Appl. Math. Inf. Sci. 9, No. 1, 501-506 (2015)

Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090158

Group Classification, Symmetry Reductions and Exact Solutions of a Generalized Korteweg-de Vries-Burgers Equation

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Received: 22 Jan. 2014, Revised: 26 Jul. 2014, Accepted: 27 Jul. 2014

Abstract: Lie group classification is performed on the generalized Korteweg-de Vries-Burgers equation $u_t + \delta u_{xxx} + g(u)u_x - \nu u_{xx} + \gamma u = f(x)$, which occurs in many applications of physical phenomena. We show that the equation admits a four-dimensional equivalence Lie algebra. It is also shown that the principal Lie algebra consists of a single translation symmetry. Several possible extensions of the principal Lie algebra are computed and their associated symmetry reductions and exact solutions are obtained. Also, one-dimensional optimal system of subalgebras is obtained for the case when the principal Lie algebra is extended by two symmetries.

Keywords: Generalized Korteweg-de Vries-Burgers equation, Group classification, Symmetry reductions, Exact solutions

1 Introduction

Many differential equations of physical interest involve parameters, arbitrary elements or functions, which need to be determined. Usually, these arbitrary parameters are determined experimently. However, the Lie symmetry approach through the method of group classification has proven to be a versatile tool in specifying the forms of these parameters systematically [1,2,3,4,6,5].

The first group classification problem was investigated by Sophus Lie [7] in 1881 for linear second-order partial differential equations (PDEs) with two independent variables. The main idea of group classification of a differential equation involving an arbitrary element(s), say, for example, $g(u)$ and $f(x)$, consists of finding the Lie point symmetries of the differential equation with arbitrary functions $g(u)$ and $f(x)$, and then computing systematically all possible forms of $g(u)$ and $f(x)$ for which the principal Lie algebra can be extended.

In this paper we study one such differential equation, namely, the generalized Korteweg-de Vries-Burgers equation [8]

$$u_t + \delta u_{xxx} + g(u)u_x - \nu u_{xx} + \gamma u = f(x), \tag{1}$$

which contains two arbitrary functions $g(u)$ and $f(x)$. We perform Lie group classification of (1) and then find symmetry reductions and exact solutions. This equation arises from many physical scenarios such as the propagation of undular bores in shallow water, the flow of liquids containing gas bubbles, weakly nonlinear plasma waves with certain dissipative effect, theory of ferro electricity, nonlinear circuit, and the propagation of waves in an elastic tube filled with a viscous fluid [9].

2 Equivalence transformations

An equivalence transformation (see for example [1]) of (1) is an invertible transformation involving the variables t, x and u that map (1) into itself. The operator

$$\begin{aligned} Y = {} & \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u \\ & + \mu^1(t,x,u,f,g)\partial_f + \mu^2(t,x,u,f,g)\partial_g, \end{aligned} \tag{2}$$

is the generator of the equivalence group for (1) provided it is admitted by the extended system

$$u_t + \delta u_{xxx} + g(u)u_x - \nu u_{xx} + \gamma u = f(x), \tag{3a}$$

$$f_t = 0, \quad f_u = 0, \quad g_t = 0, \quad g_x = 0. \tag{3b}$$

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The prolonged operator for the extended system (3) has the form

$$\widetilde{Y} = Y^{[3]} + \omega_1^1 \partial_{f_t} + \omega_2^1 \partial_{f_x} + \omega_3^1 \partial_{f_u} + \omega_1^2 \partial_{g_t} + \omega_2^2 \partial_{g_x} + \omega_3^2 \partial_{g_u}, \tag{4}$$

where $Y^{[3]}$ is the third-prolongation of (2) given by

$$\begin{aligned} Y^{[3]} = & \tau\partial_t + \xi\partial_x + \eta\partial_u + \mu^1\partial_f + \mu^2\partial_g + \zeta_1\partial_{u_t} + \zeta_2\partial_{u_x} \\ & + \zeta_{22}\partial_{u_{xx}} + \zeta_{222}\partial_{u_{xxx}}. \end{aligned}$$

The variables ζ's and ω's are defined by the prolongation formulae

$$\begin{aligned} \zeta_1 &= D_t(\eta) - u_x D_t(\tau) - u_t D_t(\xi), \\ \zeta_2 &= D_x(\eta) - u_x D_x(\tau) - u_t D_x(\xi), \\ \zeta_{22} &= D_x(\zeta_2) - u_{xx} D_x(\tau) - u_{xt} D_x(\xi), \\ \zeta_{222} &= D_x(\zeta_{22}) - u_{xxx} D_x(\tau) - u_{xxt} D_x(\xi) \end{aligned}$$

and

$$\begin{aligned} \omega_1^1 &= \widetilde{D}_t(\mu^1) - f_t\widetilde{D}_t(\tau) - f_x\widetilde{D}_t(\xi) - f_u\widetilde{D}_t(\eta), \\ \omega_2^1 &= \widetilde{D}_x(\mu^1) - f_t\widetilde{D}_x(\tau) - f_x\widetilde{D}_x(\xi) - f_u\widetilde{D}_x(\eta), \\ \omega_3^1 &= \widetilde{D}_u(\mu^1) - f_t\widetilde{D}_u(\tau) - f_x\widetilde{D}_u(\xi) - f_u\widetilde{D}_u(\eta), \\ \omega_1^2 &= \widetilde{D}_t(\mu^2) - g_t\widetilde{D}_t(\tau) - g_x\widetilde{D}_t(\xi) - g_u\widetilde{D}_t(\eta), \\ \omega_2^2 &= \widetilde{D}_x(\mu^2) - g_t\widetilde{D}_x(\tau) - g_x\widetilde{D}_x(\xi) - g_u\widetilde{D}_x(\eta), \\ \omega_3^2 &= \widetilde{D}_u(\mu^2) - g_t\widetilde{D}_u(\tau) - g_x\widetilde{D}_u(\xi) - g_u\widetilde{D}_u(\eta), \end{aligned}$$

respectively, where $[D_t = \partial_t + u_t\partial_u + \cdots, \quad D_x = \partial_x + u_x\partial_u + \cdots]$ are the total derivative operators and

$$\begin{aligned} \widetilde{D}_x &= \partial_x + f_x\partial_f + g_x\partial_g + \cdots, \\ \widetilde{D}_t &= \partial_t + f_t\partial_f + g_t\partial_g + \cdots, \\ \widetilde{D}_u &= \partial_u + f_u\partial_f + g_u\partial_g + \cdots \end{aligned}$$

are the total derivative operators for the extended system. The application of the prolongation (4) and the invariance conditions of system (3) leads to following equivalent generators

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t}, \\ Y_2 &= \frac{\partial}{\partial x}, \\ Y_3 &= \frac{\partial}{\partial u} + \gamma\frac{\partial}{\partial f}, \\ Y_4 &= u\frac{\partial}{\partial u} + f\frac{\partial}{\partial f}. \end{aligned}$$

Thus the four-parameter equivalence group is given by

$Y_1 : \bar{t} = a_1 + t,\ \bar{x} = x,\ \bar{u} = u,\ \bar{f} = f,\ \bar{g} = g,$

$Y_2 : \bar{t} = t,\ \bar{x} = a_2 + x,\ \bar{u} = u,\ \bar{f} = f,\ \bar{g} = g,$

$Y_3 : \bar{t} = t,\ \bar{x} = x,\ \bar{u} = a_3 + u,\ \bar{f} = \gamma a_3 + f,\ \bar{g} = g,$

$Y_4 : \bar{t} = t,\ \bar{x} = x,\ \bar{u} = e^{a_4}u,\ \bar{f} = e^{a_4}f,\ \bar{g} = g$

and their composition gives

$$\begin{aligned} \bar{t} &= a_1 + t, \\ \bar{x} &= a_2 + x, \\ \bar{u} &= (a_3 + u)e^{a_4}, \\ \bar{f} &= (\gamma a_3 + f)e^{a_4}, \\ \bar{g} &= g. \end{aligned}$$

3 Principal Lie algebra

The symmetry group of equation (1) will be generated by the vector field of the form

$$\Gamma = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial u} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{5}$$

Applying the third prolongation of Γ to (1) yields the following overdetermined system of linear PDEs:

$$\begin{aligned} & \tau_u = 0, \tau_x = 0, \xi_u = 0, \eta_{uu} = 0, \\ & 2\nu\xi_x - \nu\tau_t + 3\delta\eta_{xu} - 3\delta\xi_{x,x} = 0, 3\xi_x - \tau_t = 0, \\ & \eta g_u - \xi_t - g\xi_x + g\tau_t - 2\nu\eta_{xu} + \nu\xi_{xx} + 3\delta\eta_{xxu} - \delta\xi_{xxx} = 0, \\ & \gamma\eta - \xi f_x + \eta_t + f\eta_u - u\gamma\eta_u + g\eta_x - f\tau_t + u\gamma\tau_t - \nu\eta_{xx} \\ & + \delta\eta_{xxx} = 0. \end{aligned} \tag{6}$$

Solving the above system for arbitrary f and g we find that the principal Lie algebra consists of one translation symmetry, namely

$$\Gamma_1 = \frac{\partial}{\partial t}.$$

4 Lie group classification

Solving the system (6), we obtain the following classifying relations:

$$\begin{aligned} & \frac{2ga_t}{3} - \frac{2a_t\nu^2}{9\delta} + \left(B + u\left(k + \frac{x\nu a_t}{9\delta}\right)\right)g_u - q_t - \frac{1}{3}xa_{tt} = 0, \\ & + u\gamma a_t - fa_t + f\left(k + \frac{x\nu a_t}{9\delta}\right) - u\gamma\left(k + \frac{x\nu a_t}{9\delta}\right) \\ & + \gamma\left(B + u\left(k + \frac{x\nu a_t}{9\delta}\right)\right) + B_t + g\left(\frac{u\nu a_t}{9\delta} + B_x\right) \\ & - \left(q + \frac{xa_t}{3}\right)f_x + u\left(k_t + \frac{x\nu a_{tt}}{9\delta}\right) - \nu B_{xx} + \delta B_{xxx} = 0. \end{aligned}$$

Using the equivalence transformations obtained in Section 2, these classifying relations lead to the following four cases for the functions g and f and for each case we also provide the associated extended symmetries.

Case (A): $f(x)$ arbitrary, $g(u) = g_0$, where f_0, g_0 are nonzero constants.

$$\Gamma_2 = 9\delta t\frac{\partial}{\partial t} - (9\delta\gamma tu + g_0\nu tu - \nu ux)\frac{\partial}{\partial u}$$
$$-\left(2\nu^2 t - 6\delta g_0 t - 3\delta x\right)\frac{\partial}{\partial x},$$
$$\Gamma_3 = u\frac{\partial}{\partial u},$$
$$\Gamma_4 = \frac{\partial}{\partial x},$$
$$\Gamma_5 = F(t,x)\frac{\partial}{\partial u},$$

where $F(t,x)$ is any solution of

$$9\delta(6\delta g_0 t - 2\nu^2 t + 3\delta x)f'(x) + 9\delta(9\gamma t + \nu g_0 t - \nu x$$
$$+9\delta)f(x)C_2 - 9C_4\delta f'(x) + 9C_3\delta f(x) - 9\nu\delta F_{xx}$$
$$+9\delta^2 F_{xxx} + 9g_0\delta F_x + 9\gamma\delta F + 9\delta F_t = 0.$$

Case (B): $f(x) = f_0$, $g(u) = g_0 - g_1 \ln u$, where f_0, g_0, g_1 are nonzero constants.

$$\Gamma_2 = \frac{\partial}{\partial x}.$$

Case (C): $f(x) = f_0$, $g(u) = u^2 + \bar{g}_0 u + \bar{g}_1$, where $\bar{g}_0 \neq 0$ is an arbitrary constant.

$$\Gamma_2 = \frac{\partial}{\partial x}.$$

Case (D): $f(x) = f_0 + f_1 x$, $g(u) = g_0 + \tilde{g}_1 u$, where $f_0, f_1, g_0, \tilde{g}_1$ are nonzero constants.

$$\Gamma_2 = e^{(-1/2)tR_1}R_1\frac{\partial}{\partial u} - 2\tilde{g}_1 e^{(-1/2)tR_1}\frac{\partial}{\partial x},$$
$$\Gamma_3 = e^{(-1/2)tR_2}R_2\frac{\partial}{\partial u} - 2\tilde{g}_1 e^{(-1/2)tR_2}\frac{\partial}{\partial x},$$

where

$$R_1 = \gamma - \sqrt{4f_1\tilde{g}_1 + \gamma^2} \neq 0, \quad R_2 = \gamma + \sqrt{4f_1\tilde{g}_1 + \gamma^2} \neq 0$$

are arbitrary constants.

5 Symmetry reductions and exact solutions

In order to obtain symmetry reductions and exact solutions, one has to solve the associated Lagrange equations

$$\frac{dt}{\tau(t,x,u,)} = \frac{dx}{\xi(t,x,u)} = \frac{du}{\eta(t,x,u)}.$$

For symmetry reductions purposes we consider only those cases in which the equation (1) is nonlinear.

5.1 Case (B).

The linear combination of $\Gamma_1 + c\Gamma_2$ gives rise to the group-invariant solution

$$u = F(z) \tag{7}$$

where c is a non-zero constant, $z = x - ct$ is an invariant of the symmetry $\Gamma_1 + c\Gamma_2$ and $F(z)$ satisfies the third-order nonlinear ODE

$$\delta F'''(z) - \nu F''(z) - cF'(z) + g_0 F'(z) - g_1 F'(z)\ln(F(z))$$
$$+\gamma F(z) - f_0 = 0.$$

5.2 Case (C).

The symmetry $\Gamma_1 + c\Gamma_2$ gives rise to the group-invariant solution

$$u = F(z) \tag{8}$$

where $z = x - ct$ is an invariant of $\Gamma_1 + c\Gamma_2$ and $F(z)$ satisfies

$$\delta F'''(z) - \nu F''(z) - cF'(z) + \left(g_0 F(z) + F(z)^2 + g_1\right)F'(z)$$
$$+\gamma F(z) - f_0 = 0.$$

5.3 Case (D). *One-dimensional optimal system of subalgebras*

In this case we have three symmetries for the corresponding equation (1) and so we first obtain the optimal system of one-dimensional subalgebras and then present the optimal system of group-invariant solutions. We use the method given in [10]. The adjoint transformations are given by

$$\mathrm{Ad}(\exp(\varepsilon\Gamma_i))\Gamma_j = \Gamma_j - \varepsilon[\Gamma_i,\Gamma_j] + \frac{1}{2}\varepsilon^2[\Gamma_i,[\Gamma_i,\Gamma_j]] - \cdots,$$

where $[\Gamma_i, \Gamma_j]$ denotes the commutator of Γ_i and Γ_j defined as

$$[\Gamma_i,\Gamma_j] = \Gamma_i\Gamma_j - \Gamma_j\Gamma_i.$$

In Table 1 and Table 2, we give, respectively, the commutator table of the Lie point symmetries of the system (1) and the adjoint representations of the symmetry group of (1). These tables are then used to construct the optimal system of one-dimensional subalgebras for system (1).

Table 1. Commutator table of the Lie algebra of system (1)

	Γ_1	Γ_2	Γ_3
Γ_1	0	$-\frac{1}{2}R_1\Gamma_2$	$-\frac{1}{2}R_2\Gamma_3$
Γ_2	$\frac{1}{2}R_1\Gamma_2$	0	0
Γ_3	$\frac{1}{2}R_2\Gamma_3$	0	0

Table 2. Adjoint table of the Lie algebra of system (1)

Ad	Γ_1	Γ_2	Γ_3
Γ_1	Γ_1	$e^{(1/2)R_1\varepsilon}\Gamma_2$	$e^{(1/2)R_2\varepsilon}\Gamma_3$
Γ_2	$\Gamma_1 - \frac{1}{2}R_1\varepsilon\Gamma_2$	Γ_2	Γ_3
Γ_3	$\Gamma_1 - \frac{1}{2}R_2\varepsilon\Gamma_3$	Γ_3	Γ_3

Thus, from Tables 1 and 2 one can obtain an optimal system of one-dimensional subalgebras given by $\{\Gamma_1, \Gamma_3+\Gamma_2, \Gamma_3-\Gamma_2, \Gamma_3\}$.

5.3.1 Symmetry reductions and exact solutions based on the one-dimensional optimal system of subalgebras

Here we use the optimal system of one-dimensional subalgebras calculated above to obtain symmetry reductions that transform (1) into ordinary differential equations (ODEs). We then look for exact solutions of the ODEs.

Case (D.1) The symmetry Γ_1 gives rise to the group-invariant solution

$$u = F(z) \tag{9}$$

where $z = x$ is an invariant of the symmetry Γ_1 and $F(z)$ satisfies the ODE

$$\delta F'''(z) + g_0 F'(z) + \tilde{g}_1 F'(z) F(z) - \nu F''(z) + \gamma F(z) - f_1 z - f_0 = 0.$$

Case (D.2) The symmetry $\Gamma_3 + \Gamma_2$ gives us the group-invariant solution

$$\begin{aligned} u(t,x) = {} & \frac{1}{2\tilde{g}_1(e^{(-1/2)tP_1} + e^{(1/2)tP_1})}\Big\{2F(z)\tilde{g}_1 e^{(-1/2)tP_1} \\ & +2F(z)\tilde{g}_1 e^{(1/2)tP_1} - e^{(-1/2)tP_1}P_1 x - e^{(-1/2)tP_1}\gamma x \\ & +P_1 e^{(1/2)tP_1}x - e^{(1/2)tP_1}\gamma x\Big\}, \end{aligned} \tag{10}$$

where $P_1 = \sqrt{4f_1\tilde{g}_1 + \gamma^2}$ is a non-zero arbitrary constant, $z = t$ is an invariant of $\Gamma_3 + \Gamma_2$ and the function $F(z)$ satisfies the ODE

$$\begin{aligned} & -F(z)e^{-(1/2)zP_1}P_1\tilde{g}_1 + \gamma F(z)e^{-(1/2)zP_1}\tilde{g}_1 \\ & +F(z)P_1 e^{(1/2)zP_1}\tilde{g}_1 + \gamma F(z)e^{(1/2)zP_1}\tilde{g}_1 \\ & +2(F'(z))e^{-(1/2)zP_1}\tilde{g}_1 - g_0 e^{-(1/2)zP_1}P_1 \\ & -g_0 e^{-(1/2)zP_1}\gamma - 2e^{-(1/2)zP_1} f_0\tilde{g}_1 \\ & +2(F'(z))e^{(1/2)zP_1}\tilde{g}_1 + g_0 P_1 e^{(1/2)zP_1} - g_0 e^{(1/2)zP_1}\gamma \\ & -2e^{(1/2)zP_1} f_0\tilde{g}_1 = 0 \end{aligned}$$

whose solution is

$$\begin{aligned} F(z) = {} & \Big\{\Big[\frac{(-P_1 g_0 + 2f_0\tilde{g}_1 + \gamma g_0)e^{(1/2)(\gamma-P_1)z}}{\tilde{g}_1(\gamma+P_1)} \\ & + \frac{(P_1 g_0 + 2f_0\tilde{g}_1 + \gamma g_0)e^{(1/2)(\gamma-3P_1)z}}{\tilde{g}_1(\gamma-P_1)}\Big]e^{zP_1} \\ & +C_1\Big\}e^{-(1/2)z(\gamma+P_1)}(e^{-zP_1}+1)^{-1}, \end{aligned}$$

where $P_1 \neq \pm\gamma$ and C_1 is an arbitrary constant. Consequently the required group invariant solution is completed by (10).

Case (D.3) The symmetry $\Gamma_3 - \Gamma_2$ gives rise to the group-invariant solution of the form

$$\begin{aligned} u(t,x) = {} & \frac{1}{2g_1(e^{-(1/2)tP_1} - e^{(1/2)tP_1})}\Big\{2F(z)g_1 e^{-(1/2)tP_1} \\ & -2F(z)g_1 e^{(1/2)tP_1} - e^{-(1/2)tP_1}P_1 x - e^{-(1/2)tP_1}\gamma x \\ & -P_1 e^{(1/2)tP_1}x + e^{(1/2)tP_1}\gamma x\Big\}, \end{aligned} \tag{11}$$

where $z = t$ is an invariant of $\Gamma_3 - \Gamma_2$ and the function $F(z)$ satisfies

$$\begin{aligned} & -F(z)e^{-(1/2)zP_1}P_1 g_1 + \gamma F(z)e^{-(1/2)zP_1}g_1 \\ & -F(z)P_1 e^{(1/2)zP_1}g_1 - \gamma F(z)e^{(1/2)zP_1}g_1 \\ & +2(F'(z))e^{-(1/2)zP_1}g_1 - g_0 e^{-(1/2)zP_1}P_1 \\ & -g_0 e^{-(1/2)zP_1}\gamma - 2e^{-(1/2)zP_1} f_0 g_1 \\ & -2(F'(z))e^{(1/2)zP_1}g_1 - g_0 P_1 e^{(1/2)zP_1} \\ & +g_0 e^{(1/2)zP_1}\gamma + 2e^{(1/2)zP_1} f_0 g_1 = 0 \end{aligned}$$

whose solution is

$$\begin{aligned} F(z) = {} & \Big\{\Big[\frac{(g_0 P_1 + 2f_0\tilde{g}_1 + \gamma g_0)\mathrm{e}^{(1/2)(\gamma-3P_1)z}}{\tilde{g}_1(\gamma-P_1)} \\ & - \frac{(-g_0 P_1 + 2f_0\tilde{g}_1 + \gamma g_0)\mathrm{e}^{(1/2)(\gamma-P_1)z}}{\tilde{g}_1(\gamma+P_1)}\Big]\mathrm{e}^{zP_1} \\ & +B_1\Big\}\mathrm{e}^{-(1/2)z(\gamma+P_1)}(\mathrm{e}^{-zP_1}-1)^{-1} \end{aligned}$$

where $P_1 \neq \pm\gamma$ and B_1 is an arbitrary constant. Consequently the group-invariant solution is completed by (11).

Case (D.4) The symmetry Γ_3 gives the group-invariant solution

$$u(t,x) = \frac{2F(z)\tilde{g}_1 - P_1 x - x\gamma}{2\tilde{g}_1} \tag{12}$$

where $z = t$ is an invariant of Γ_3 and the function $F(z)$ satisfies

$$F(z)\gamma\tilde{g}_1 - F(z)P_1\tilde{g}_1 + 2\left(F'(z)\right)\tilde{g}_1 - g_0 P_1 - g_0\gamma - 2f_0\tilde{g}_1 = 0$$

whose solution is given by

$$F(z) = e^{-(1/2)(\gamma-P_1)z}C_1 + \frac{g_0 P_1 + 2f_0\tilde{g}_1 + g_0\gamma}{\tilde{g}_1(\gamma-P_1)}$$

and consequently the group-invariant solution is completed by (12).

6 Conclusion

Lie group classification was performed on a generalized Korteweg-de Vries-Burgers equation (1). The functional forms of (1) of the type linear, quadratic, exponential and logarithmic were obtained. The Lie algebra obtained was of dimension two, three and infinite. For the case when the principal Lie algebra was extended by two symmetries, one-dimensional optimal system of subalgebras was obtained and the corresponding group-invariant solutions were derived. The functional forms obtained in this paper, can be chosen to suit physical phenomena modelled by the resulting equations. The symmetry reductions and exact solutions found in this work can be used to model practical problems of physical interest and also serve as benchmarks against numerical integrators.

Acknowledgement

ARA is grateful to the National Research Foundation of South Africa for the generous financial support. MM acknowledges the financial support from the North-West University, Mafikeng Campus through the post-doctoral fellowship.

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