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Regional Constrained Observability for Distributed Hyperbolic Systems

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Abstract: The aim of this paper is to develop the question of the regional constrained observability for distributed hyperbolic system evolving in spatial domain Ω. It consists in the reconstruction of the initial conditions, in a subregion ω of Ω, knowing that the initial position is between two prescribed functions in ω and also the initial speed is between two others functions also prescribed in ω. We give some definitions and proprieties concerning this concept and then we describe two approaches for solving this problem. The first is based on subdifferential technics and the second uses the Lagrangian multiplier method. This last approach leads to an algorithm for the reconstruction of the initial conditions. The obtained results are illustrated by numerical simulations which lead to some conjectures.

Keywords: Distributed systems, Hyperbolic systems, Regional observability, Regional constrained observability, Strategic sensors, Subdifferential approach, Lagrangian approach.

1 Introduction

In the distributed systems analysis, one of the interesting problems is the knowledge of the initial conditions of a such system, this is called observation problem. Many works have been devoted to this problem in the global case where the aim is to reconstruct the initial conditions in the whole system evolution domain Ω([5], [6], [7]).
The concept of regional observability was introduced by El Jai and al. in the nineties, and studied, for many class of distributed systems, in various works ([1], [11]). It concerns the reconstruction of the initial conditions only in a given subregion $\omega \subset \Omega$. The regional constrained observability problems were considered and studied for parabolic systems, its consist in reconstructing the initial state of such a system and the reconstructed state is between two prescribed functions given only in a subregion $\omega \subset \Omega$([3]).
Here we present an extension of the results on regional constrained observability to hyperbolic ones. Our interest is to reconstruct the initial conditions for an hyperbolic system knowing that these conditions are between certain prescribed functions given only on a subregion ω. There are many reasons for introducing this concept : Firstly, the mathematical model of a real system is obtained either from the measurements, or from approximation techniques and is very often affected by perturbations. Consequently the solution of such a system is only approximately known. Secondly, the observation error is smaller than in general case and the initial conditions to be reconstructed are to be between some bounds.
The remainder of the paper is organized as follows: Section 2 is devoted for introduce the notion of regional constrained observability for hyperbolic systems, in this section we give definitions and proprieties related to this notion. In section 3, we give a characterization of the notion using a subdifferential technics. In the last section, we describe a reconstruction method based on the Lagrangian multiplier approach which leads to a practice algorithm, then we give numerical simulations which show the efficiency of the obtained algorithm.

2 Problem statement.

Let Ω be an open bounded of $\mathbb{R}^n (n = 1,2,3)$, with a regular boundary $\partial\Omega$. For $T > 0$ we denote $Q = \Omega \times]0,T[$, $\Sigma = \partial\Omega \times [0,T]$, and consider the system

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described by the hyperbolic equation:

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} = Ay(x,t) & Q \\ y(x,0) = y^0(x);\ \dfrac{\partial y(x,0)}{\partial t} = y^1(x) & \Omega \\ y(\xi,t) = 0 & \Sigma, \end{cases} \quad (1)$$

where A is a second order differential linear and elliptic operator such that $\bar{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ admits a compact resolvent and generates a strongly continuous semi-group $(\bar{S}(t))_{t\geq 0}$ on a subspace of a state Hilbert space $L^2(\Omega) \times L^2(\Omega)$. We assume that $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, then the system (1) admits a unique solution $y \in C(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega))$ ([9]).
The measurements are given by the output function:

$$z(t) = Cy(.,t), \quad t \in [0,T], \quad (2)$$

with $C : L^2(\Omega) \longrightarrow \mathbb{R}^q$ denotes the observation operator depending on the structure and the number q of sensors considered.

If we denote by $\bar{y} = \begin{bmatrix} y \\ \dfrac{\partial y}{\partial t} \end{bmatrix}$ and $\bar{y}^0 = \begin{bmatrix} y^0 \\ y^1 \end{bmatrix}$, then the system (1) can be written as follows:

$$\begin{cases} \dfrac{\partial \bar{y}(x,t)}{\partial t} = \bar{A}\bar{y}(x,t) & Q \\ \bar{y}(x,0) = \bar{y}^0(x) & \Omega. \end{cases} \quad (3)$$

The system (3) is autonomous, then it admits a unique solution given by:

$$\bar{y}(t) = \bar{S}(t)\bar{y}^0.$$

With the assumption that the operator A admits a basis orthogonal eigenfunctions (ϕ_{nj}) associated with the eigenvalues λ_n of multiplicity r_n, the semigroup $(\bar{S}(t))_{t\geq 0}$ can be written,
for all $(y_1, y_2) \in H_0^1(\Omega) \times L^2(\Omega)$, as:

$$\bar{S}(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty}\sum_{j=1}^{r_n} [\langle y_1, \phi_{nj}\rangle \cos\sqrt{-\lambda_n}t \\ +(-\lambda_n)^{-\frac{1}{2}}\langle y_2, \phi_{nj}\rangle \sin\sqrt{-\lambda_n}t]\phi_{nj}(.) \\ \\ \sum_{n=1}^{\infty}\sum_{j=1}^{r_n} [-\sqrt{-\lambda_n}\langle y_1, \phi_{nj}\rangle \sin\sqrt{-\lambda_n}t \\ +\langle y_2, \phi_{nj}\rangle \cos\sqrt{-\lambda_n}t]\phi_{nj}(.) \end{pmatrix} \quad (4)$$

The system (3) is augmented by the output function:

$$\bar{z}(t) = \bar{C}\bar{y}(.,t), \quad (5)$$

where $\bar{C} = (C, 0)$.
In the sequel we denote $\mathscr{F} = L^2(\Omega) \times L^2(\Omega)$ and $\mathscr{O} = L^2(0,T;\mathbb{R}^q)$.
We consider the observability operator defined by:

$$\begin{aligned} K : \mathscr{F} &\longrightarrow \mathscr{O} \\ (y_1, y_2) &\longmapsto \bar{C}\bar{S}(.)(y_1, y_2), \end{aligned}$$

which is linear bounded with the adjoint given by:

$$\begin{aligned} K^* : \mathscr{O} &\longrightarrow \mathscr{F} \\ z &\longmapsto \int_0^T \bar{S}^*(t)\bar{C}^* z\,dt. \end{aligned}$$

Let ω be a subregion of Ω with positive Lesbegue measure, $\mathscr{F}_\omega = L^2(\omega) \times L^2(\omega)$ and χ_ω be the restriction operator defined by:

$$\begin{aligned} \chi_\omega : \mathscr{F} &\longrightarrow \mathscr{F}_\omega \\ (y_1, y_2) &\longmapsto (y_1, y_2)_{|\omega}, \end{aligned}$$

with the adjoint χ_ω^* given by:

$$\chi_\omega^*(y_1, y_2)(x) = \begin{cases} (y_1, y_2)(x), & x \in \omega \\ 0, & x \in \Omega \setminus \omega. \end{cases}$$

As it is well known, a sensor is conventionally defined by a couple (D, f), where $D \subset \bar{\Omega}$ is the geometric support of the sensor and f is the spatial distribution of the information on the support D.
In the case of a pointwise sensor (internal or boundary) $D = \{b\}$ and $f = \delta_b(.)$, where δ_b is the Dirac mass concentrated in b, and the sensor is then denoted by (b, δ_b). For definitions and properties of strategic sensors we refer to ([7]).

We recall that the system (1)-(2) is said to be exactly (respectively weakly) observable in ω
if $Im\chi_\omega K^* = \mathscr{F}_\omega$ (respectively $\ker K\chi_\omega^* = \{0\}$). For more details, we refer to ([11]).

Here, Let $\alpha_i(.)$ and $\beta_i(.)$ $(i = 1,2)$ be functions in $L^2(\omega)$ such that $\alpha_i(.) \leq \beta_i(.)$ *a.e in* ω, $i = 1,2$.
Throughout the paper we set:

$$\begin{aligned} G :&= [\alpha_1(.), \beta_1(.)] \times [\alpha_2(.), \beta_2(.)] \\ &= \{(y_1(.), y_2(.)) \in \mathscr{F}_\omega \,|\alpha_1(.) \leq y_1(.) \leq \beta_1(.) \\ &\quad \textit{and } \alpha_2(.) \leq y_2(.) \leq \beta_2(.) \textit{ a.e in } \omega\}. \end{aligned}$$

Then the problem of regional constrained observability of the system (1)-(2) concerns the possibility of reconstructing (y^0, y^1) provided that the initial position $y^0 \in [\alpha_1(.), \beta_1(.)]$ and the initial speed $y^1 \in [\alpha_2(.), \beta_2(.)]$ in the subregion ω.

Definition 1.
The system (1)-(2) is said to be G-observable in ω if

$$(Im\chi_\omega K^*) \cap G \neq \emptyset.$$

Definition 2.
A sensor is said to be G-strategic in ω if the observed system is G-observable in ω.

Remark.

1. If the system (1)-(2) is exactly(resp. weakly) observable in ω then it is G-observable in ω. Indeed, if the system (1)-(2) is exactly(resp. weakly)

observable in ω then $Im\chi_\omega K^* = \mathscr{F}_\omega$ (resp. $\overline{Im\chi_\omega K^*} = \mathscr{F}_\omega$) which results

$$(Im\chi_\omega K^*) \cap G \neq \emptyset$$

this means that the system (1)-(2) is G-observable in ω.

2.There exist system which are not weakly observable in Ω but are G-observable in ω. This is illustrated by the following example:

Exapmle. Let's consider the one dimensional wave equation evolving in $\Omega =]0,1[$

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} = \dfrac{\partial^2 y(x,t)}{\partial x^2} &]0,1[\times]0,T[\\ y(x,0) = y^0(x);\ \dfrac{\partial y(x,0)}{\partial t} = y^1(x) &]0,1[\\ y(0,t) = y(1,t) = 0 &]0,T[, \end{cases} \tag{6}$$

augmented with the pointwise measurements

$$z(t) = y(b,t), \tag{7}$$

where $b = \dfrac{1}{2} \in]0,1[$.

Let $y^0(x) = sin(2\pi x)$ and $y^1(x) = sin(\pi x)$ the initial conditions to be observed. Then for $\omega =]\dfrac{2}{6}, \dfrac{5}{6}[$ we have the following result:

Lemma 1. *The system (6)-(7) is not weakly observable in $]0,1[$ but it is G-observable in ω.*

Proof. To show that the system (6)-(7) is not weakly observable it is sufficient to verify that $(y^0,y^1) \in Ker(K)$.

Since the operator $\Delta = \dfrac{\partial^2}{\partial x^2}$ has a complete set of eigenfunctions (ϕ_n) in $L^2(\Omega)$ associated to the eigenvalues λ_n given by:

$$\phi_n(x) = \sqrt{2}\sin(n\pi x) \quad and \quad \lambda_n = -n^2\pi^2,$$

then, from (4) we have

$$\begin{aligned} K\begin{pmatrix} y^0 \\ y^1 \end{pmatrix} &= \bar{C}\bar{S}(t)\begin{pmatrix} y^0 \\ y^1 \end{pmatrix} \\ &= \sum_{n=1}^{+\infty}\left[\begin{array}{l} \langle y^0,\phi_n\rangle_{L^2(\Omega)}\cos(\sqrt{-\lambda_n t}) \\ +\frac{1}{\sqrt{-\lambda_n}}\langle y^1,\phi_n\rangle_{L^2(\Omega)}\sin(\sqrt{-\lambda_n t}) \end{array}\right]\phi_n(b) \\ &= \sqrt{2}\sum_{n\in 2\mathbb{N}+1}^{+\infty}\left[\begin{array}{l} \langle y^0,\phi_n\rangle_{L^2(\Omega)}\cos(\sqrt{-\lambda_n t}) \\ +\frac{1}{\sqrt{-\lambda_n}}\langle y^1,\phi_n\rangle_{L^2(\Omega)}\sin(\sqrt{-\lambda_n t}) \end{array}\right]\sin(\frac{n\pi}{2}). \end{aligned}$$

We have $\langle y^0,\phi_1\rangle_{L^2(\Omega)} = 0$, $\langle y^1,\phi_1\rangle_{L^2(\Omega)} = 0$, and $\forall n \in 2\mathbb{N}^*+1$

$$\langle y^0,\phi_n\rangle_{L^2(\Omega)} = \frac{\sqrt{2}}{2}\left[\begin{array}{l} \frac{1}{(n-2)\pi}\sin((n-2)\pi) \\ -\frac{1}{(n+2)\pi}\sin((n+2)\pi) \end{array}\right] = 0,$$

$$\langle y^1,\phi_n\rangle_{L^2(\Omega)} = -\frac{\sqrt{2}}{2}\left[\begin{array}{l} \frac{1}{(n+1)\pi}(\cos((n+1)\pi)-1) \\ +\frac{1}{(n-1)\pi}(\cos((n-1)\pi)-1) \end{array}\right] = 0.$$

Hence $K\begin{pmatrix} y^0 \\ y^1 \end{pmatrix} = 0$, and then the system (6)-(7) is not weakly observable in Ω.

On the other hand, we show that (y^0,y^1) is G-observable in ω, indeed, suppose that $K\chi_\omega^*\chi_\omega(y^0,y^1) = 0$, then

$$\sum_{n=1}^{+\infty}\left[\begin{array}{l} \langle y^0,\phi_n\rangle_{L^2(\omega)}\cos(\sqrt{-\lambda_n t}) \\ +\frac{1}{\sqrt{-\lambda_n}}\langle y^1,\phi_n\rangle_{L^2(\omega)}\sin(\sqrt{-\lambda_n t}) \end{array}\right]\phi_n(b) = 0.$$

Since for T so large, the set $\{\sin(\sqrt{-\lambda_n t}), \cos(\sqrt{-\lambda_n t})\}_{n\geq 1}$ forms a complete orthonormal set of $L^2(0,T)$, then

$$\langle y^0,\phi_n\rangle_{L^2(\omega)}\phi_n(b) = \langle y^1,\phi_n\rangle_{L^2(\omega)}\phi_n(b) = 0, \quad \forall n \geq 1.$$

But for $n \in 2\mathbb{N}+1$ we have

$$\phi_n(b) = \sqrt{2}\sin(n\frac{\pi}{2}) \neq 0,$$

which gives necessary

$$\langle y^0,\phi_n\rangle_{L^2(\omega)} = \langle y^1,\phi_n\rangle_{L^2(\omega)} = 0, \quad \forall n \in 2\mathbb{N}+1. \tag{8}$$

But for $n = 5$, we have

$$\langle y^1,\phi_5\rangle_{L^2(\omega)} = \sqrt{2}\int_{\frac{2}{6}}^{\frac{5}{6}}\cos(\pi x)\sin(5\pi x)dx = \frac{\sqrt{2}}{6\pi},$$

which contradicts (8), thus $K\chi_\omega^*\chi_\omega(y^0,y^1) \neq 0$ and then (y^0,y^1) is weakly observable in ω.

Moreover, for $\alpha_1(x) = |y^0| - 1$, $\beta_1(x) = |y^0| + 1$ and $\alpha_2(x) = |y^1| - 1$, $\beta_2(x) = |y^1| + 1$, we have $\chi_\omega(y^0,y^1) \in G$ and then the system (6)-(7) is G-observable in ω.

The following result is a characterization of the G-observability in ω.

Proposition 1. *The system (1)-(2) is G-observable in ω if and only if*

$$(Ker\chi_\omega + ImK^*) \cap G \neq \emptyset.$$

Proof. Suppose that the system (1)-(2) is G-observable in ω, that is to say

$$Im\chi_\omega K^* \cap G \neq \emptyset,$$

then there exists $\bar{y} \in G$ and $\theta \in \mathscr{O}$ such that $\chi_\omega\bar{y} = \bar{y} = \chi_\omega K^*\theta$, which implies that $\chi_\omega(\bar{y} - K^*\theta) = 0$. If we set $\bar{y}_1 = \bar{y} - K^*\theta$ and $\bar{y}_2 = K^*\theta$, then $\bar{y} = \bar{y}_1 + \bar{y}_2$ with $\bar{y}_1 \in Ker\chi_\omega$ and $\bar{y}_2 \in ImK^*$, which means that $\bar{y} \in Ker\chi_\omega + ImK^*$ and consequently

$$(Ker\chi_\omega + ImK^*) \cap G \neq \emptyset.$$

Inversely, if $(Ker\chi_\omega + ImK^*) \cap G \neq \emptyset$ then there exists $\bar{y} \in G$ such that $\bar{y} \in Ker\chi_\omega + ImK^*$, so $\bar{y} = \bar{y}_1 + \bar{y}_2$ where $\chi_\omega\bar{y}_1 = 0$ and $\bar{y}_2 \in ImK^*$, therefore $\chi_\omega\bar{y} = \chi_\omega\bar{y}_2$, then $\bar{y} = \chi_\omega\bar{y} \in Im\chi_\omega K^*$, thus

$$(Im\chi_\omega K^*) \cap G \neq \emptyset,$$

which means that the system (1)-(2) is G-obsrvable in ω.

In the following, we present two approaches to solve the problem of regional constrained observability of the system (1)-(2).

3 Subdifferential approach

Solve the problem of G-observability is equivalent to minimizing the reconstruction error given by

$$\begin{cases} min\|K\overline{y}-\overline{z}\|^2_{\mathscr{O}} \\ \overline{y}\in Y, \end{cases} \quad (9)$$

where $Y=\{\overline{y}\in\mathscr{F}\mid \chi_\omega\overline{y}\in[\alpha_1(.),\beta_1(.)]\times[\alpha_2(.),\beta_2(.)]\}$. we will solve the problem (9) by the sub-differential approach([2]).
Let us denote by:
• $\Gamma_0(\mathscr{F})$ the set of functions $f:\mathscr{F}\longrightarrow\tilde{\mathbb{R}}=]-\infty,+\infty]$ proper, lower semi-continuous (l.s.c) and convexe in $\mathscr{F}$.
• For $f\in\Gamma_0(\mathscr{F})$
$dom(f)=\{\overline{y}\in\mathscr{F}\mid f(\overline{y})<+\infty\}$ and f^* the polar function of f given by:

$$f^*(\overline{y}^*)=\sup_{\overline{y}\in dom(f)}\{\langle\overline{y}^*,\overline{y}\rangle-f(\overline{y})\},\quad \forall\overline{y}^*\in\mathscr{F}.$$

• For $\overline{y}^0\in dom(f)$ the set:

$$\partial f(\overline{y}^0)=\{\overline{y}^*\in\mathscr{F}\mid f(\overline{y})\geq f(\overline{y}^0)+\langle\overline{y}^*,\overline{y}-\overline{y}^0\rangle,\ \forall\overline{y}\in\mathscr{F}\},$$

denotes the subdifferential of f at $\overline{y}^0$, then we have the following property

$$\overline{y}^*\in\partial f(\overline{y}^0)\quad \text{if and only if}\quad f(\overline{y}^0)+f^*(\overline{y}^*)=\langle\overline{y}^0,\overline{y}^*\rangle.$$

With these notations the problem (9) is equivalent to the problem without constraints:

$$\begin{cases} \inf(\|K\overline{y}-\overline{z}\|^2_{\mathscr{O}}+\Psi_Y(\overline{y})) \\ \overline{y}\in\mathscr{F}, \end{cases} \quad (10)$$

where Ψ_Y denotes the indicator function of Y, given by:

$$\Psi_Y(\overline{y})=\begin{cases} 0 & if\ \overline{y}\in Y \\ +\infty & otherwise \end{cases}$$

The solution of the problem (10) is characterized by the following result:

Proposition 2.*If the system (1)-(2) is G-observable in ω, then the following assertions are equivalent:*

1.$\overline{y}^*$ is a solution of (10).
2.$\overline{y}^*\in Y$ and
$\Psi_Y^*(-2K^*(K\overline{y}^*-\overline{z}))=-2\|Ky^*\|^2_{\mathscr{O}}+2\langle K^*\overline{z},\overline{y}^*\rangle.$

Proof. Let $f(\overline{y})=\|K\overline{y}-\overline{z}\|^2_{\mathscr{O}}$. $\overline{y}^*$ is a solution of (10) if and only if $0\in\partial(f+\Psi_Y)(\overline{y}^*)$.
It is clear that $f\in\Gamma_0(\mathscr{F})$ and since Y is closed, convex and non empty, then $\Psi_Y\in\Gamma_0(\mathscr{F})$. Moreover under the hypothesis of the G-observability in ω, we have $Dom(f)\cap Dom(\Psi_Y)\neq\emptyset$, but f is continuous then

$$\partial(f+\Psi_Y)(\overline{y}^*)=\partial f(\overline{y}^*)+\partial\Psi_Y(\overline{y}^*),$$

it follows that $\overline{y}^*$ is a solution of (10) if and only if $0\in(\partial f(\overline{y}^*)+\partial\Psi_Y(\overline{y}^*))$.
Moreover f is Frechet-differentiable, then

$$\partial f(\overline{y}^*)=\{\nabla f(\overline{y}^*)\}=\{2K^*(K\overline{y}^*-\overline{z})\},$$

thus $\overline{y}^*$ is a solution of (10) if and only if

$$-2K^*(K\overline{y}^*-\overline{z})\in\partial\Psi_Y(\overline{y}^*)$$

which equivalent to

$$\overline{y}^*\in Y,\Psi_Y(\overline{y}^*)+\Psi_Y^*(-2K^*(K\overline{y}^*-\overline{z}))=\langle\overline{y}^*,-2K^*K\overline{y}^*+2K^*\overline{z}\rangle,$$

which implies that

$$\overline{y}^*\in Y,\Psi_Y^*(-2K^*(K\overline{y}^*-\overline{z}))=-2\|K\overline{y}^*\|^2_{\mathscr{O}}+2\langle K^*\overline{z},\overline{y}^*\rangle.$$

Remark.
This approach can not be exploited numerically.

We will give a second approach giving an algorithm that is usable numerically.

4 Lagrangian multiplier approach

If we suppose that the system (1)-(2) is exactly observable in Ω, then any state $\overline{y}\in\mathscr{F}$ can be written in the form $K^*\theta$ with $\theta\in\mathscr{O}$. As a result, the problem (9) can be rewritten as follow:

$$\begin{cases} \min\|KK^*\theta-\overline{z}\|^2_{\mathscr{O}} \\ \theta\in V=\{\hat{\theta}\in\mathscr{O}\mid\chi_\omega K^*\hat{\theta}\in G\}, \end{cases} \quad (11)$$

Then we have the following result:

Proposition 3.*If the system (1)-(2) is exactly observable in Ω, then the solution of (11) is given by*

$$\theta^*=(KK^*KK^*)^{-1}KK^*\overline{z}-\frac{1}{2}(KK^*KK^*)^{-1}K\chi^*_\omega(\lambda^*_1,\lambda^*_2),$$

and the solution of the problem (9) is given by:

$$\overline{y}^*=R_\omega K^*\overline{z}-\frac{1}{2}R_\omega\chi^*_\omega(\lambda^*_1,\lambda^*_2),$$

where $(\lambda^*_1,\lambda^*_2)$ is the solution of

$$\begin{cases} \frac{1}{2}R_\omega\chi^*_\omega(\lambda^*_1,\lambda^*_2)=-\overline{y}^*+R_\omega K^*\overline{z} \\ \overline{y}^*=P_G(\rho(\lambda^*_1,\lambda^*_2)+\overline{y}^*), \end{cases} \quad (12)$$

while $P_G:\mathscr{F}_\omega\longrightarrow G$ denotes the projection operator, $\rho>0$ and $R_\omega=\chi_\omega K^*(KK^*KK^*)^{-1}K$.

Proof. If the system (1)-(2) is exactly observable in Ω then it is G-observable in ω, thus $V\neq\emptyset$ and the problem (11) has a solution. The constraint problem (11) is equivalent to saddle point problem

$$\begin{cases} \min\|KK^*\theta-\overline{z}\|^2_{\mathscr{O}} \\ (\theta,\overline{y})\in W, \end{cases} \quad (13)$$

where

$$W=\{(\theta,\bar{y})\in\mathscr{O}\times G \mid \chi_\omega K^*\theta-\bar{y}=0\}.$$

To the problem (13) we associate the Lagrangian functional L defined by:

$$\forall(\theta,\bar{y},\lambda_1,\lambda_2)\in\mathscr{O}\times G\times\mathscr{F}_\omega,$$

$$L(\theta,\bar{y},\lambda_1,\lambda_2)=\|KK^*\theta-\bar{z}\|^2_{\mathscr{O}}+\langle(\lambda_1,\lambda_2),\chi_\omega K^*\theta-\bar{y}\rangle.$$

Let us recall that $(\theta^*,\bar{y}^*,\lambda_1^*,\lambda_2^*)$ is a saddle point of L if:

$$\max_{(\lambda_1,\lambda_2)\in\mathscr{F}_\omega} L(\theta^*,\bar{y}^*,(\lambda_1,\lambda_2))=L(\theta^*,\bar{y}^*,(\lambda_1^*,\lambda_2^*)) = \min_{\substack{\theta\in\mathscr{O}\\ \bar{y}\in G}} L(\theta,\bar{y},(\lambda_1^*,\lambda_2^*)).$$

The proof is divided into three steps:

• Step 1

The set $\mathscr{O}\times G$ is non empty, closed and convex, moreover the function $(\lambda_1,\lambda_2)\longmapsto L(\theta,\bar{y},(\lambda_1,\lambda_2))$ is concave, upper semi-continuous and differentiable. The same $(\theta,\bar{y})\longmapsto L(\theta,\bar{y},(\lambda_1,\lambda_2))$ is convex, lower semi-continuous and differentiable. Moreover, there exists $(\lambda_1^0,\lambda_2^0)\in\mathscr{F}_\omega$ such that $\lim_{\|(\theta,\bar{y})\|\to+\infty} L(\theta,\bar{y},(\lambda_1^0,\lambda_2^0))=+\infty$, and there exists $(\theta^0,\bar{y}^0)\in\mathscr{O}\times G$ such that $\lim_{\|(\lambda_1,\lambda_2)\|\to+\infty} L(\theta^0,\bar{y}^0,(\lambda_1,\lambda_2))=-\infty$.

This shows that L admits a saddle point.

• Step 2

Let $(\theta^*,\bar{y}^*,(\lambda_1^*,\lambda_2^*))$ be a saddle point of L. We will prove that $\bar{y}^*=\chi_\omega K^*\theta^*$ is the restriction in ω of the solution of (9). We have

$$L(\theta^*,\bar{y}^*,(\lambda_1,\lambda_2))\le L(\theta^*,\bar{y}^*,(\lambda_1^*,\lambda_2^*))\le L(\theta,\bar{y},(\lambda_1^*,\lambda_2^*)) \quad \forall(\theta,\bar{y},(\lambda_1,\lambda_2))\in\mathscr{O}\times G\times\mathscr{F}_\omega. \tag{14}$$

From the first inequality of (14) we have
$\|KK^*\theta^*-\bar{z}\|^2_{\mathscr{O}}+\langle(\lambda_1,\lambda_2),\chi_\omega K^*\theta^*-\bar{y}^*\rangle \le \|KK^*\theta^*-\bar{z}\|^2_{\mathscr{O}}+\langle(\lambda_1^*,\lambda_2^*),\chi_\omega K^*\theta^*-\bar{y}^*\rangle$, $\forall(\lambda_1,\lambda_2)\in\mathscr{F}_\omega$.

So
$\langle(\lambda_1,\lambda_2),\chi_\omega K^*\theta^*-\bar{y}^*\rangle\le\langle(\lambda_1^*,\lambda_2^*),\chi_\omega K^*\theta^*-\bar{y}^*\rangle$, $\forall(\lambda_1,\lambda_2)\in\mathscr{F}_\omega$,
which implies that $\chi_\omega K^*\theta^*=\bar{y}^*$, hence $\chi_\omega K^*\theta^*\in G$.

From the second inequality of (14) it follows that

$$L(\theta^*,\bar{y}^*,(\lambda_1^*,\lambda_2^*))\le L(\theta,\bar{y},(\lambda_1^*,\lambda_2^*)), \quad \forall(\theta,\bar{y})\in\mathscr{O}\times G,$$

this means that
$\|KK^*\theta^*-\bar{z}\|^2_{\mathscr{O}}+\langle(\lambda_1^*,\lambda_2^*),\chi_\omega K^*\theta^*-\bar{y}^*\rangle \le \|KK^*\theta-\bar{z}\|^2_{\mathscr{O}}+\langle(\lambda_1^*,\lambda_2^*),\chi_\omega K^*\theta-\bar{y}\rangle$, $\forall(\theta,\bar{y})\in\mathscr{O}\times G$.

Since $\bar{y}^*=\chi_\omega K^*\theta^*$, we have
$\|KK^*\theta^*-\bar{z}\|^2_{\mathscr{O}}\le\|KK^*\theta-\bar{z}\|^2_{\mathscr{O}}+\langle(\lambda^*,\lambda_2^*),\chi_\omega K^*\theta-\bar{y}\rangle$, $\forall(\theta,\bar{y})\in\mathscr{O}\times G$.

Taking $\bar{y}=\chi_\omega K^*\theta$, we obtain:

$$\|KK^*\theta^*-\bar{z}\|^2_{\mathscr{O}}\le\|KK^*\theta-\bar{z}\|^2_{\mathscr{O}}, \quad \forall\theta\in\mathscr{O},$$

which implies that θ^* is a solution of (11), and so $\bar{y}_0^*=K^*\theta^*$ whose the restriction $\bar{y}^*=\chi_\omega K^*\theta^*$ is solution of (9).

• Step 3

Let $(\theta^*,\bar{y}^*,(\lambda_1^*,\lambda_2^*))$ is a saddle point of L, then the following assumptions are hold

$$2\langle KK^*\theta^*-\bar{z},KK^*(\theta-\theta^*)\rangle+\langle(\lambda_1^*,\lambda_2^*),\chi_\omega K^*(\theta-\theta^*)\rangle=0 \quad \forall\theta\in\mathscr{O} \tag{15}$$

$$-\langle(\lambda_1^*,\lambda_2^*),(\bar{y}-\bar{y}^*)\rangle\ge 0, \quad \forall\bar{y}\in G \tag{16}$$

$$\langle(\lambda_1,\lambda_2)-(\lambda_1^*,\lambda_2^*),\chi_\omega K^*\theta^*-\bar{y}^*\rangle=0, \quad \forall(\lambda_1,\lambda_2)\in\mathscr{F}_\omega \tag{17}$$

For details on the saddle point theory and its applications we refer to ([4], [8], [10]).

From (15) we deduce

$$2\langle(KK^*)^*(KK^*\theta^*-\bar{z}),(\theta-\theta^*)\rangle+\langle(\chi_\omega K^*)^*(\lambda_1^*,\lambda_2^*),(\theta-\theta^*)\rangle=0, \quad \forall\theta\in\mathscr{O},$$

then

$$-2(KK^*)^*KK^*\theta^*+2(KK^*)^*\bar{z}=(\chi_\omega K^*)^*(\lambda_1^*,\lambda_2^*),$$

since the system is observable in Ω, then KK^*KK^* is invertible, and consequently

$$\theta^*=(KK^*KK^*)^{-1}KK^*\bar{z}-\frac{1}{2}(KK^*KK^*)^{-1}K\chi_\omega^*(\lambda_1^*,\lambda_2^*),$$

so $\bar{y}^*$ is given by

$$\bar{y}^*=\chi_\omega K^*(KK^*KK^*)^{-1}KK^*\bar{z}-\frac{1}{2}\chi_\omega K^*(KK^*KK^*)^{-1}K\chi_\omega^*(\lambda_1^*,\lambda_2^*),$$

then

$$\bar{y}^*=R_\omega K^*\bar{z}-\frac{1}{2}R_\omega\chi_\omega^*(\lambda_1^*,\lambda_2^*),$$

with $R_\omega=\chi_\omega K^*(KK^*KK^*)^{-1}K$.

Using (16), we have

$$-\langle(\lambda_1^*,\lambda_2^*),(\bar{y}-\bar{y}^*)\rangle\ge 0, \quad \forall\bar{y}\in G$$

so $\langle\rho((\lambda_1^*,\lambda_2^*)+\bar{y}^*)-\bar{y}^*,\bar{y}-\bar{y}^*\rangle\le 0$, $\forall\bar{y}\in G$ and $\forall\rho>0$ then

$$\bar{y}^*=P_G(\rho(\lambda_1^*,\lambda_2^*)+\bar{y}^*).$$

Corollary 1. *If the system (1)-(2) is exactly observable in Ω and the function*

$$L_\omega=[(K\chi_\omega^*)^*K\chi_\omega^*]^{-1}(K\chi_\omega^*)^*KK^*KK^*[(\chi_\omega K^*)^*\chi_\omega K^*]^{-1}(\chi_\omega K^*)^*,$$

is coercive, then for ρ suitably chosen, the system (12) has a unique solution $((\lambda_1^*,\lambda_2^*),\bar{y}^*)$.

Proof. We have

$$\bar{y}^*=\chi_\omega K^*(KK^*KK^*)^{-1}KK^*\bar{z}-\frac{1}{2}\chi_\omega K^*(KK^*KK^*)^{-1}K\chi_\omega^*(\lambda_1^*,\lambda_2^*),$$

then

$$(\lambda_1^*,\lambda_2^*)=-2L_\omega\bar{y}^*+2[(K\chi_\omega^*)^*K\chi_\omega^*]^{-1}(K\chi_\omega^*)^*KK^*\bar{z}.$$

So if $(\theta^*, \overline{y}^*, (\lambda_1^*, \lambda_2^*))$ is a saddle point of L then the system (12) is equivalent to

$$\begin{cases}(\lambda_1^*, \lambda_2^*) = -2L_\omega \overline{y}^* + 2[(K\chi_\omega^*)^* K\chi_\omega^*]^{-1}(K\chi_\omega^*)^* KK^* \overline{z} \\ \overline{y}^* = P_G(-2\rho L_\omega \overline{y}^* + 2\rho[(K\chi_\omega^*)^* K\chi_\omega^*]^{-1}(K\chi_\omega^*)^* KK^* \overline{z} + \overline{y}^*).\end{cases}$$

It follows that $\overline{y}^*$ is a fixed point of the function defined by:

$$\begin{aligned} F_\rho : G &\longrightarrow G \\ \overline{y} &\longmapsto P_G(-2\rho L_\omega \overline{y} + 2\rho[(K\chi_\omega^*)^* K\chi_\omega^*]^{-1}(K\chi_\omega^*)^* KK^* \overline{z} + \overline{y}).\end{aligned}$$

Since the operator L_ω is coercive, then $\exists m > 0$ such that

$$\langle L_\omega \overline{y}, \overline{y} \rangle \geq m \|\overline{y}\|^2, \quad \forall \overline{y} \in \mathscr{F}_\omega$$

It follows that $\forall \overline{y}_1, \overline{y}_2 \in G$

$$\begin{aligned} \|F_\rho(\overline{y}_2) - F_\rho(\overline{y}_1)\| &\leq \| -2\rho L_\omega(\overline{y}_2 - \overline{y}_1) + (\overline{y}_2 - \overline{y}_1)\| \\ &\leq 4\rho^2 \|L_\omega\|^2 \|\overline{y}_2 - \overline{y}_1\|^2 + \|\overline{y}_2 - \overline{y}_1\|^2 \\ &\quad -4\rho \langle L_\omega(\overline{y}_2 - \overline{y}_1), (\overline{y}_2 - \overline{y}_1)\rangle \\ &\leq 4\rho^2 \|L_\omega\|^2 \|\overline{y}_2 - \overline{y}_1\|^2 + \|\overline{y}_2 - \overline{y}_1\|^2 \\ &\quad -4\rho m \|\overline{y}_2 - \overline{y}_1\|^2 \\ &\leq (4\rho^2 \|L_\omega\|^2 + 1 - 4m)\|\overline{y}_2 - \overline{y}_1\|^2.\end{aligned}$$

if we choose

$$0 < \rho < \frac{m}{\|L_\omega\|^2}$$

then F_ρ is contractant, which implies the uniqueness of $\overline{y}^*$ and $(\lambda_1^*, \lambda_2^*)$.

4.1 Numerical approach

From proposition (3) it follows that the solution of the problem (9) arises to compute the saddle points of L, which is equivalent to solving the problem

$$\inf_{(\theta,\overline{y}) \in \mathscr{O} \times G} \left(\sup_{(\lambda_1,\lambda_2) \in \mathscr{F}_\omega} L(\theta, \overline{y}, (\lambda_1, \lambda_2)) \right)$$

To accomplish this we use the following algorithm of Uzawa type ([8]):

Step 1: Choose

- ⊖ the precision threshold ε small enough.
- ⊖ the subregion ω, the sensor (D, f).
- ⊖ the functions $\overline{y}_0 \in G$ and $(\lambda_1^1, \lambda_2^1) \in \mathscr{F}_\omega$.

Step 2: Repeat

- ⊖ Solve $KK^*KK^*(\theta_n) = KK^*\overline{z} - \frac{1}{2}K\chi_\omega^*(\lambda_1^n, \lambda_2^n)$, $n \geq 1$.
- ⊖ Calculate $\overline{y}_n = P_G(\rho(\lambda_1^n, \lambda_2^n) + \overline{y}_{n-1})$, $n \geq 1$.
- ⊖ Calculate $(\lambda_1^{n+1}, \lambda_2^{n+1}) = (\lambda_1^n, \lambda_2^n) + (\chi_\omega K^* \theta_n - \overline{y}_n)$, $n \geq 1$.

Until $\|\overline{y}_{n+1} - \overline{y}_n\|_{\mathscr{F}_\omega} \leq \varepsilon$.

Step 3 : Let $(\theta^*, \overline{y}^*, \lambda_1^*, \lambda_2^*)$ be a saddle point of L, then the sequence θ_n converges to θ^* solution of the problem (13) and $\overline{y}_n$ lead to the initial condition $\overline{y}^*$ to be reconstructed in ω ([8]).

4.2 Simulation results

Here we give a numerical example that leads to some results related to the choice of the subregion, the initial conditions and the sensor location. In $\Omega =]0,1[$, let's consider the one-dimensional system:

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} = \dfrac{\partial^2 y(x,t)}{\partial x^2} &]0,1[\times]0,T[\\ y(x,0) = y^0(x); \ \dfrac{\partial y(x,0)}{\partial t} = y^1(x) &]0,1[\\ y(0,t) = y(1,t) = 0 &]0,T[, \end{cases} \tag{18}$$

augmented with the pointwise measurements given by:

$$z(t) = y(b,t), \ b \in \Omega. \tag{19}$$

The initial conditions to be reconstructed are

$$y^0(x) = (x^2(x-1)^2 - 2x(x-1))/2$$

$$y^1(x) = (\frac{1}{2}x^2(x-1) - \frac{2}{3}x(x-1))/3$$

We take $T = 2$ and $G = [\alpha_1(.), \beta_1(.)] \times [\alpha_2(.), \beta_2(.)]$ with

$$\alpha_1(x) = x^2(x-1)^2 \ , \quad \beta_1(x) = -2x(x-1)$$

$$\alpha_2(x) = \frac{1}{2}x^2(x-1)^2 \ , \quad \beta_2(x) = -\frac{2}{3}x(x-1)$$

Applying the previous algorithm, we obtain the following results:

• Global case: $\omega = \Omega$

− If the sensor is located in $b = 0.4$, we have

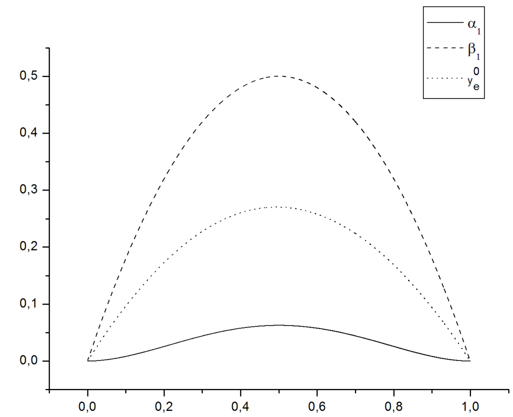

Fig. 1: The estimated initial position y_e^0.

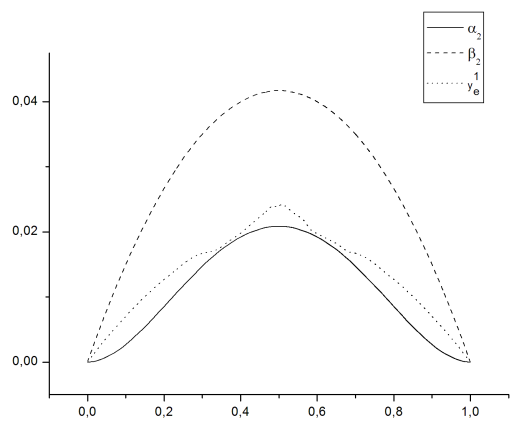

Fig. 2: The estimated initial speed y_e^1.

From figure 1 (resp. figure 2), we note that the initial estimated position y_e^0 (resp. initial estimated speed y_e^1) is between $\alpha_1(.)$ and $\beta_1(.)$ (resp. $\alpha_2(.)$ and $\beta_2(.)$), which shows that the sensor (b,δ_b) is G-strategic in ω. The estimated position and speed are obtained with reconstruction error

$$\|(y^0,y^1)-(y_e^0,y_e^1)\|^2 = 8.76\times 10^{-3}.$$

− If the sensor is located in $b=0.3$, we have

Figure 3 (resp. figure 4) shows that the initial estimated position y_e^0 (resp. initial estimated speed y_e^1) is not between $\alpha_1(.)$ and $\beta_1(.)$ (resp. $\alpha_2(.)$ and $\beta_2(.)$) and then the sensor (b,δ_b) is not G-strategic in ω.

• Regional case: $\omega =]0.4, 0.6[$

− If the sensor is located in $b=0.5$, we have

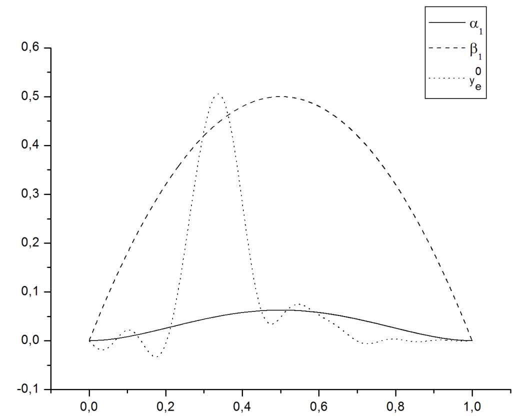

Fig. 3: The estimated initial position y_e^0.

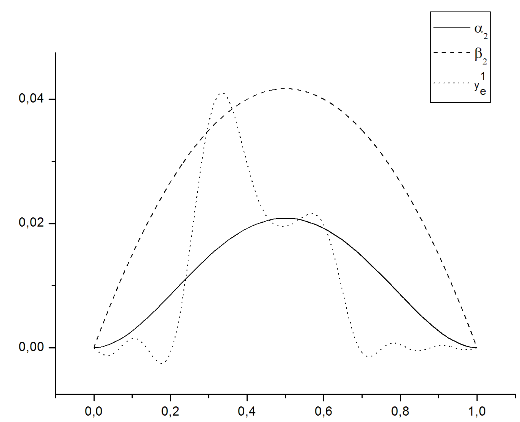

Fig. 4: The estimated initial speed y_e^1.

Figure 5 (resp. figure 6) shows that the initial estimated position y_e^0 (resp. initial estimated speed y_e^1) is between $\alpha_1(.)$ and $\beta_1(.)$ (resp. $\alpha_2(.)$ and $\beta_2(.)$) and then the sensor (b,δ_b) is G-strategic in ω. The estimated position and speed are obtained with reconstruction error

$$\|(y^0,y^1)-(y_e^0,y_e^1)\|^2 = 1.43\times 10^{-3}.$$

We note that there exists a best location of the sensor allowing a good reconstruction of the initial conditions. This is illustrated by the following figure:

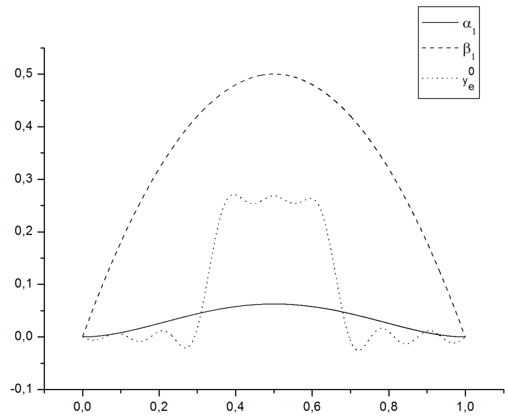

Fig. 5: The estimated initial position y_e^0.

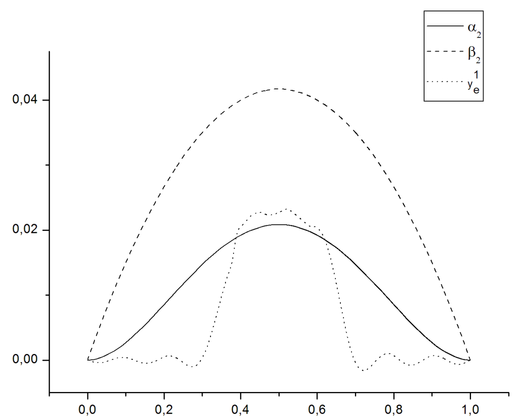

Fig. 6: The estimated initial speed y_e^1.

5 Conclusion

In this work, we have studied the problem of regional constrained observability for hyperbolic distributed systems. We explored two approaches to solve this problem, the first was based on the subdifferential technics and the second on the Lagrangian multipliers. This one leads to an algorithm which is implemented numerically. Many questions remain open, this is the case where the subregion ω is a part of the boundary $\partial\Omega$ of the evolution domain Ω. This questions is under consideration and it's will be the subject of the future works.

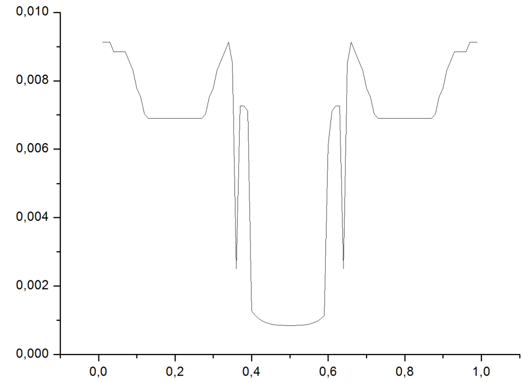

Fig. 7: The reconstruction error with respect to the sensor location b.

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