\sqrt{x^2-1} \pm i$. For $\arctan x$, choose $z = \pm 1 + x\,i$, as in the previous proof. For $\text{arccot}\, x$, choose $z = x \pm i$. In all these cases, the ratio $z/|z|$ is an algebraic function of x, so it is an algebraic number for all $x \in \mathscr{A} \bigcap \mathbb{R}$. The last expression in Eq. (2) is then a ratio of two logs with algebraic arguments, so Lemma 4 applies. □

Conversely, if $x \in \mathbb{R}$ then it will be transcendental whenever $\text{arctrig}(x)/\pi \in \mathscr{A} \setminus \mathbb{Q}$. This implies, e.g., the transcendence of any $\text{trig}\left(\sqrt{2}\,\pi\right)$.

The following extension of Lemma 3 was conjectured by Gelfond and proved by Baker in 1966, becoming the definitive result in this area.

Lemma 5 (Baker) *Given non-zero algebraic numbers $\alpha_1, \ldots, \alpha_n$ such that $\ln\alpha_1, \ldots, \ln\alpha_n$ are linearly independent over $\mathbb{Q}$, then the numbers $1, \ln\alpha_1, \ldots, \ln\alpha_n$ are linearly independent over $\mathscr{A}$. That is, for any $\beta_0, \ldots, \beta_n \in \mathscr{A}$ not all zero, we have*

$$\beta_0 + \sum_{k=1}^{n} \beta_k \ln\alpha_k \neq 0 \, .$$

For a proof, see Theorem 2.1 of Ref. [1].[8] This lemma has several interesting consequences.

Corollary 4 *Given non-zero algebraic numbers $\alpha_1, \ldots, \alpha_n$, for any $\beta_1, \ldots, \beta_n \in \mathscr{A}$ the number $\beta_1 \ln\alpha_1 + \ldots + \beta_n \ln\alpha_n$ is either null or transcendental. It is transcendental when $\ln\alpha_1, \ldots, \ln\alpha_n$ are linearly independent over $\mathbb{Q}$ and $\beta_1, \ldots, \beta_n$ are not all zero.*

For a proof, see Theorem 2.2 of Ref. [1].

Corollary 5 *Let $\alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots, \beta_n$ be non-zero algebraic numbers. Then the product $e^{\beta_0}\, {\alpha_1}^{\beta_1} \ldots {\alpha_n}^{\beta_n}$ is a transcendental number.*

For a proof, see Theorem 2.3 of Ref. [1].

Corollary 6 *For any algebraic numbers $\alpha_1, \ldots, \alpha_n$ other than 0 or 1, let $1, \beta_1, \ldots, \beta_n$ be algebraic numbers linearly independent over $\mathbb{Q}$. Then the number ${\alpha_1}^{\beta_1} \ldots {\alpha_n}^{\beta_n}$ is transcendental.*

For a proof, see Theorem 2.4 of Ref. [1].

Corollary 7 *The number $e^{\alpha+\pi\beta}$ is transcendental for all algebraic values of α and β, $\alpha \neq 0$.*

For a proof, see Corollary 2 of Ref. [8]. Note that $e^{\alpha+\pi\beta}$ is transcendental even if $\alpha = 0$, as long as $i\beta \notin \mathbb{Q}$ (see Footnote 5). Note also that Corollary 7 implies the transcendence of $(\alpha + \ln\beta)/\pi$ for any non-zero $\alpha, \beta \in \mathscr{A}$.

All this said, it is embarrassing that the numbers $\ln\pi$, $\pi + e$ and πe are still not known to be transcendental. In fact, not even an irrationality proof is known, though it is easy to show that at least one of $\pi + e$ and πe must be irrational. This is proved, e.g., in a nice survey on irrational numbers by Ross in Ref. [14], but let us present a short proof for completeness. Let us call *quadratic* any algebraic number which is a root of a 2nd-order polynomial equation with rational coefficients.[9] From the fact that π is not a quadratic number (since it is not even an algebraic number), it follows that

Lemma 6 (Harmless irrationality) *At least one of the numbers $\pi + e$ and πe is irrational.*

Proof. Consider the quadratic equation $(x - \pi)\cdot(x - e) = 0$, whose roots are π and e. By expanding the product, one has $x^2 - (\pi + e)\,x + \pi e = 0$. Assume, towards a contradiction, that both coefficients $\pi + e$ and πe are rational numbers. Then, our quadratic equation would have rational coefficients and both roots would be quadratic numbers. However, π is *not* a quadratic number.

□

Since this proof does not make use of any property of e, it is clear that Lemma 6 can be generalized.

Lemma 7 (General irrationality) *Given any irrational number u which is not quadratic and any complex number v, at least one of the numbers $u + v$ and $u\,v$ is irrational.*

Proof. The proof is identical to the previous one, being enough to substitute π by u and e by v.

□

In particular, this lemma applies when $u = t$ is a transcendental number, so at least one of the numbers $t + v$ and $t\,v$ is irrational. Of course, for any algebraic $v \neq 0$ *both* $t + v$ and $t\,v$ are *transcendental* numbers,[10] so the interesting case is when v is also a transcendental number. This leads us to the following result.

Theorem 2 (Transcendence of sums and products) *Given two transcendental numbers t_1 and t_2, at least one of the numbers $t_1 + t_2$ and $t_1\, t_2$ is transcendental.*

[8] Baker also gave a quantitative lower bound for these linear forms in logs, which had profound consequences for diophantine equations. This work won him a Fields medal in 1970.

[9] All rational numbers are quadratic because they are a root of $x\,(x - p/q) = 0$.

[10] This basic transcendence rule is easily proved by contradiction.

Proof. Given $t_1, t_2 \notin \mathscr{A}$, consider the quadratic equation $(x-t_1)(x-t_2)=0$, whose roots are t_1 and t_2. As it is equivalent to $x^2-(t_1+t_2)x+t_1t_2=0$, assume, towards a contradiction, that both $s=t_1+t_2$ and $p=t_1t_2$ are algebraic numbers. The equation then reads $x^2-sx+p=0$, so, by completing the square, one finds

$$x^2-sx+\frac{s^2}{4}=\frac{s^2}{4}-p$$
$$\implies \left(x-\frac{s}{2}\right)^2=\frac{s^2}{4}-p\,. \tag{3}$$

This implies that $(x-s/2)^2$ is algebraic (see Footnote 6), which is impossible because, being x one of the roots t_1 and t_2, the number $x-s/2$ must be transcendental. □

This theorem implies, in particular, that at least one of $\pi+e$ and πe is *transcendental*. However, we are in a position to prove a stronger result.

Theorem 3 (Transcendence of two numbers) *Given any non-zero complex number $z \neq 1/e$, at least two of the numbers $z+e$, ze, and $\ln z$ are transcendental.*

Proof. If $z \neq 0$ is an algebraic number, then both $z+e$ and ze are transcendental numbers, so let us restrict our attention to $z \notin \mathscr{A}$. If $\ln z$ is transcendental then we are done because we know, from Theorem 2, that at least one of $z+e$ and ze is transcendental. All that remains is to check whether $\ln z \in \mathscr{A}$ implies that both $z+e$ and ze are transcendental numbers. Since $\ln z=\alpha \implies z=e^{\alpha}$, then $z+e=e^{\alpha}+e$ is a transcendental number for all $\alpha \in \mathscr{A}$, $\alpha \neq 1$, according to Corollary 1.[11] Also, since $\ln z \in \mathscr{A}$, then $1+\ln z=\ln(ez)$ is also algebraic, and then, according to Lemma 1, the number ze has to be transcendental for all z such that $\ln(ze) \neq 0$, i.e. $z \neq 1/e$.

□

In particular, this theorem implies that at least two of $\pi+e$, πe, and $\ln \pi$ are transcendental numbers.

Indeed, we can make suitable choices of t_1 and t_2 in Theorem 2 in order to get further transcendence results.

Corollary 8 *For any transcendental number t and algebraic numbers α and β not both zero, the numbers $\alpha t+\beta/t$ and $t(\alpha-t)$ are both transcendental.*

Proof. For any $t \notin \mathscr{A}$ and $\alpha,\beta \in \mathscr{A}$, not both zero, take $t_1=\alpha t$ and $t_2=\beta/t$ in Theorem 2. If exactly one of α,β is null, then the proof is immediate. Otherwise, since $t_1t_2=\alpha\beta \in \mathscr{A}$ (see Footnote 6), then $t_1+t_2=\alpha t+\beta/t$ has to be a transcendental number. Finally, for any $\alpha \in \mathscr{A}$, take $t_1=t$ and $t_2=\alpha-t$ in Theorem 2. Since $t_1+t_2=\alpha \in \mathscr{A}$, then the number $t_1t_2=t(\alpha-t)$ has to be transcendental. □

[11] Note that $\alpha=1$ implies $z=e$, a case in which our theorem also holds since both $z+e=2e$ and $ze=e^2$ are transcendental.

Given any transcendental number t, t^r is transcendental for all rational $r \neq 0$, which can be readily proved by contradiction, writing $r=p/q$, p and q being non-zero integers. What about t^{α}, α being an irrational algebraic? On taking $t=e^{\pi}$, we know that $t^i \in \mathscr{A}$ whereas $t^{\sqrt{2}} \notin \mathscr{A}$ (see Footnote 5). The next theorem sheds some light on this question.

Theorem 4 (Existence of an irrational algebraic exponent) *Given any transcendental number t, there is an irrational algebraic α such that t^{α} is also transcendental.*

Proof. Given any transcendental number t, assume, towards a contradiction, that $t^{\alpha}=\beta$ is algebraic for all $\alpha \in \mathscr{A}\setminus\mathbb{Q}$. From Corollary 8, $t^r(\beta-t^r)$ is transcendental for all $r \in \mathbb{Q}$, $r \neq 0$, which means that $t^{r+\alpha}-t^{2r}=t^{r+\alpha}(1-t^{r-\alpha})$ is transcendental. Clearly, $\alpha_{1,2}:=r\pm\alpha$ is an irrational algebraic, so $t^{\alpha_1}(1-t^{\alpha_2})=\beta_1(1-\beta_2)$ should also be transcendental, which is false because it is the product of two algebraic numbers.

□

Note that the similar proposition "for all transcendental number t, there is an irrational algebraic α such that t^{α} is algebraic" is *false*, as follows from Lemma 1, taking $t=e$. For $t=\pi$, however, it remains open the question if there is some algebraic $\alpha \neq 0$ for which π^{α} is algebraic.[12]

Another consequence of Corollary 8 is as follows.

Theorem 5 (Linear independence of hyperbolic functions) *For any transcendental number t and any rational $r \neq 0$, the numbers 1, $\cosh(r\ln t)$, and $\sinh(r\ln t)$ are linearly independent over $\mathscr{A}$. In particular, both $\cosh(r\ln t)$ and $\sinh(r\ln t)$ are transcendental numbers.*

Proof. Since t^r is transcendental for any $t \notin \mathscr{A}$ and any $r \in \mathbb{Q}$, $r \neq 0$, then it follows from Corollary 8 that, for any $\alpha,\beta \in \mathscr{A}$ not both zero,

$$\alpha t^r+\frac{\beta}{t^r}=\alpha t^r+\beta t^{-r} \notin \mathscr{A}$$
$$\implies \alpha e^{r\ln t}+\beta e^{-r\ln t} \notin \mathscr{A}$$
$$\implies (\alpha+\beta)\cosh(r\ln t)+(\alpha-\beta)\sinh(r\ln t) \notin \mathscr{A}\,. \tag{4}$$

Since α and β are arbitrary algebraic numbers, then

$$\widetilde{\alpha}\cosh(r\ln t)+\widetilde{\beta}\sinh(r\ln t) \notin \mathscr{A}\,, \tag{5}$$

where $\widetilde{\alpha}=\alpha+\beta$ and $\widetilde{\beta}=\alpha-\beta$ are also algebraic numbers (not both zero), so

$$\widetilde{\alpha}\cosh(r\ln t)+\widetilde{\beta}\sinh(r\ln t) \neq \gamma, \quad \forall\,\gamma \in \mathscr{A}\,. \tag{6}$$

[12] Note that this question is relevant for the transcendence of $\ln\pi$, because, given non-null $\alpha,\beta \in \mathscr{A}$, $\beta \neq 1$, $\pi^{\alpha}=\beta \Rightarrow \alpha\ln\pi=\ln\beta$ is transcendental, according to the log-version of Lemma 1.

Therefore, $-\gamma + \widetilde{\alpha}\, \cosh(r\ln t) + \widetilde{\beta}\, \sinh(r\ln t) \neq 0$, which shows that 1, $\cosh(r\ln t)$ and $\sinh(r\ln t)$ are linearly independent over $\mathscr{A}$.

The transcendence of $\cosh(r\ln t)$ follows on taking $\widetilde{\alpha} \neq 0$ and $\widetilde{\beta} = 0$ in Eq. (6), whereas that of $\sinh(r\ln t)$ follows on taking $\widetilde{\alpha} = 0$ and $\widetilde{\beta} \neq 0$.

□

In addition, it is easy to prove the transcendence of $\tanh(r\ln t)$.

Theorem 6 (Transcendence of $\tanh(r\ln t)$**)** *For any transcendental number* t *and any* $r \in \mathbb{Q}$, $r \neq 0$, *the number* $\tanh(r\ln t)$ *is transcendental.*

Proof. For any transcendental number t and any $r \in \mathbb{Q}$, $r \neq 0$, we have $\tanh(r\ln t) := \sinh(r\ln t)/\cosh(r\ln t) = (t^r - t^{-r})/(t^r + t^{-r}) = (t^{2r} - 1)/(t^{2r} + 1) \neq 1$. Now, assume, towards a contradiction, that $\tanh(r\ln t) = \alpha$, for some $\alpha \in \mathscr{A}$, $\alpha \neq 0, 1$. Then

$$\begin{aligned} &\frac{t^{2r}-1}{t^{2r}+1} = \alpha \\ \Longrightarrow\; &t^{2r} - 1 = \alpha\left(t^{2r}+1\right) = \alpha\, t^{2r} + \alpha \\ \Longrightarrow\; &(1-\alpha)\, t^{2r} = \alpha + 1 \\ \Longrightarrow\; &t^{2r} = \frac{1+\alpha}{1-\alpha}, \end{aligned} \tag{7}$$

which is impossible since the quotient of two algebraic numbers is also algebraic, whereas $t^{2r} \notin \mathscr{A}$.

□

It follows, in particular, that $\cosh(\ln\pi)$, $\sinh(\ln\pi)$, and $\tanh(\ln\pi)$ are transcendental numbers. Similar results can be derived for the basic trigonometric functions.

Theorem 7 (Linear independence of trig values) *For any algebraic numbers* α, β, $\alpha \neq 0, 1$ *and* $i\beta \notin \mathbb{Q}$, *the numbers* 1, $\cos(\beta\ln\alpha)$ *and* $\sin(\beta\ln\alpha)$ *are linearly independent over* $\mathscr{A}$. *In particular, both* $\cos(\beta\ln\alpha)$ *and* $\sin(\beta\ln\alpha)$ *are transcendental numbers.*

Proof. Since $i\beta \in \mathscr{A} \setminus \mathbb{Q}$, then, from Lemma 3, $t = \alpha^{i\beta}$ is a transcendental number. From Corollary 8, the sum $a\,t + b/t$ is also transcendental for all algebraic a and b not both zero, so

$$\begin{aligned} &a\,\alpha^{i\beta} + b\,\alpha^{-i\beta} = a\,e^{i\beta\ln\alpha} + b\,e^{-i\beta\ln\alpha} \notin \mathscr{A} \\ \Longrightarrow\; &(a+b)\cos(\beta\ln\alpha) + i\,(a-b)\sin(\beta\ln\alpha) \notin \mathscr{A}, \end{aligned} \tag{8}$$

which is equivalent to $\widetilde{a}\cos(\beta\ln\alpha) + \widetilde{b}\sin(\beta\ln\alpha) \neq c$, for all $c \in \mathscr{A}$. Therefore, for all $\widetilde{a}, \widetilde{b} \in \mathscr{A}$ not both zero, $\widetilde{a}\cos(\beta\ln\alpha) + \widetilde{b}\sin(\beta\ln\alpha) - c \neq 0$, for all $c \in \mathscr{A}$. The transcendence of $\cos(\beta\ln\alpha)$ follows by taking $\widetilde{a} \neq 0$, $\widetilde{b} = 0$ and that of $\sin(\beta\ln\alpha)$ follows by taking $\widetilde{a} = 0$, $\widetilde{b} \neq 0$.

□

Theorem 8 (Transcendence of $\tan(\beta\ln\alpha)$**)** *For any algebraic numbers* α, β, $\alpha \neq 0, 1$ *and* $i\beta \notin \mathbb{Q}$, *the number* $\tan(\beta\ln\alpha)$ *is transcendental.*

Proof. Given non-zero algebraic numbers α, β, $\alpha \neq 1$, assume, towards a contradiction, that $\tan(\beta\ln\alpha) = \gamma$ for some $\gamma \in \mathscr{A}$. Then

$$\begin{aligned} \gamma &= \frac{\sin(\beta\ln\alpha)}{\cos(\beta\ln\alpha)} = \frac{1}{i}\,\frac{e^{i\beta\ln\alpha} - e^{-i\beta\ln\alpha}}{e^{i\beta\ln\alpha} + e^{i\beta\ln\alpha}} \\ \Longrightarrow\; i\gamma &= \frac{\alpha^{i\beta} - \alpha^{-i\beta}}{\alpha^{i\beta} + \alpha^{-i\beta}} = \frac{\alpha^{2i\beta} - 1}{\alpha^{2i\beta} + 1}. \end{aligned} \tag{9}$$

The last equality implies that $i\gamma \neq 1$. From Lemma 3, we know that $t = \alpha^{i\beta}$ is transcendental for all algebraic values of β such that $i\beta \notin \mathbb{Q}$, therefore

$$\begin{aligned} &i\gamma = \widetilde{\gamma} = \frac{t^2-1}{t^2+1} \\ \Longrightarrow\; &\widetilde{\gamma}\, t^2 + \widetilde{\gamma} = t^2 - 1 \\ \Longrightarrow\; &(1-\widetilde{\gamma})\, t^2 = 1 + \widetilde{\gamma} \\ \Longrightarrow\; &t^2 = \frac{1+\widetilde{\gamma}}{1-\widetilde{\gamma}} = \frac{1+i\gamma}{1-i\gamma}, \end{aligned} \tag{10}$$

which should be algebraic since it is a quotient of two algebraic numbers. However, this is impossible because t^2 is transcendental for all $t \notin \mathscr{A}$.

□

Theorems 7 and 8 imply, for instance, that all numbers $\mathrm{trig}(\ln 2)$ and $\mathrm{trig}\left(\sqrt{2}\,\pi\right)$ are transcendental.

3 Conclusion

Summarizing, we reviewed in this note the main transcendence results presently known involving the basic trigonometric and hyperbolic functions, as well as e^z and $\ln z$. Since not even an irrationality proof is known for some numbers like $\ln\pi$, $\pi + e$ and $\pi\, e$,[13] we decided to explore a generalization of a well-known 'harmless' irrationality theorem, our Lemma 6, towards the derivation of conditional transcendence results for those numbers. Hopefully, the results put forward here in this paper should be useful for those researchers who are investigating the irrationality and/or transcendence of such numbers.

[13] There is a recent work on modular functions by Nesterenko (1996) [12], in which he shows that π and $e^{\sqrt{n}\,\pi}$ are algebraically independent over $\mathbb{Q}$ for all integer $n > 0$. This implies that, for any rational $r \neq 0$, $q\ln\pi \neq \sqrt{n}\; p\,\pi - \ln r$ for all non-negative integers p and q (not both zero). For $r = 1$, e.g., one concludes that $\ln\pi$ and $\sqrt{n}\,\pi$ are linearly independent over $\mathbb{Q}$. In particular, $\ln\pi$ is not a rational multiple of π.

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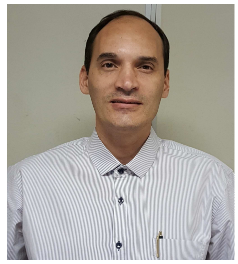
References

[1] A. Baker, *Transcendental number theory*, 2nd ed., Cambridge Univ. Press, Cambridge, 1990.
[2] A. Baker and G. Wüstholz, *Logarithmic Forms and Diophantine Geometry*, Cambridge Univ. Press, New York, 2007.
[3] P. J. Cameron, *Introduction to Algebra*, 2nd ed., Oxford Univ. Press, New York, 2008.
[4] A. O. Gelfond, *Sur le septième problème de Hilbert*, Izv. Akad. Nauk SSSR **7** (1934), 623–630.
[5] Ch. Hermite, *Sur la fonction exponentielle*, C. R. Math. Acad. Sci. Paris (Sér. A) **77** (1873), 18–24.
[6] D. Hilbert, *Mathematical problems*, Göttinger Nachrichten (1900), 253–297. See also: Bull. Amer. Math. Soc. **8** (1902), 437–479.
[7] D. H. Lehmer, *A note on trigonometric algebraic numbers*, Amer. Math. Monthly **40** (1933), 165–166.
[8] F. M. S. Lima and D. Marques, *Some transcendental functions with an empty exceptional set*. Available at arXiv: 1004.1668v2 (2010).
[9] F. Lindemann, *Über die zahl* π, Math. Ann. **20** (1882), 213–225.
[10] J. Liouville, *Sur des classes très-étendue de quantités dont la valeur n'est ni algébrique, ni mêne reductibles à des irrationnelles algébriques*, C. R. Math. Acad. Sci. Paris **18** (1844), 883–885.
[11] B. H. Margolius, *Plouffe's constant is transcendental*. Pre-print (2003). Available at: http://www.plouffe.fr/simon/articles/plouffe.pdf
[12] Yu. V. Nesterenko, *Modular functions and transcendence questions* (Russian), Mat. Sb. **187** (1996), no. 9, 65–96; English transl.: Sb. Math. **187** (1996), 1319–1348.
[13] I. Niven, *Irrational numbers*, 2nd print, The Carus mathematical monographs, MAA, New York, 1963.
[14] M. Ross, *Irrational thoughts*, Math. Gaz. **88** (2000), 68–78.
[15] T. Schneider, *Transzendenzuntersuchungen periodischer Funktionen: I. Transzendenz von Potenzen; II. Transzendenzeigenschaften elliptischer Funktionen*, J. Reine Angew. Math. **172** (1934), 65–74.



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