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Some Common Fixed Point Results in C^*-Algebra-Valued Metric Spaces

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Abstract: This work is intended as an attempt to improve and simplify some recent fixed point theorems for two pairs of mappings using generalized contractive conditions in the framework of C^*-algebra-valued metric spaces. Also, we provide an application and examples to illustrate our results.

Keywords: C^*-algebra, C^*-algebra-valued metric space, Generalized contractive condition ,Common fixed point.

1 Introduction

The theory of fixed point is a very active area of research despite having a history of more than hundred years. The strength of fixed point theory lies in its application, which is spread throughout the existing literature fixed point theory. Banach [7] in 1922 , introduced the first important and significant result in the field of metric fixed point theory. Later, Banach contraction principle has been extended and generalized in numerous different ways (see [1, 3-6, $8-9, 12-13$]). In the similar way, various common fixed point results were proved in different types of spaces i.e cone metric spaces [10], fuzzy metric spaces [2], uniform spaces [15], noncommutative Banach spaces [16] and so on.The concept of C^*-algebra valued metric spaces was introduced by Ma et al. [11] in 2014 which was more general than metric space, replacing the set of real numbers by C^*-algebras and some fixed point results for mappings under contractive or expensive conditions were also proved.There are many examples of C^*-algebra, such as the set of complex numbers, the set of all bounded linear operators on a Hilbert space $H, L(H)$, and the set of $n \times n$ matrices, $M_n(\mathrm{C})$. If a normed algebra $\mathring{A}$ admits a unit $I, aI = Ia = a$ for all $a \in \mathring{A}$, and $\|I\| = 1$, then we say that $\mathring{A}$ is a unital normed algebra. A complete unital normed algebra $\mathring{A}$ is called unital Banach algebra. We say that $a \in \mathring{A}$ is invertible if there is an element $b \in \mathring{A}$ such that $ab = ba = I$. In this case, b is unique and written as a^{-1}. The set $\mathrm{Inv}(\mathring{A}) = \{a \in \mathring{A} \mid a \text{ is invertible }\}$ is a group under multiplication. We define spectrum of an element a to be the set $(a) = \{\lambda \in \mathrm{C} \mid \lambda I - a \notin \mathrm{Inv}(\mathring{A})\}$.
Let $\mathring{A}$ be a unital C^*-algebra with a unit I, then
(i) $I^* = I$,
(ii) For any $a \in \mathrm{Inv}(\mathring{A}), (a^*)^{-1} = (a^{-1})^*$.
(iii) For any $a \in \mathring{A}\, (a^*) = (a)^* = \{\lambda \in \mathrm{C} : \lambda \in (a)\}$.
(iv) The sum of two positive elements in a C^*-algebra is a positive element.
(v) If a is an arbitrary element of a C^*-algebra $\mathring{A}\mathscr{A}$, then a^*a is positive.
(vi) Let $\mathring{A}$ be a C^*-algebra. If $a, b \in \mathring{A}^+$and $a \le b$, then for any $x \in \mathring{A}$ both x^*ax and x^*bx are positive elements and $x^*ax \le x^*bx$.
All over this paper, $\mathring{\mathscr{A}}$ means a unital C^*-algebra with a unit $I, \mathbb{R}$ is set of real numbers and $\mathbb{R}^+$ is the set of non-negative real numbers, $M_n(\mathbb{R})$ is $n \times n$ matrix with entries in $\mathbb{R}, \Delta$ is C^*-algebra valued Metric Space. In the next definition, we will define C^*-algebra valued Metric Space.

Definition 1.[11] *Let $\mho$ be a nonempty set. Suppose that the mapping $\Delta : \mho \times \mho \to \mathring{\mathscr{A}}$ is defined, with the following properties:*
(i) $0_{\mathring{\mathscr{A}}} \le \Delta(\ell, \ell')$ for all ℓ and ℓ' in $\mho$,
(ii) $\Delta(\ell, \ell') = 0_{\mathring{\mathscr{A}}}$ if and only if $\ell = \ell'$,
(iii) $\Delta(\ell, \ell') = \Delta(\ell', \ell)$ for all ℓ and ℓ' in $\mho$,
(iv) $\Delta(\ell, \ell') \le \Delta(\ell, z) + \Delta(z, \ell')$ for all ℓ, ℓ' and z in $\mho$.
Then Δ is said to be a C^*-algebra-valued metric on $\mho$, and $(\mho, \mathring{\mathscr{A}}, \Delta)$ is said to be a C^*-algebra-valued metric

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space. We know that range of mapping Δ in metric space is the set of real numbers which is C^*-algebra; then $*$-algebra-valued metric space generalizes the concept of metric spaces, replacing the set of real numbers by $\mathring{\mathscr{A}}_+$

Definition 2.[11] *Let $(\mho, \mathring{\mathscr{A}}, \Delta)$ is a C^*-algebra-valued metric space and let $\{\ell_n\}$ be a sequence in $\mho$. If*
(i) for any $\varepsilon > 0$, there is N such that $\|\Delta(\ell_n, \ell)\| \leq \varepsilon$ for all $n > N$, then the sequence $\{\ell_n\}$ is said to be convergent, and we denote it as $\lim_{n\to\infty} \ell_n = \ell$
(ii) for any $\varepsilon > 0$, there is N such that $\|\Delta(\ell_n, \ell_m)\| \leq \varepsilon$ for all $m, n > N$, then the sequence $\{\ell_n\}$ is said to be Cauchy sequence.
(iii) C^*-algebra-valued metric space is said to be complete if every Cauchy sequence in $\mho$ with respect to $\mathring{\mathscr{A}}$ is convergent.

Lemma 1.[14] *Let $\mathring{\mathscr{A}}$ be a C^*-algebra then*
(i) If $\{b_n\} \subseteq \mathring{\mathscr{A}}$ and $\lim_{n\to\infty} b_n = 0_{\mathring{\mathscr{A}}}$, then for any $a \in \mathring{\mathscr{A}}$, $\lim_{n\to\infty} a^* b_n a = 0_{\mathring{\mathscr{A}}}$. Set $\mathring{\mathscr{A}}_h = \{\ell \in \mathring{\mathscr{A}} : a = a^*\}$.
(ii) If $a, b \in \mathring{\mathscr{A}}_h$ and $c \in \mathring{\mathscr{A}}'_+$, then $a \leq b$ deduces $ca \leq cb$, where $\mathring{\mathscr{A}}'_+ = \mathring{\mathscr{A}}_+ \cap \mathring{\mathscr{A}}'$.
(iii) Let $\{\ell_n\}$ be a sequence in X. If $\{\ell_n\}$ converges to ℓ and ℓ', respectively, then $\ell = \ell'$.
That is, the limit of a convergent sequence in a C^*-algebra-valued metric space is unique.

Lemma 2.[14] *Let $\mathring{\mathscr{A}}$ be a C^*-algebra then*
(i) the set $\mathring{\mathscr{A}}^+$ is closed cone in $\mathring{\mathscr{A}}$ [a cone C in a real or complex vector space is a subset closed under addition and under scalar multiplication by $\mathbb{R}^+$],
(ii) the set $\mathring{\mathscr{A}}^+$ is equal to $\{a^*a : a \in \mathring{\mathscr{A}}\}$,
(iii) if $0_{\mathring{\mathscr{A}}} \preceq a \leq b$, then $\|a\| \leq \|b\|$,
(iv) if $\mathring{\mathscr{A}}$ is unital and a and b are positive invertible elements, Then $a \leq b$ implies $0_{\mathring{\mathscr{A}}} \leq b^{-1} \leq a^{-1}$.

Definition 3.[11] *Suppose that $(\mho, \mathring{\mathscr{A}}, \Delta)$ is a C^*-algebra-valued metric space. A mapping $\mathscr{Q} : \mho \to X$ is called C^*-algebra-valued contractive mapping on $\mho$, if there is an $\lambda \in \mathring{\mathscr{A}}$ with $\|\lambda\| < 1$ such that*

$$\Delta(\mathscr{Q}\ell, \mathscr{Q}\ell') \leq \lambda^* \Delta(\ell, \ell')\lambda \text{ for all } \ell, \ell' \in \mho.$$

2 Main results

Theorem 1.*Let $(\mho, \mathring{\mathscr{A}}, \Delta)$ be a complete C^*-algebra-valued metric space. Let $\mathscr{Q}_1, \mathscr{Q}_2 : \mho \to \mho$ be two self mappings. Suppose that there exists $\lambda \in \mathring{\mathscr{A}}$ with $\|\lambda\| \in [0, 1)$ such that*

$$\max\{\Delta(\mathscr{Q}_1(\ell), \mathscr{Q}_2\mathscr{Q}_1(\ell)), \Delta(\mathscr{Q}_2(\ell), \mathscr{Q}_1\mathscr{Q}_2(\ell))\} \preceq \lambda^*(\min\{\Delta(\ell, \mathscr{Q}_1(\ell)), \Delta(\ell, \mathscr{Q}_2(\ell))\})\lambda \qquad (2.1)$$

for every $\ell \in \mho$ and

$$\gamma(\ell') = inf\{\Delta(\ell, \ell') + \min\{\Delta(\ell, \mathscr{Q}_1(\ell))\} : \ell \in \mho\} \succ 0_{\mathring{\mathscr{A}}} \qquad (2.2)$$

for every $\ell' \in \mho$ with ℓ' is not a common fixed point of $\mathscr{Q}_1$ and $\mathscr{Q}_2$.
Then $\mathscr{Q}_1$ and $\mathscr{Q}_2$ has a unique fixed point.

*Proof.*Let $\ell_0 \in \mho$ be arbitrary and define a sequence $\{\ell_n\}$ by

$$\ell_n = \begin{cases} Q_1(\ell_{n-1}) & \text{if } n \text{ is odd} \\ Q_2(\ell_{n-1}) & \text{if } n \text{ is even} \end{cases}$$

Then if $n \in \mathbb{N}$ is odd, from (2.1), we have
$\Delta(\ell_n, \ell_{n+1}) = \Delta(\mathscr{Q}_1(\ell_{n-1}), \mathscr{Q}_2(\ell_n))$
$= \Delta(\mathscr{Q}_1(\ell_{n-1}), \mathscr{Q}_2\mathscr{Q}_1(\ell_{n-1}))$
$\preceq$
$\max\{\Delta(\mathscr{Q}_1(\ell_{n-1}), \mathscr{Q}_2\mathscr{Q}_1(\ell_{n-1})), \Delta(\mathscr{Q}_2(\ell_{n-1}), \mathscr{Q}_1\mathscr{Q}_2(\ell_{n-1}))\}$
$\preceq \lambda^*(\min\{\Delta(\ell_{n-1}, \mathscr{Q}_1(\ell_{n-1})), \Delta(\ell_{n-1}, \mathscr{Q}_2(\ell_{n-1}))\})\lambda$
$\preceq \lambda^* \Delta(\ell_{n-1}, \mathscr{Q}_1(\ell_{n-1}))\lambda$
$= \lambda^* \Delta(\ell_{n-1}, \ell_n)\lambda.$

If $n \in \mathbb{N}$ is even, then by (2.1), we have
$\Delta(\ell_n, \ell_{n+1}) = \Delta(\mathscr{Q}_2(\ell_{n-1}), \mathscr{Q}_1(\ell_n))$
$= \Delta(\mathscr{Q}_2(\ell_{n-1}), \mathscr{Q}_1\mathscr{Q}_2(\ell_{n-1}))$
$\preceq$
$\max\{\Delta(\mathscr{Q}_2(\ell_{n-1}), \mathscr{Q}_1\mathscr{Q}_2(\ell_{n-1})), \Delta(\mathscr{Q}_1(\ell_{n-1}), \mathscr{Q}_2\mathscr{Q}_1(\ell_{n-1}))\}$
$\preceq \lambda^*(\min\{\Delta(\ell_{n-1}, \mathscr{Q}_2(\ell_{n-1})), \Delta(\ell_{n-1}, \mathscr{Q}_1(\ell_{n-1}))\})\lambda$
$\preceq \lambda^* \Delta(\ell_{n-1}, \mathscr{Q}_2(\ell_{n-1}))\lambda$
$= \lambda^* \Delta(\ell_{n-1}, \ell_n)\lambda$, for any positive integer n, it must be the case that
$\Delta(\ell_n, \ell_{n+1}) \preceq \lambda^* \Delta(\ell_{n-1}, \ell_n)\lambda. \qquad (2.3)$
By repeated application of (2.3), we obtain
$\Delta(\ell_n, \ell_{n+1}) \preceq \lambda^* \Delta(\ell_{n-1}, \ell_n)\lambda$
$\preceq (\lambda^*)^2 \Delta(\ell_{n-2}, \ell_{n-1})(\lambda)^2$
$\vdots$
$\preceq (\lambda^*)^n \Delta(\ell_0, \ell_1)(\lambda)^n.$
For $m, n \in N$ with $m > n$, and by triangular inequality in C^*-algebra-valued metric spaces, we have
$\Delta(\ell_n, \ell_m) \leq \Delta(\ell_n, \ell_{n+1}) + \ldots + \Delta(\ell_{m-1}, \ell_m)$
$\preceq (\lambda^*)^n \Delta(\ell_0, \ell_1)(\lambda)^n + \cdots + (\lambda^*)^{m-1} \Delta(\ell_0, \ell_1)(\lambda)^{m-1}$
$\preceq \left[(\lambda^*)^n (\lambda)^n + \cdots (\lambda^*)^{m-1} (\lambda)^{m-1}\right] (\Delta(\ell_0, \ell_1))$
$\preceq \left[(\lambda^n)^* \lambda^n + \cdots (\lambda^{m-1})^* \lambda^{m-1}\right] (\Delta(\ell_0, \ell_1))$
$\preceq \sum_{i=n}^{m-1} |\lambda^i|^2 \Delta(\ell_0, \ell_1)$
$\leq \left\| \sum_{i=n}^{m-1} |\lambda^i|^2 \Delta(\ell_0, \ell_1) \right\| I$
$\leq \left\| \sum_{i=n}^{m-1} |\lambda^i|^2 \right\| \|\Delta(\ell_0, \ell_1)\| I$
$\leq \sum_{i=n}^{m-1} \|\lambda\|^{2i} \|\Delta(\ell_0, \ell_1)\| I$
$\leq \frac{\|\lambda\|^{2n}}{1 - \|\lambda\|} \|\Delta(\ell_0, \ell_1)\| I.$
Thus $\Delta(\ell_n, \ell_m)$ tends to $0_{\mathring{\mathscr{A}}}$ as $n \to \infty$, which further implies $\lim_{m,n\to\infty} \|\Delta(\ell_n, \ell_m)\| = 0$. Thus sequence $\{\ell_n\}$ is a Cauchy sequence in $\mho$. Since $\mho$ is complete, so $\ell_n \to z$ in $\mho$. Assume that z is not a common fixed point of $\mathscr{Q}_1$ and $\mathscr{Q}_2$.
Then by (2.2), we have
$0_{\mathring{\mathscr{A}}} = inf\{\Delta(\ell, z) + \min\{\Delta(\ell, \mathscr{Q}_1(\ell))\} : \ell \in \mho\}$
$\preceq inf\{\Delta(\ell_n, z) + \min\{\Delta(\ell_n, \mathscr{Q}_1(\ell_n))\} : n \in \mathbb{N}\}$
$\preceq inf\left\{\frac{\|\lambda\|^{2n}}{1 - \|\lambda\|} \|\Delta(\ell_0, \ell_1)\| I + \Delta(\ell_n, \ell_{n+1}) : n \in \mathbb{N}\right\}$

$\preceq$
$inf\left\{\frac{\|\lambda\|^{2n}}{1-\|\lambda\|}\|\Delta(\ell_0,\ell_1)\|I+(\|\lambda\|)^{2n}\|\Delta(\ell_0,\ell_1)\|I : n\in\mathbb{N}\right\}=$
$0_{\mathring{\mathscr{A}}}$, which is a contradiction.
Therefore, $z=\mathscr{Q}_1(z)=\mathscr{Q}_2(z)$.
Uniqueness:
If $v=\mathscr{Q}_1(v)=\mathscr{Q}_2(v)$ for some $v\in\mho$, then

$$\begin{aligned}\Delta(v,z)&=\max\{\Delta(\mathscr{Q}_1(v),\mathscr{Q}_2\mathscr{Q}_1(v)),\Delta(\mathscr{Q}_2(v),\mathscr{Q}_1\mathscr{Q}_2(v))\}\\&\preceq\lambda^*(\min\{\Delta(v,\mathscr{Q}_1(v)),\Delta(v,\mathscr{Q}_2(v))\})\lambda\\&\preceq\lambda^*(\min\{\Delta(v,v),\Delta(v,v)\})\lambda\\&\preceq\lambda^*\Delta(v,v)\lambda,\end{aligned}$$

which implies that $\Delta(v,z)=0_{\mathring{\mathscr{A}}}$.
Hence the maps $\mathscr{Q}_1$ and $\mathscr{Q}_2$ has a unique fixed point.

Example 1. Let $\mho=\{\frac{1}{n}\}_{n=1}^{\infty}\cup\{0\}$ and $\mathring{\mathscr{A}}=M_2(\mathbb{R})$ be endowed with the norm $\|A\|=\left(\sum_{i,j=1}^{2}|a_{i,j}|^2\right)^{\frac{1}{2}}$, and $^*: M_2(\mathbb{R})\to M_2(\mathbb{R})$ involution given by $A^*=A$. Clearly, each matrix of type $A=\begin{bmatrix}a&0\\0&b\end{bmatrix}$ belongs to $\mathring{\mathscr{A}}_+$ if $a,b\geq 0$.

This implies that $\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\leq\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}$ if and only if $a_{ij}-b_{ij}\leq 0$ for all $i,j=1,2$.
Define $\Delta:\mho\times\mho\to\mathring{\mathscr{A}}$ by

$$\Delta(\ell,\ell')=\begin{bmatrix}|\ell-\ell'|^2&0\\0&|\ell-\ell'|^2\end{bmatrix}=|\ell-\ell'|^2\begin{bmatrix}1&0\\0&1\end{bmatrix}=|\ell-\ell'|^2 I_{\mathring{\mathscr{A}}}$$

Then $\left(M_2(\mathbb{R}),\mathring{\mathscr{A}},\Delta\right)$ is a complete C^*-algebra-valued metric space.
Now, we define $\mathscr{Q}_1,\mathscr{Q}_2:\mho\to\mho$ by
$\mathscr{Q}_1(0)=0, \mathscr{Q}_1\left(\frac{1}{2n}\right)=\frac{1}{4n}, \mathscr{Q}_1\left(\frac{1}{2n-1}\right)=0$ and $\mathscr{Q}_1\left(\frac{1}{4n+2}\right)=\frac{1}{4n+3}$ and
$\mathscr{Q}_2(0)=0, \mathscr{Q}_2\left(\frac{1}{4n}\right)=\frac{1}{4n+1}, \mathscr{Q}_2\left(\frac{1}{2n-1}\right)=\frac{1}{4n+2}$ and $\mathscr{Q}_2\left(\frac{1}{2n}\right)=0$.
Then for $\ell=\frac{1}{2n}$, we have

$$\begin{aligned}&\max\{\Delta(\mathscr{Q}_1(\ell),\mathscr{Q}_2\mathscr{Q}_1(\ell)),\Delta(\mathscr{Q}_2(\ell),\mathscr{Q}_1\mathscr{Q}_2(\ell))\}\\&=\max\left\{\Delta\left(\mathscr{Q}_1\left(\tfrac{1}{2n}\right),\mathscr{Q}_2\mathscr{Q}_1\left(\tfrac{1}{2n}\right)\right),\Delta\left(\mathscr{Q}_2\left(\tfrac{1}{2n}\right),\mathscr{Q}_1\mathscr{Q}_2\left(\tfrac{1}{2n}\right)\right)\right\}\\&=\max\left\{\Delta\left(\tfrac{1}{4n},\tfrac{1}{4n+1}\right),\Delta(0,0)\right\}\\&=\max\left\{\begin{bmatrix}\left|\frac{1}{4n}-\frac{1}{4n+1}\right|^2&0\\0&\left|\frac{1}{4n}-\frac{1}{4n+1}\right|^2\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix}\right\}\\&=\max\left\{\left|\tfrac{1}{4n}-\tfrac{1}{4n+1}\right|^2 I_{\mathscr{A}},0\right\}\\&\leq\max\left\{\tfrac{1}{4n}I_{\mathring{\mathscr{A}}},0\right\}\\&=\tfrac{1}{4n}I_{\mathring{\mathscr{A}}}\\&\leq\lambda^*(\min\{\Delta(\ell,\mathscr{Q}_1(\ell)),\Delta(\ell,\mathscr{Q}_2(\ell))\})\lambda\\&\leq\lambda^*\left(\min\left\{\tfrac{1}{2n},\tfrac{1}{2n}\right\}\right)\lambda\\&=\lambda^*\left(\tfrac{1}{2n}\right)\lambda.\end{aligned}$$

It is easy to see that the above inequality is true for $\ell=\frac{1}{2n-1}$ and $\lambda=\begin{bmatrix}\frac{1}{2}&0\\0&\frac{1}{2}\end{bmatrix}$. Also,

$$\gamma(\ell')=inf\{\Delta(\ell,\ell')+\min\{\Delta(\ell,\mathscr{Q}_1(\ell))\}:\ell\in\mho>0_{\mathring{\mathscr{A}}},\tag{2.4}$$

for every $\ell'\in\mho$ with ℓ' is not a common fixed point of $\mathscr{Q}_1$ and $\mathscr{Q}_2$ for $\frac{1}{3}\leq\|\lambda\|<1$.
This shows that the all conditions of Theorem 1 are satisfied and 0 is a fixed point for $\mathscr{Q}_1$ and $\mathscr{Q}_2$.

Corollary 1. *Let $(\mho,\mathring{\mathscr{A}},\Delta)$ be a complete C^*-algebra-valued metric space. Let $\mathscr{Q}_2:\mho\to\mho$ be self -mapping. Suppose that there exists $\lambda\in\mathring{\mathscr{A}}$ with $\|\lambda\|\in[0,1)$ such that*

$$\Delta\left(\mathscr{Q}_2(\ell),\mathscr{Q}_2^2(\ell)\right)\preceq\lambda^*(\Delta(\ell,\mathscr{Q}_2(\ell)))\lambda,\tag{2.5}$$

for every $\ell\in\mho$ and that

$$\gamma(\ell')=inf\{\Delta(\ell,\ell')+\Delta(\ell,\mathscr{Q}_2(\ell)):\ell\in\mho\}\succ 0_{\mathring{\mathscr{A}}},\tag{2.6}$$

for every $\ell'\in\mho$ with ℓ' is not a fixed point of $\mathscr{Q}_2$. Then $\mathscr{Q}_2$ has a unique fixed point.

Proof. Taking $\mathscr{Q}_1=\mathscr{Q}_2$ in Theorem 1, we get the required result.

Theorem 2. *Let $(\mho,\mathring{\mathscr{A}},\Delta)$ be a complete C^*-algebra-valued metric space. Let $\mathscr{Q}_1$ and $\mathscr{Q}_2$ be mappings from $\mho$ onto itself. Suppose that there exists $\lambda\in\mathring{\mathscr{A}}$ with $\|\lambda\|>1$ such that*

$$\min\{\Delta(\mathscr{Q}_2\mathscr{Q}_1(\ell),\mathscr{Q}_1(\ell)),\Delta(\mathscr{Q}_1\mathscr{Q}_2(\ell),\mathscr{Q}_2(\ell))\}\succcurlyeq\lambda^*(\max\{\Delta(\mathscr{Q}_1(\ell),\ell),\Delta(\mathscr{Q}_2(\ell),\ell)\})\lambda,\tag{2.7}$$

for every $\ell\in\mho$ and

$$\gamma(\ell')=inf\{\Delta(\ell,\ell')+\min\{\Delta(\ell,\mathscr{Q}_1(\ell)),\Delta(\ell,\mathscr{Q}_2(\ell))\}:\ell\in\mho\}\succ 0_{\mathring{\mathscr{A}}},\tag{2.8}$$

for every $\ell'\in\mho$ with ℓ' is not a common fixed point of $\mathscr{Q}_1$ and $\mathscr{Q}_2$. Then $\mathscr{Q}_1$ and $\mathscr{Q}_2$ has a unique fixed point.

Proof. Let $\ell_0\in\mho$ be arbitrary. Since $\mathscr{Q}_1$ is onto, there is an element ℓ_1 such that $\ell_1\in\mathscr{Q}_1^{-1}(\ell_0)$. Since $\mathscr{Q}_2$ is onto, there is an element ℓ_2 such that $\ell_2\in\mathscr{Q}_2^{-1}(\ell_1)$. Continue like this, we can find $\ell_{2n+1}\in\mathscr{Q}_1^{-1}(\ell_{2n})$ and $\ell_{2n+2}\in\mathscr{Q}_2^{-1}(\ell_{2n+1})$ for $n=1,2,3,\ldots$.
Therefore, $\ell_{2n}=\mathscr{Q}_1(\ell_{2n+1})$ and $\ell_{2n+1}=\mathscr{Q}_2(\ell_{2n+2})$ for $n=0,1,2,\ldots$
If $n=2m$, then using (2.7)

$$\begin{aligned}\Delta(\ell_{n-1},\ell_n)&=\Delta(\ell_{2m-1},\ell_{2m})=\Delta(\mathscr{Q}_2(\ell_{2m}),\mathscr{Q}_1(\ell_{2m+1}))=\Delta(\mathscr{Q}_2\mathscr{Q}_1(\ell_{2m+1}),\mathscr{Q}_1(\ell_{2m+1}))\\&\succcurlyeq\min\{\Delta(Q_2Q_1(\ell_{2m+1})\ Q_1(\ell_{2m+1})\ \Delta(Q_1Q_2(\ell_{2m+1}),Q_2(\ell_{2m+1}))\}\\&\succcurlyeq\lambda^*(\max\{\Delta(\mathscr{Q}_1(\ell_{2m+1}),\ell_{2m+1},\Delta(\mathscr{Q}_2(\ell_{2m+1}),\ell_{2m+1}\})\lambda\\&\succcurlyeq\lambda^*\Delta(\mathscr{Q}_1(\ell_{2m+1}),\ell_{2m+1})\lambda=\lambda^*\Delta(\ell_{2m},\ell_{2m+1})\lambda\\&=\lambda^*\Delta(\ell_n,\ell_{n+1})\lambda.\end{aligned}$$

If $n=2m+1$, then using (2.7)

$$\begin{aligned}\Delta(\ell_{n-1},\ell_n)&=\Delta(\ell_{2m},\ell_{2m+1})\\&=\Delta(\mathscr{Q}_1(\ell_{2m+1}),\mathscr{Q}_2(\ell_{2m+2}))\\&=\Delta(\mathscr{Q}_1\mathscr{Q}_2(\ell_{2m+2}),\mathscr{Q}_2(\ell_{2m+2}))\\&\succcurlyeq\min\{\Delta(Q_2Q_1(\ell_{2\ m+2})\ Q_1(\ell_{2\ m+2}),\ \Delta(Q_1Q_2(\ell_{2\ m+2})_2 Q_2(\ell_{2\ m+2}))\}\\&\succcurlyeq\lambda^*(\max\{\Delta(\mathscr{Q}_1(\ell_{2m+2}),\ell_{2m+2},\Delta(\mathscr{Q}_2(\ell_{2m+2}),\ell_{2m+2}\})\lambda\end{aligned}$$

$\succcurlyeq \lambda^*\Delta(\mathscr{Q}_2(\ell_{2m+2}),\ell_{2m+2})\lambda$
$= \lambda^*\Delta(\ell_{2m+1},\ell_{2m+2})\lambda$
$= \lambda^*\Delta(\ell_n,\ell_{n+1})\lambda.$
Thus for any positive integer n, it must be the case that $\Delta(\ell_{n-1},\ell_n) \succcurlyeq \lambda^*\Delta(\ell_n,\ell_{n+1})\lambda$, which implies that, $\|\Delta(\ell_n,\ell_{n+1})\| \leq \frac{1}{\|\lambda\|^2}\|\Delta(\ell_{n-1},\ell_n)\| \leq \dots \leq \frac{1}{\|\lambda\|^{2n}}\|\Delta(\ell_0,\ell_1)\|,$ (2.9)
$R = \frac{1}{\|\lambda\|^2}$, then $0 < R < 1$ since $\|\lambda\| > 1$.
Now, (2.9) becomes $\|\Delta(\ell_n,\ell_{n+1})\| \leq R^n\|\Delta(\ell_0,\ell_1)\|$.
Also, we have For $m,n \in N$ with $m > n$, and by triangular inequality in C^*-algebra-valued metric space, we have

$$\begin{aligned}\|\Delta(\ell_n,\ell_m)\| &\leq \|\Delta(\ell_n,\ell_{n+1})\| + \ldots + \|\Delta(\ell_{m-1},\ell_m)\| \\ &\leq R^n\|\Delta(\ell_0,\ell_1)\| + \ldots + R^{m-1}\|\Delta(\ell_0,\ell_1)\| \\ &\leq \frac{R^n}{1-R}\|\Delta(\ell_0,\ell_1)\|.\end{aligned}$$

Therefore $\lim_{m,n\to\infty}\|\Delta(\ell_n,\ell_m)\| = 0$ and so sequence $\{\ell_n\}$ is a Cauchy sequence in $\mho$. Since $\mho$ is complete, so $\ell_n \to z$ in $\mho$. Assume that z is not a common fixed point of $\mathscr{Q}_1$ and $\mathscr{Q}_2$. Then by hypothesis, we have
$0 < inf\{\|\Delta(\ell,z)\| +$
$\min\{\|\Delta(\ell,\mathscr{Q}_1(\ell))\|,\|\Delta(\ell,\mathscr{Q}_2(\ell))\|\} : \ell \in \mho\}$
$\leq \inf\{\|\Delta(\ell_n,z)\| +$
$\min\{\|\Delta(\ell_n,Q_1(\ell_n))\| + \|\Delta(\ell_n,Q_2(\ell_n))\|\} : n \in N\}$
$\leq inf\left\{\frac{R^n}{1-R}\|\Delta(\ell_0,\ell_1)\| + \|\Delta(\ell_{n-1},\ell_n)\| : n \in \mathbb{N}\right\}$
$\leq inf\left\{\frac{R^n}{1-R}\|\Delta(\ell_0,\ell_1)\| + (R)^{n-1}\|\Delta(\ell_0,\ell_1)\| : n \in \mathbb{N}\right\} =$
0, which is a contradiction.
Therefore, $z = \mathscr{Q}_1(z) = \mathscr{Q}_2(z)$. Uniqueness: If $v = \mathscr{Q}_1(v) = \mathscr{Q}_2(v)$ for some $v \in \mho$, then

$$\begin{aligned}\Delta(v,z) &= \min\{\Delta(\mathscr{Q}_2\mathscr{Q}_1(v),\mathscr{Q}_1(v)),\Delta(\mathscr{Q}_1\mathscr{Q}_2(v),\mathscr{Q}_2(v))\} \\ &\succcurlyeq \lambda^\star(\max\{\Delta(\mathscr{Q}_1(v),v),\Delta(\mathscr{Q}_2(v),v,)\})\lambda \\ &\succcurlyeq \lambda^*(\max\{\Delta(v,v),\Delta(v,v)\})\lambda \\ &= \lambda^*\Delta(v,v) = 0_{\mathring{\mathscr{A}}},\end{aligned}$$

which implies that $\Delta(v,z) = 0_{\mathring{\mathscr{A}}}$.
Hence the maps $\mathscr{Q}_1$ and $\mathscr{Q}_2$ has a unique fixed point.

Corollary 2.*Let $(\mho,\mathring{\mathscr{A}},\Delta)$ be a complete C^*-algebra-valued metric space and let $\mathscr{Q}_2$ be an onto self-mapping. Suppose that there exists $\lambda \in \mathring{\mathscr{A}}$ with $\|\lambda\| > 1$ such that*

$$\Delta\left(\mathscr{Q}_2^2(\ell),\mathscr{Q}_2(\ell)\right) \succcurlyeq \lambda^*\Delta(\mathscr{Q}_2(\ell),\ell)\lambda, \quad (2.10)$$

for every $\ell \in \mho$ and that

$$\gamma(\ell') = inf\{\Delta(\ell,\ell') + \Delta(\mathscr{Q}_2(\ell),\ell) : \ell \in \mho\} \succ 0_{\mathring{\mathscr{A}}}, \quad (2.11)$$

for every $\ell' \in \mho$ with ℓ' is not a fixed point of $\mathscr{Q}_2$. Then $\mathscr{Q}_2$ has a unique fixed point.

*Proof.*Taking $\mathscr{Q}_1 = \mathscr{Q}_2$ in Theorem 2, we have the desired result.

Theorem 3.*Let $(\mho,\mathring{\mathscr{A}},\Delta)$ be a complete C^*-algebra-valued metric space. Let $\mathscr{Q}_2 : \mho \to \mho$ be self -mapping. Suppose that there exists $\lambda \in \mathring{\mathscr{A}}$ with $\|\lambda\| > 1$ such that $\Delta\left(\mathscr{Q}_2^2(\ell),\mathscr{Q}_2(\ell)\right) \geqslant \lambda^*\Delta(\mathscr{Q}_2(\ell),\ell)\lambda$ for every $\ell \in \mho$ and $\mathscr{Q}_2$ is onto continuous. Then $\mathscr{Q}_2$ has a fixed point.*

*Proof.*Assume that there exists $\ell' \in \mho$ with ℓ' is not a fixed point of $\mathscr{Q}_2$ and $\gamma(\ell') = inf\{\Delta(\ell,\ell') + \Delta(\mathscr{Q}_2(\ell),\ell) : \ell \in \mho\} = 0_{\mathring{\mathscr{A}}}$.
Then there exists a sequence $\{\ell_n\}$ such that

$$\lim_{n\to\infty}\{\Delta(\ell_n,\ell') + \Delta(\mathscr{Q}_2(\ell_n),\ell_n)\} = 0_{\mathring{\mathscr{A}}}.$$

So, we have $\Delta(\ell_n,\ell') \to 0_{\mathring{\mathscr{A}}}$ and $\Delta(\mathscr{Q}_2(\ell_n),\ell_n) \to 0_{\mathring{\mathscr{A}}}$ as $n \to \infty$.
Hence

$\Delta(\mathscr{Q}_2(\ell_n),\ell') \leq \Delta(\mathscr{Q}_2(\ell_n),\ell_n) + \Delta(\ell_n,\ell')$ implies that $\mathscr{Q}_2(\ell_n) = \ell'$.

Since $\mathscr{Q}_2$ is continuous, we have
$\mathscr{Q}_2(\ell') = \mathscr{Q}_2(\lim_{n\to\infty}\ell_n) = \lim_{n\to\infty}\mathscr{Q}_2(\ell_n) = y$.
This is a contradiction.
Hence if ℓ' is not a fixed point of $\mathscr{Q}_2$, then

$$\gamma(\ell') = inf\{\Delta(\ell,\ell') + \Delta(\mathscr{Q}_2(\ell),\ell) : \ell \in X\} \succ 0_{\mathring{\mathscr{A}}},$$

which is condition (2.11) of Corollary 2.
By Corollary 2, there exists $z \in \mho$ such that $z = \mathscr{Q}_2(z)$.

*Example 2.*Let $\mho = [0,1)$ and $\mathring{\mathscr{A}} = M_2(\mathbb{R})$ be endowed with the norm

$$\|A\| = \left(\sum_{i,j=1}^{2}|a_{i,j}|^2\right)^{\frac{1}{2}},$$

and $^* : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ involution given by $A^* = A$.
This implies that $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} \leq \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}$ if and only if $a_{ij} - b_{ij} \leq 0$ for all $i,j = 1,2$.
Define $\Delta : \mho \times \mho \to \mathring{\mathscr{A}}$ by

$$\Delta(\ell,\ell') = \begin{bmatrix} |\ell-\ell'|^2 & 0 \\ 0 & |\ell-\ell'|^2\end{bmatrix} = |\ell-\ell'|^2\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} = |\ell-\ell'|^2 I_{\mathring{\mathscr{A}}}.$$

Then $\left(M_2(R),\mathring{\mathscr{A}},\Delta\right)$ is a complete C^*-algebra-valued metric space. Define $\mathscr{Q}_2 : \mho \to \mho$ by $\mathscr{Q}_2(\ell) = 8\ell$. Obviously $\mathscr{Q}_2$ is onto and continuous.
Also for each $\ell,\ell' \in \mho$ we have

$$\begin{aligned}\Delta\left(\mathscr{Q}_2^2\ell,\mathscr{Q}_2\ell\right) &= \Delta(64\ell,8\ell) \\ &= \begin{bmatrix} |64\ell-8\ell|^2 & 0 \\ 0 & |64\ell-8\ell|^2\end{bmatrix} \\ &= |56\ell|^2 I_{\mathring{\mathscr{A}}} \\ &\succcurlyeq \lambda^*\Delta(\mathscr{Q}_2(\ell),\ell)\lambda \text{ where } \|\lambda\| = 7.\end{aligned}$$

Thus $\mathscr{Q}_2$ satisfy the conditions given in Corollary 2 and 0 is the unique common fixed point of $\mathscr{Q}_2$.

Theorem 4. *Let $(\mho, \mathring{\mathscr{A}}, \Delta)$ be a complete C^*-algebra-valued metric space. Let $\mathscr{Q}_2 : \mho \to \mho$ be self -mapping.Suppose that there exists $\lambda \in \mathring{\mathscr{A}}$ with $\|\lambda\| > 1$ such that*

$$\Delta(\mathscr{Q}_2(\ell), \mathscr{Q}_2(\ell')) \succcurlyeq \lambda^* \min\{\Delta(\ell, \mathscr{Q}_2(\ell)), \Delta(\mathscr{Q}_2(\ell'), \ell'), \Delta(\ell, \ell')\}\lambda, \quad (2.12)$$

for every $\ell, \ell' \in \mho$ and $\mathscr{Q}_2$ is onto continuous, then $\mathscr{Q}_2$ has a fixed point.

Proof. Replacing ℓ' by $\mathscr{Q}_2(\ell)$ in (2.12), we obtain

$$\Delta\left(\mathscr{Q}_2(\ell), \mathscr{Q}_2^2(\ell)\right) \succcurlyeq \lambda^* \min\left\{\Delta(\ell, \mathscr{Q}_2(\ell)), \Delta\left(\mathscr{Q}_2^2(\ell), \mathscr{Q}_2(\ell)\right), \Delta(\ell, \mathscr{Q}_2(\ell))\right\}\lambda, \quad (2.13)$$

for all $x \in \mho$.
Without loss of generality, we may assume that $\mathscr{Q}_2(\ell) \neq \mathscr{Q}_2^2(\ell)$.
Otherwise, $\mathscr{Q}_2$ has a fixed point. Since $\|\lambda\| > 1$, it follows from (2.12) that

$$\Delta\left(\mathscr{Q}_2(\ell), \mathscr{Q}_2^2(\ell)\right) \succcurlyeq \lambda^* \Delta(\mathscr{Q}_2(\ell), \ell)\lambda,$$

for every $\ell \in \mho$. By the argument similar to that used in Theorem 3, we can prove that, if $\ell' \neq \mathscr{Q}_2(\ell')$, then $\gamma(\ell') = inf\{\Delta(\ell, \ell') + \Delta(\mathscr{Q}_2(\ell), \ell) : \ell \in \mho\} > 0_{\mathring{\mathscr{A}}}$, which is condition (2.10) of Corollary 2.
So, Corollary 2 applies to get a fixed point of $\mathscr{Q}_2$.

3 Application

As an application of Corollary 1, we find an existence and uniqueness result for a type of following Fredholm integral equation:

$$\ell(s) = \int_E K(s, t, \ell(t))dt + h(s) \quad (1)$$

for all $s, t \in E$ where E is a Lebesgue measurable set. Suppose that $K : E^2 \times \mathbb{R} \to \mathbb{R}$ and $h \in L^\infty(E)$. Let $\mho = L^\infty(E)$, $H = L^2(E)$ and $L(H) = A$. Define $\Delta : \mho \times \mho \to \mathring{\mathscr{A}}$ by

$$\Delta(\ell, \mathscr{Q}_2(\ell)) = M_{|\ell - \mathscr{Q}_2(\ell)|}$$

for all $\ell \in \mho$, where $M : H \to H$ is the multiplicative operator defined by:

$$M_\varphi(\psi) = \varphi.\psi$$

Then $(\mho, A, \Delta)$ is a C^*-algebra-valued metric space.
Now we consider the following assumption:
for all $\ell, \ell' \in \mho$ there exist $\gamma \in (0,1)$ such that

$$|K(s, t, \ell(t)) - K(s, t, \ell'(t))| \leq \gamma|(\ell(t) - \ell'(t))|.$$

Theorem 5. *If the above assumption holds. Then the integral equation (1) has a unique solution in $\mho$.*

Proof. We define $\mathscr{Q}_2 : \mho \to \mho$ by

$$\mathscr{Q}_2(\ell)(s) = \int_E K(s, t, \ell(t))dt + h(s),$$
$$\mathscr{Q}_2^2(\ell)(s) = \int_E K(s, t, \mathscr{Q}_2(\ell)(t))dt + h(s), \quad \forall s, t \in E.$$

Set $\lambda = \gamma I$, then $\lambda \in L(H)_+$ and $\| \lambda \| = \gamma < 1$. For every $\varphi \in H$, we have

$$\begin{aligned}
&\| \Delta(\mathscr{Q}_2(\ell), \mathscr{Q}_2^2(\ell)) \| = \| \mathscr{M}_{|\mathscr{Q}_2(\ell) - \mathscr{Q}_2^2(\ell)|} \| \\
&= \sup_{\|\varphi\|=1} (\mathscr{M}_{|\mathscr{Q}_2(\ell) - \mathscr{Q}_2^2(\ell)|}\varphi, \varphi) \\
&= \sup_{\|\varphi\|=1} \int_E \left[| \int_E K(s, t, \ell(t)) - K(s, t, \mathscr{Q}_2(\ell)(t))dt| \right] \varphi(s)\overline{\varphi(s)}ds \\
&\leq \sup_{\|\varphi\|=1} \int_E \left[\int_E |K(s, t, \ell(t)) - K(s, t, \mathscr{Q}_2(\ell)(t))|dt \right] |\varphi(s)|^2 ds \\
&\leq \gamma \sup_{\|\varphi\|=1} \int_E |\varphi(s)|^2 ds \,.\, \| \ell - \mathscr{Q}_2(\ell) \|_\infty \\
&\leq \gamma \| \ell - \mathscr{Q}_2(\ell) \|_\infty \\
&= \| \lambda \| \| \Delta(\ell, \mathscr{Q}_2(\ell)) \| .
\end{aligned}$$

Since $\| \lambda \| < 1$, Hence, the Fredholm integral Equation (1) has a unique solution.

4 Conclusion

In this paper,we have proved some recent fixed point theorems for two pairs of mappings using generalized contractive conditions in the framework of C^*-algebra-valued metric spaces. An example and application is also given to support the result. This work can further be extended and generalised in various metric spaces also.

Competing interests

The authors declare that they have no competing interests.

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