

Classical and Bayesian Statistical Inference of Extropy Measures for the Inverse Garhy Distribution

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Received: 15 May 2026, Revised: 13 Jul. 2026, Accepted: 19 Aug. 2026

Published online: 1 Nov. 2026

Abstract: In this article, we develop and investigate a new one-parameter distribution called the inverse Garhy distribution (IGD). One of the cooler things in this article is the treatment of extropy based on uncertainty measures. Extropy, weighted extropy, residual extropy, weighted residual extropy, past extropy, and weighted past extropy are computed in closed-form. Some important statistical and mathematical properties of the newly defined IGD, including mode, quantile function, moments, mean, incomplete moments, inverse moments, and stress-strength reliability, are explored and computed. Statistical inference for unknown parameter and extropy measures is performed using maximum likelihood, maximum product spacing, and Bayesian methods. The IGD is trained on a medical dataset and compared against some well-known inverse statistical distributions using likelihood-based criteria and goodness-of-fit statistics. The IGD outperformed all other models. The IGD is analytically tractable, performs well on real data, and supplies a valuable new tool for modeling positively skewed lifetime data and investigating extropy-based uncertainty in survival and reliability analysis.

Keywords: Garhy distribution; stress-strength reliability; Extropy; Weighted extropy; Bayesian; Maximum likelihood estimation; Survival analysis

1 Introduction

Entropy has long been the workhorse of information theory. Since Shannon's foundational work, we have used it to measure uncertainty, compress data, and reason about probability distributions in almost every corner of statistics. But entropy is not the whole narrative, and I think it is important to be upfront about that. In 2015, Lad et al. proposed extropy [39], which they called the dual of entropy. The concept is simple: where Shannon entropy measures the average surprise of the results of a distribution, extropy measures what the pdf itself adds. For the continuous random variable X with PDF $f(x)$, the extropy is said to be $\mathfrak{W}_1(X) = -\frac{1}{2} \int_0^\infty f^2(x; \alpha) dx$. Extropy is more sensitive to the tail behavior of a distribution, which makes it genuinely useful in contexts where the tails are the whole point: reliability engineering, survival

analysis, extreme-value modeling. It also has clean theoretical properties—it is non-negative, it is maximized by the uniform distribution, and it connects naturally to divergence measures. Recent work has shown that extropy performs well in statistical inference, pattern recognition, and risk assessment, particularly when one cares both about variability and about where the probability mass is concentrated. Some authors found it especially useful as a criterion for model selection and goodness-of-fit testing, where entropy-based measures sometimes fail to distinguish distributions that differ mainly in tail shape.

In the other hand, this paper regards inverse distributions. The concept is ancient and basic: if Y is a positive random variable, then set $X = 1/Y$, the distribution of which is known as the inverse distribution of Y . What makes this transformation interesting to revisit is its utility. No new parameters are introduced. The nice

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mathematical properties of the original distribution remain in tractable forms. But its resulting family can fit data that the original distribution handles poorly—specifically, data that have heavy right tails or bathtub-shaped hazard functions, both of which occur naturally in lifetime and reliability studies. Table 1 summarizes a few representative examples of distributions constructed in this manner. Not an exhaustive list, but it demonstrates how widely the inverse transformation has been applied. Inverted lifetime models have also been used in step-stress and stress-strength reliability studies [6, 11], alongside other recent one- and two-parameter lifetime models [7, 34, 49].

Table 1: A literature of some Inverse Distributions

Distribution	Citation
Inverse power Gompertz distribution	[3]
Inverse Gompertz distribution	[26]
Inverse power Muth distribution	[22]
Inverse Kumaraswamy distribution	[2]
Inverted Topp-Leone distribution	[43]
Inverse Kies distribution	[38]
Inverse power Akash distribution	[30]
Inverse Sushila distribution	[5]
Inverse Unit Teissier distribution	[15]
Inverse Unit Exponential distribution	[14]
Inverse Gompertz Fréchet distribution	[8]
Inverted Exponential Weibull distribution	[40]
Inverse Nakagami-m distribution	[41]
Inverted Nadarajah-Haghighi distribution	[53]
Inverse x-gamma distribution	[54]
Inverse Power Cauchy distribution	[51]
Inverse Weighted Lindley distribution	[48]
Inverse Power Ramos-Louzada distribution	[13]
Inverse Exponential Power distribution	[21]
Inverted Power Ishita distribution	[31]
Inverted Power Rama distribution	[46]
Inverse Power Maxwell distribution	[12]
Inverse Power Two-Parameter Weighted Lindley distribution	[28]
Inverse-Power Logistic-Exponential distribution	[16]
Inverse Power Zeghdoudi model	[23]
Inverse Maxwell distribution	[45]

The aim of the present article is to bring these two threads together. We study a new statistical distribution constructed by applying the inverse transformation to a standard baseline family, and we examine its entropy along with the classical information-theoretic and inferential properties. The article is organized and structured as follows: Section 2 introduces the distribution and derives its fundamental characteristics. Section 3 develops the entropy and related information measures. Section 4 studies some important statistical properties of the IGD. Section 5 considers bayesian and classical estimation approaches. Some simulation results are provided in Section 6. Section 7 presents applications to real data. Finally, some concluding remarks are mentioned in Section 8.

2 Model Formulation

Elgarhy [25] introduced the Garhy distribution (GD), and its stress-strength and progressively censored reliability inference were subsequently developed in [9, 10]. The cumulative distribution function (CDF) and probability density function (PDF) of the GD are defined, respectively, as below

$$G(y; \alpha) = 1 - \left[\frac{3\alpha^2 y^2}{6 + \alpha^3} + (\alpha y + 1) \right] e^{-\alpha y} \quad y > 0, \alpha > 0, \quad (1)$$

and

$$g(y; \alpha) = \frac{\alpha^3 y}{6 + \alpha^3} (\alpha^2 + 3y) e^{-\alpha y} \quad y > 0, \alpha > 0, \quad (2)$$

Applying the inverse transformation $X = Y^{-1}$ to the CDF (1), we have a new one-parameter distribution called inverse GD (IGD). The CDF and PDF of IGD are provided via

$$F(x; \alpha) = e^{-\frac{\alpha}{x}} \left(\alpha x^{-2} \left(\frac{3\alpha}{\alpha^3 + 6} + x \right) + 1 \right), \quad (3)$$

$x > 0, \alpha > 0$, and

$$f(x; \alpha) = \frac{\alpha^3 x^{-4} (\alpha^2 x + 3)}{\alpha^3 + 6} e^{-\frac{\alpha}{x}}. \quad (4)$$

The survival function (SF), hazard rate function (HRF), reversed HRF and cumulative HRF of the IGD are provided via

$$S(x; \alpha) = 1 - e^{-\frac{\alpha}{x}} \left(\alpha x^{-2} \left(\frac{3\alpha}{\alpha^3 + 6} + x \right) + 1 \right),$$

$$h(x; \alpha) = \frac{\frac{\alpha^3 x^{-4} (\alpha^2 x + 3)}{\alpha^3 + 6}}{e^{-\frac{\alpha}{x}} - \left(\alpha x^{-2} \left(\frac{3\alpha}{\alpha^3 + 6} + x \right) + 1 \right)}$$

$$\tau(x; \alpha) = \frac{\alpha^3 x^{-4} (\alpha^2 x + 3)}{(\alpha^3 + 6) \left(\alpha x^{-2} \left(\frac{3\alpha}{\alpha^3 + 6} + x \right) + 1 \right)},$$

and

$$H(x; \alpha) = -\ln \left[1 - e^{-\frac{\alpha}{x}} \left(\alpha x^{-2} \left(\frac{3\alpha}{\alpha^3 + 6} + x \right) + 1 \right) \right].$$

Figures 1 and 2 show the PDF and HRF for IGD. From these figures it can be observe that the PDF can be unimodal or right skewed but the HRF cab be up-side-down.

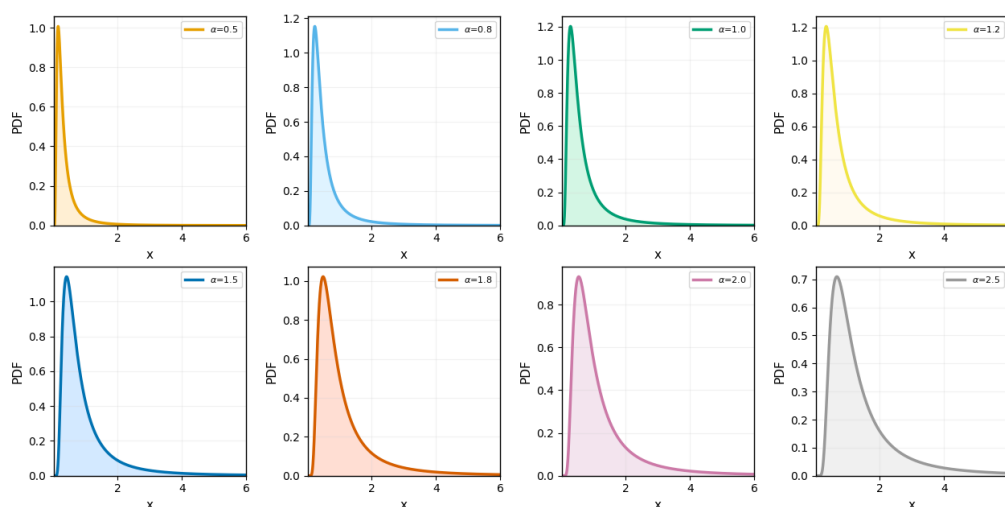


Fig. 1: PDF plots for the IGD

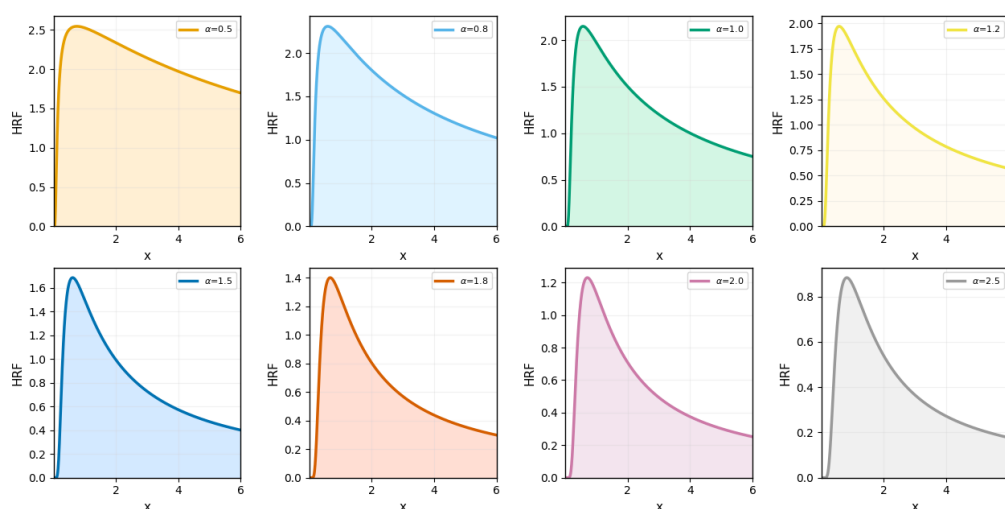


Fig. 2: HRF plots for the IGD

3 Measures of Extropy

Extropy is another significant uncertainty measure in information theory. It also an excellent complements entropy by quantifying what entropy tends to overlook—peakedness behavior, where lifetime data typically tell its most interesting stories. The measure has impotent uses in reliability engineering, survival analysis, lifetime statistical distributions more broadly, as well as dynamic notions of uncertainty that evolve over time or upon conditioning. In this section, we derive six extropy-based functionals for the proposed distribution.

These are: Extropy (EX), Weighted Extropy (WEX), Residual Extropy (REX), Weighted Residual Extropy (WREX), Past Extropy (PEX), Weighted Past Extropy (WPEX).

3.1 Extropy

Lad et al. [39] proposed EX, a double complement of traditional entropy, as an alternative measure of uncertainty. The EX of X is expressed as follows

$$\varpi_1(X) = -\frac{1}{2} \int_0^\infty f^2(x; \alpha) dx. \quad (5)$$

The EX of the IGD can be calculated by inserting PDF (4) into Equation (5) as follows

$$\begin{aligned} \varpi_1(X) &= \frac{-\alpha^6}{2(\alpha^3+6)^2} \int_0^\infty x^{-8} (\alpha^4 x^2 + 6\alpha^2 x + 9) e^{-2\alpha/x} dx \\ &= \frac{-\alpha^6}{2(\alpha^3+6)^2} \int_0^\infty (\alpha^4 x^{-6} + 6\alpha^2 x^{-7} + 9x^{-8}) \\ &\quad \times e^{-2\alpha/x} dx. \end{aligned}$$

Let $z = 2\alpha/x$, then

$$\begin{aligned} \varpi_1(X) &= \frac{-1}{256\alpha(\alpha^3+6)^2} \\ &\quad \times \int_0^\infty (4\alpha^6 z^4 + 12\alpha^3 z^5 + 9z^6) e^{-z} dz \\ &= \frac{-3(2\alpha^6 + 30\alpha^3 + 135)}{16\alpha(\alpha^3+6)^2}. \end{aligned}$$

3.2 Weighted extropy

Balakrishnan et al. [17] introduced the concept of WEX as a substitute for traditional entropy. Unlike entropy, WEX provides a different perspective on uncertainty by giving higher values in a probability distribution more weight. It is depicted as

$$\varpi_2(X) = -\frac{1}{2} \int_0^\infty x f^2(x; \alpha) dx. \quad (6)$$

The WEX of the IGD can be expressed by inserting PDF (4) into Equation (6) as follows

$$\begin{aligned} \varpi_2(X) &= \frac{-\alpha^6}{2(\alpha^3+6)^2} \int_0^\infty x^{-7} (\alpha^4 x^2 + 6\alpha^2 x + 9) e^{-2\alpha/x} dx \\ &= \frac{-\alpha^6}{2(\alpha^3+6)^2} \int_0^\infty (\alpha^4 x^{-5} + 6\alpha^2 x^{-6} + 9x^{-7}) \\ &\quad \times e^{-2\alpha/x} dx. \end{aligned}$$

Let $z = 2\alpha/x$, then

$$\begin{aligned} \varpi_2(X) &= \frac{-1}{128\alpha(\alpha^3+6)^2} \\ &\quad \times \int_0^\infty (4\alpha^6 z^3 + 12\alpha^3 z^4 + 9z^5) e^{-z} dz \\ &= \frac{-3(\alpha^6 + 12\alpha^3 + 45)}{16(\alpha^3+6)^2}. \end{aligned}$$

3.3 Residual extropy

The REX defined by Qiu and Jia [47] and can be computed as follows:

$$\varpi_3(X) = -\frac{1}{2\bar{F}^2(t; \alpha)} \int_t^\infty f^2(x; \alpha) dx, \quad t \geq 0. \quad (7)$$

The REX of the IGD can be expressed by inserting PDF (4) into Equation (7) as follows

$$\begin{aligned} \varpi_3(X) &= \frac{-\alpha^6}{2(\alpha^3+6)^2 \bar{F}^2(t; \alpha)} \\ &\quad \times \int_t^\infty x^{-8} (\alpha^4 x^2 + 6\alpha^2 x + 9) e^{-2\alpha/x} dx \\ &= \frac{-\alpha^6}{2(\alpha^3+6)^2 \bar{F}^2(t; \alpha)} \\ &\quad \times \int_t^\infty (\alpha^4 x^{-6} + 6\alpha^2 x^{-7} + 9x^{-8}) e^{-2\alpha/x} dx. \end{aligned}$$

Let $z = 2\alpha/x$, then

$$\begin{aligned} \varpi_3(X) &= \frac{-1}{256\alpha(\alpha^3+6)^2 \bar{F}^2(t; \alpha)} \\ &\quad \times \int_0^{\frac{2\alpha}{t}} (4\alpha^6 z^4 + 12\alpha^3 z^5 + 9z^6) e^{-z} dz \quad (8) \\ &= \frac{-(4\alpha^6 \gamma(5, t) + 12\alpha^3 \gamma(6, t) + 9\gamma(7, t))}{256\alpha(\alpha^3+6)^2 \bar{F}^2(t; \alpha)}, \end{aligned}$$

where $\gamma(\cdot, \cdot)$ is the lower incomplete gamma function.

3.4 Weighted residual extropy

The WREX defined by Sathar and Nair [4] and can be computed as follows:

$$\varpi_4(X) = -\frac{1}{2\bar{F}^2(t; \alpha)} \int_t^\infty x f^2(x; \alpha) dx, \quad t \geq 0. \quad (9)$$

The WREX of the IGD can be expressed by inserting PDF (4) into Equation (9) as follows

$$\begin{aligned} \varpi_4(X) &= \frac{-\alpha^6}{2(\alpha^3+6)^2 \bar{F}^2(t; \alpha)} \\ &\quad \times \int_t^\infty x^{-7} (\alpha^4 x^2 + 6\alpha^2 x + 9) e^{-2\alpha/x} dx \\ &= \frac{-\alpha^6}{2(\alpha^3+6)^2 \bar{F}^2(t; \alpha)} \\ &\quad \times \int_t^\infty (\alpha^4 x^{-5} + 6\alpha^2 x^{-6} + 9x^{-7}) e^{-2\alpha/x} dx. \end{aligned}$$

Let $z = 2\alpha/x$, then

$$\begin{aligned} \varpi_4(X) &= \frac{-1}{128(\alpha^3+6)^2 \bar{F}^2(t; \alpha)} \\ &\quad \times \int_0^{\frac{2\alpha}{t}} (4\alpha^6 z^3 + 12\alpha^3 z^4 + 9z^5) e^{-z} dz \quad (10) \\ &= \frac{-(4\alpha^6 \gamma(4, t) + 12\alpha^3 \gamma(5, t) + 9\gamma(6, t))}{128(\alpha^3+6)^2 \bar{F}^2(t; \alpha)}, \end{aligned}$$

3.5 Past extropy

The PEX defined by Krishnan et al. [37] and can be computed as follows:

$$\varpi_5(X) = -\frac{1}{2F^2(t; \alpha)} \int_0^t f^2(x; \alpha) dx, \quad t \geq 0. \quad (11)$$

The PEX of the IGD can be expressed by inserting PDF (4) into Equation (11) as follows

$$\begin{aligned} \varpi_5(X) &= \frac{-\alpha^6}{2(\alpha^3 + 6)^2 F^2(t; \alpha)} \\ &\times \int_0^t x^{-8} (\alpha^4 x^2 + 6\alpha^2 x + 9) e^{-2\alpha/x} dx \\ &= \frac{-\alpha^6}{2(\alpha^3 + 6)^2 F^2(t; \alpha)} \\ &\times \int_0^t (\alpha^4 x^{-6} + 6\alpha^2 x^{-7} + 9x^{-8}) e^{-2\alpha/x} dx. \end{aligned}$$

Let $z = 2\alpha/x$, then

$$\begin{aligned} \varpi_5(X) &= \frac{-1}{256\alpha(\alpha^3 + 6)^2 F^2(t; \alpha)} \\ &\times \int_{\frac{2\alpha}{t}}^{\infty} (4\alpha^6 z^4 + 12\alpha^3 z^5 + 9z^6) e^{-z} dz \\ &= \frac{-(4\alpha^6 \Gamma(5, t) + 12\alpha^3 \Gamma(6, t) + 9\Gamma(7, t))}{256\alpha(\alpha^3 + 6)^2 F^2(t; \alpha)}, \end{aligned} \quad (12)$$

where $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function.

3.6 Weighted past extropy

The WPEX defined by Sathar and Nair [4] and can be computed as follows:

$$\varpi_6(X) = -\frac{1}{2F^2(t; \alpha)} \int_0^t x f^2(x; \alpha) dx, \quad t \geq 0 \quad (13)$$

$$\begin{aligned} \varpi_6(X) &= \frac{-\alpha^6}{2(\alpha^3 + 6)^2 F^2(t; \alpha)} \\ &\times \int_0^t x^{-7} (\alpha^4 x^2 + 6\alpha^2 x + 9) e^{-2\alpha/x} dx \\ &= \frac{-\alpha^6}{2(\alpha^3 + 6)^2 F^2(t; \alpha)} \\ &\times \int_0^t (\alpha^4 x^{-5} + 6\alpha^2 x^{-6} + 9x^{-7}) e^{-2\alpha/x} dx. \end{aligned}$$

Let $z = 2\alpha/x$, then

$$\begin{aligned} \varpi_6(X) &= \frac{-1}{128(\alpha^3 + 6)^2 F^2(t; \alpha)} \\ &\times \int_{\frac{2\alpha}{t}}^{\infty} (4\alpha^6 z^3 + 12\alpha^3 z^4 + 9z^5) e^{-z} dz \\ &= \frac{-(4\alpha^6 \Gamma(4, t) + 12\alpha^3 \Gamma(5, t) + 9\Gamma(6, t))}{128(\alpha^3 + 6)^2 F^2(t; \alpha)}, \end{aligned} \quad (14)$$

Table 2 presents numerical values of several extropy measures for the IGD distribution across different values of the parameter α , evaluated at two selected time points, $t = 10$ and $t = 12$. Both EX and WEX decrease slowly with α , showing that these measures are sensitive to the shape parameter. The conditional measures REX and WREX are fairly stable with some monotonic decrease with increasing α . By comparison, the measures PEX and WPEX decline more steeply in magnitude, exhibiting greater sensitivity to α . Comparing the two time points, the values at $t = 12$ are generally slightly more negative than those at $t = 10$, suggesting an increase in uncertainty over time. Overall, the results demonstrate consistent and coherent behavior of all extropy measures under variations in both α and time.

4 Statistical properties

In this section, we discuss and compute some important statistical and mathematical properties of the IGD.

4.1 Mode

Finding the mode means finding the value of x that maximizes the density function. For the IGD, it is easier to work with the log-density. Write

$$\ln f(x; \alpha) = \ln \left(\frac{\alpha^3}{\alpha^3 + 6} \right) - 4 \ln x + \ln(\alpha^2 x + 3) - \frac{\alpha}{x}.$$

Differentiate once:

$$\frac{d}{dx} \ln f(x; \alpha) = -\frac{4}{x} + \frac{\alpha^2}{\alpha^2 x + 3} + \frac{\alpha}{x^2}.$$

Set the derivative to zero to locate critical points:

$$-\frac{4}{x} + \frac{\alpha^2}{\alpha^2 x + 3} + \frac{\alpha}{x^2} = 0.$$

Multiply through by $x^2(\alpha^2 x + 3)$ to clear denominators:

$$-4x(\alpha^2 x + 3) + \alpha^2 x^2 + \alpha(\alpha^2 x + 3) = 0.$$

Expand and collect terms:

$$3\alpha^2 x^2 + (12 - \alpha^3)x - 3\alpha = 0.$$

Solving this quadratic and taking the positive root (since $x > 0$) yields the mode:

$$x_{\text{mode}} = \frac{-(12 - \alpha^3) + \sqrt{(12 - \alpha^3)^2 + 36\alpha^3}}{6\alpha^2}.$$

Table 2: Numerical values of entropy measures for the IGD distribution at different values of α and selected time points $t = 10$ and $t = 12$.

α	EX	WEX	$t = 10$				$t = 12$			
			REX	WREX	PEX	WPEX	REX	WREX	PEX	WPEX
5.0	-0.0768	-0.1876	-0.0339	-0.4389	-0.0917	-0.2210	-0.0290	-0.4486	-0.0874	-0.2118
5.5	-0.0694	-0.1876	-0.0333	-0.4329	-0.0858	-0.2273	-0.0286	-0.4434	-0.0810	-0.2165
6.0	-0.0634	-0.1875	-0.0327	-0.4270	-0.0811	-0.2339	-0.0282	-0.4384	-0.0759	-0.2215
6.5	-0.0583	-0.1875	-0.0321	-0.4213	-0.0774	-0.2409	-0.0278	-0.4334	-0.0718	-0.2267
7.0	-0.0540	-0.1875	-0.0315	-0.4156	-0.0744	-0.2481	-0.0274	-0.4286	-0.0685	-0.2321
7.5	-0.0504	-0.1875	-0.0310	-0.4101	-0.0720	-0.2556	-0.0270	-0.4238	-0.0657	-0.2378
8.0	-0.0471	-0.1875	-0.0304	-0.4047	-0.0701	-0.2633	-0.0266	-0.4191	-0.0635	-0.2436
8.5	-0.0443	-0.1875	-0.0299	-0.3994	-0.0686	-0.2713	-0.0262	-0.4144	-0.0615	-0.2497
9.0	-0.0418	-0.1875	-0.0293	-0.3941	-0.0674	-0.2795	-0.0258	-0.4099	-0.0599	-0.2559
9.5	-0.0396	-0.1875	-0.0288	-0.3890	-0.0664	-0.2878	-0.0254	-0.4054	-0.0586	-0.2623
10.0	-0.0376	-0.1875	-0.0283	-0.3840	-0.0657	-0.2964	-0.0250	-0.4009	-0.0575	-0.2689
10.5	-0.0358	-0.1875	-0.0278	-0.3790	-0.0651	-0.3051	-0.0247	-0.3966	-0.0566	-0.2756
11.0	-0.0342	-0.1875	-0.0273	-0.3742	-0.0647	-0.3140	-0.0243	-0.3923	-0.0558	-0.2825
11.5	-0.0327	-0.1875	-0.0268	-0.3694	-0.0645	-0.3230	-0.0239	-0.3880	-0.0552	-0.2895
12.0	-0.0313	-0.1875	-0.0263	-0.3647	-0.0644	-0.3322	-0.0236	-0.3838	-0.0547	-0.2966
12.5	-0.0300	-0.1875	-0.0259	-0.3601	-0.0643	-0.3415	-0.0232	-0.3797	-0.0543	-0.3038
13.0	-0.0289	-0.1875	-0.0254	-0.3556	-0.0644	-0.3509	-0.0229	-0.3757	-0.0540	-0.3112
13.5	-0.0278	-0.1875	-0.0250	-0.3512	-0.0646	-0.3604	-0.0225	-0.3717	-0.0538	-0.3186
14.0	-0.0268	-0.1875	-0.0245	-0.3468	-0.0648	-0.3701	-0.0222	-0.3677	-0.0537	-0.3262
14.5	-0.0259	-0.1875	-0.0241	-0.3426	-0.0651	-0.3799	-0.0219	-0.3639	-0.0536	-0.3339
15.0	-0.0250	-0.1875	-0.0236	-0.3384	-0.0655	-0.3897	-0.0216	-0.3600	-0.0536	-0.3416

Now we must check that this critical point actually gives a maximum. The second derivative of the log-density is

$$\frac{d^2}{dx^2} \ln f(x; \alpha) = \frac{4}{x^2} - \frac{\alpha^4}{(\alpha^2 x + 3)^2} - \frac{2\alpha}{x^3}.$$

Substituting the value of x_{mode} into this expression (a routine but lengthy calculation) shows that the second derivative is negative for every $\alpha > 0$. Hence the critical point is indeed a maximum, and the IGD is unimodal. For illustration, when $\alpha = 1$ we get $x_{\text{mode}} \approx 0.308$; when $\alpha = 2$ we get $x_{\text{mode}} \approx 0.200$. As α increases, the mode shifts toward zero.

4.2 Quantile Function

The p -th quantile $Q(p)$ of the IGD is defined implicitly by

$$F(Q(p); \alpha) = p, \quad 0 < p < 1. \quad (15)$$

Using the explicit form of the CDF given in (3), we have

$$e^{-\alpha/x} \left(1 + \frac{\alpha}{x} + \frac{3\alpha^2}{(\alpha^3 + 6)x^2} \right) = p, \quad (16)$$

where x is the quantile value (i.e. $x = Q(p)$). Equation (16) is the equation we need to solve for x given p and α . There is no closed-form solution for x in (16) because the variable appears both polynomially and inside the exponential. However, the left-hand side is strictly decreasing in x for $x > 0$, so a numerical root reliably. In practice, for each p we locate x by solving

$$e^{-\alpha/x} \left(1 + \frac{\alpha}{x} + \frac{3\alpha^2}{(\alpha^3 + 6)x^2} \right) - p = 0. \quad (17)$$

Once the quartiles $Q(1/4), Q(1/2), Q(3/4)$ and the octiles $Q(1/8), Q(3/8), Q(5/8), Q(7/8)$ are computed numerically, they are used in the shape coefficients below.

4.2.1 Bowley's Coefficient of Skewness

Moment-based skewness is not defined when $E[X^3]$ diverges – which happens for the IGD. Bowley's quartile skewness [20] provides a robust alternative:

$$SK_B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}. \quad (18)$$

For the IGD, the denominator (the interquartile range) is positive for all $\alpha > 0$, so SK_B is always well defined and lies in $[-1, 1]$ and is strictly positive for the IGD.

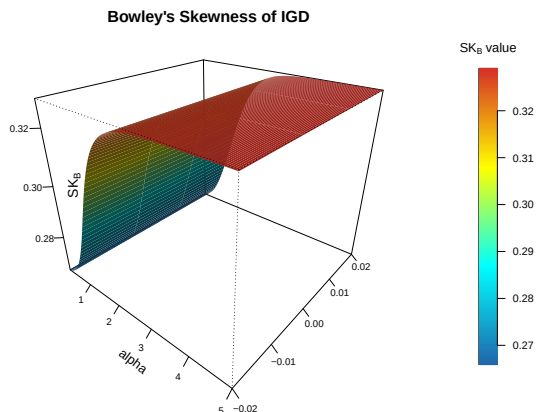


Fig. 3: Bowley's skewness of the IGD as a function of $\alpha \in (0.2, 5]$.

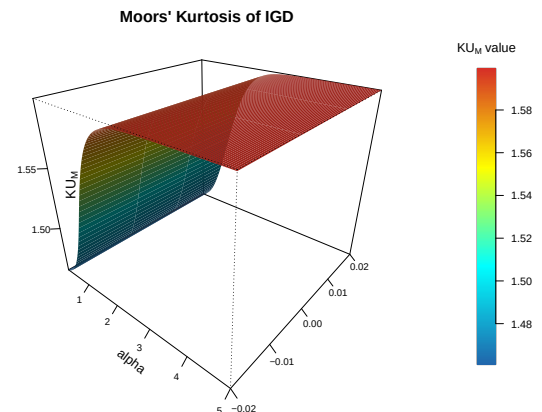


Fig. 4: Moors' kurtosis of the IGD as a function of $\alpha \in (0.2, 5]$.

The coefficient is positive for every $\alpha > 0$, confirming that the IGD is right-skewed over its entire parameter space. As α increases, SK_B approaches zero, meaning the distribution becomes more nearly symmetric for large α .

4.2.2 Moors' Coefficient of Kurtosis

The same moment problem affects the conventional kurtosis. Moors [42] proposed an octile-based measure:

$$KU_M = \frac{[Q(7/8) - Q(5/8)] + [Q(3/8) - Q(1/8)]}{Q(3/4) - Q(1/4)}. \quad (19)$$

The numerator sums the widths of two outer octile intervals, capturing tail weight. The denominator (IQR) normalises the scale. For the normal distribution $KU_M \approx 1.233$; larger values indicate heavier tails.

All computed values lie above 1.233, showing that the IGD has heavier tails than the normal distribution for any $\alpha > 0$. The coefficient decreases monotonically with α , indicating that tail heaviness diminishes – but never vanishes – as α grows.

4.3 Moments

Using the PDF of the IGD given in (4), the r -th moment $\mu'_r = E(X^r)$ can be derived. For $r < 2$, the moment exists and takes the form:

$$\mu'_r = \frac{\alpha^r ((\alpha^3 + 6) - 3r) \Gamma(2-r)}{\alpha^3 + 6}, \quad r < 2, \quad (20)$$

where $\Gamma(\cdot)$ is the gamma function. To obtain this, we start from the definition of the r -th moment and substitute the

density:

$$\begin{aligned} \mu'_r &= \int_0^\infty x^r f(x; \alpha) dx \\ &= \frac{\alpha^3}{\alpha^3 + 6} \int_0^\infty x^{r-4} (\alpha^2 x + 3) e^{-\alpha/x} dx. \end{aligned}$$

Splitting the integral into two parts gives

$$\begin{aligned} \mu'_r &= \frac{\alpha^5}{\alpha^3 + 6} \int_0^\infty x^{r-3} e^{-\alpha/x} dx \\ &\quad + \frac{3\alpha^3}{\alpha^3 + 6} \int_0^\infty x^{r-4} e^{-\alpha/x} dx. \end{aligned}$$

Now set $u = \alpha/x$. The limits flip, and after a little algebra we obtain

$$\begin{aligned} \int_0^\infty x^{r-3} e^{-\alpha/x} dx &= \alpha^{r-2} \int_0^\infty u^{-r+1} e^{-u} du \\ &= \alpha^{r-2} \Gamma(2-r), \end{aligned}$$

provided $2-r > 0$ (i.e. $r < 2$). Similarly,

$$\begin{aligned} \int_0^\infty x^{r-4} e^{-\alpha/x} dx &= \alpha^{r-3} \int_0^\infty u^{-r+2} e^{-u} du \\ &= \alpha^{r-3} \Gamma(3-r). \end{aligned}$$

Since $\Gamma(3-r) = (2-r)\Gamma(2-r)$, the second integral becomes $\alpha^{r-3}(2-r)\Gamma(2-r)$. Substituting back,

$$\begin{aligned} \mu'_r &= \frac{\alpha^5}{\alpha^3 + 6} \alpha^{r-2} \Gamma(2-r) \\ &\quad + \frac{3\alpha^3}{\alpha^3 + 6} \alpha^{r-3} (2-r) \Gamma(2-r). \end{aligned}$$

Factoring $\frac{\alpha^r \Gamma(2-r)}{\alpha^3 + 6}$ yields

$$\mu'_r = \frac{\alpha^r \Gamma(2-r)}{\alpha^3 + 6} [\alpha^3 + 3(2-r)].$$

Finally, $\alpha^3 + 3(2-r) = \alpha^3 + 6 - 3r$, which leads directly to (20). Setting $r = 1$ in the general formula gives the mean. Since $\Gamma(1) = 1$, we obtain:

$$\begin{aligned}\mu = E(X) &= \frac{\alpha((\alpha^3 + 6) - 3)}{\alpha^3 + 6} \\ &= \frac{\alpha(\alpha^3 + 3)}{\alpha^3 + 6}.\end{aligned}$$

The mean is always positive and increases with α . For $r = 2$, the moment formula does not apply because the integral diverges. Hence, $E(X^2)$ is not finite, and the variance of the IGD is infinite. This behavior is characteristic of distributions with tails that are not sufficiently light to guaranty a finite second moment.

4.4 Incomplete Moments

Often in reliability and economics, we need the incomplete moment instead of the ordinary one. This is defined in the usual way as

$$m_r(t) = \int_0^t x^r f(x; \alpha) dx, \quad t > 0.$$

If we plug the IGD density from (4) into this, we get

$$m_r(t) = \frac{\alpha^3}{\alpha^3 + 6} \int_0^t x^{r-4} (\alpha^2 x + 3) e^{-\alpha/x} dx.$$

We can split the integral into two parts:

$$\begin{aligned}m_r(t) &= \frac{\alpha^5}{\alpha^3 + 6} \int_0^t x^{r-3} e^{-\alpha/x} dx \\ &\quad + \frac{3\alpha^3}{\alpha^3 + 6} \int_0^t x^{r-4} e^{-\alpha/x} dx.\end{aligned}$$

Now change variables by letting $u = \alpha/x$. Then $x = \alpha/u$. When x goes from 0 to t , u goes from ∞ to α/t . After some algebra, the first integral becomes

$$\begin{aligned}\int_0^t x^{r-3} e^{-\alpha/x} dx &= \alpha^{r-2} \int_{\alpha/t}^{\infty} u^{-r} e^{-u} du \\ &= \alpha^{r-2} \Gamma(1-r, \alpha/t),\end{aligned}$$

where $\Gamma(s, z) = \int_z^{\infty} u^{s-1} e^{-u} du$ is the incomplete gamma function. The second integral works out similarly:

$$\int_0^t x^{r-4} e^{-\alpha/x} dx = \alpha^{r-3} \Gamma(2-r, \alpha/t).$$

Putting everything together, the incomplete moment is

$$\begin{aligned}m_r(t) &= \frac{\alpha^{r+3}}{\alpha^3 + 6} \Gamma(1-r, \alpha/t) \\ &\quad + \frac{3\alpha^r}{\alpha^3 + 6} \Gamma(2-r, \alpha/t), \quad r < 2.\end{aligned}$$

4.5 Inverse Moments

For the IGD, these quantities follow directly from the general moment expression in Section 4.3 by setting $r = -k$ with $k > 0$. Because $-k < 2$, the existence condition $r < 2$ is always satisfied, so all inverse moments are finite.

Substituting $r = -k$ into $\mu_r = \frac{\alpha^r((\alpha^3+6)-3r)\Gamma(2-r)}{\alpha^3+6}$ gives

$$E(X^{-k}) = \frac{\alpha^{-k}(\alpha^3 + 6 + 3k)\Gamma(2+k)}{\alpha^3 + 6}.$$

For example, taking $k = 1$ yields the first inverse moment

$$E(X^{-1}) = \frac{\alpha^{-1}(\alpha^3 + 9)\Gamma(3)}{\alpha^3 + 6} = \frac{2(\alpha^3 + 9)}{\alpha(\alpha^3 + 6)},$$

and for $k = 2$ we obtain

$$E(X^{-2}) = \frac{\alpha^{-2}(\alpha^3 + 12)\Gamma(4)}{\alpha^3 + 6} = \frac{6(\alpha^3 + 12)}{\alpha^2(\alpha^3 + 6)}.$$

4.6 Stress-Strength Reliability

Stress-Strength reliability measures the probability that a randomly selected component with strength Y withstands a randomly applied stress X . A system is considered reliable when $R > 0.5$. Recent developments of this measure under censored, hybrid-censored and record data can be found in [6, 9, 27, 50].

Let $X \sim \text{IGD}(\alpha_1)$ represent the stress variable and $Y \sim \text{IGD}(\alpha_2)$ represent the strength variable, assumed to be independent. The reliability parameter is defined as:

$$R = P(X < Y) = \int_0^{\infty} f_X(x; \alpha_1) \bar{F}_Y(x; \alpha_2) dx,$$

where $f_X(\cdot)$ is the probability density function given in (4) and $\bar{F}_Y(\cdot)$ is the survival function given from (3). By substitution the expressions yields:

$$\begin{aligned}R &= \frac{\alpha_1^3}{(\alpha_1^3 + 6)(\alpha_2^3 + 6)} \int_0^{\infty} x^{-4} (\alpha_1^2 x + 3) \\ &\quad \times (3\alpha_2^4 x^{-2} + \alpha_2 x + x^2) e^{-(\alpha_1 + \alpha_2)/x} dx.\end{aligned}$$

Let $y = (\alpha_1 + \alpha_2)/x$ see [32]. Then, the reliability stress-strength simplifies to the closed form:

$$\begin{aligned}R &= \frac{\alpha_1^3}{(\alpha_1^3 + 6)(\alpha_2^3 + 6)(\alpha_1 + \alpha_2)^3} \\ &\quad \times \left[\alpha_2^3 (3\alpha_1^2 + \alpha_1 \alpha_2 + \alpha_2^2 + 6) \right. \\ &\quad \left. + 18\alpha_2^2 + 6\alpha_1 \alpha_2 + 6\alpha_1^2 \right].\end{aligned}$$

5 Statistical Inference

Each of the six extropy measures derived earlier is a function of the single parameter α , so before any of them can be reported on real data, we first have to estimate α . In this article, we work with three estimators: maximum likelihood (MLE), the method of maximum product of spacings (MPS), and a Bayesian estimator obtained by Markov chain Monte Carlo (MCMC). We kept all three on purpose: they approach the problem from genuinely different directions, and as the simulation shows, they behave a little differently in small samples, even though they agree asymptotically. Once a point estimate $\hat{\alpha}$ is available, the extropy measures follow by substitution, and an interval for any measure is obtained from the interval for α through the monotone-transformation argument used before (see [1, 10, 11, 24, 27, 29, 33, 35, 44]).

5.1 Maximum likelihood estimation of α

Given a random sample $x_1, \dots, x_n$ from the IGD with density eq.(4), the log-likelihood is

$$\begin{aligned} \ell(\alpha) = & 3n \ln \alpha - n \ln(\alpha^3 + 6) - 4 \sum_{i=1}^n \ln x_i \\ & + \sum_{i=1}^n \ln(\alpha^2 x_i + 3) - \alpha \sum_{i=1}^n \frac{1}{x_i}. \end{aligned} \quad (21)$$

Setting $d\ell/d\alpha = 0$ gives the score equation

$$\begin{aligned} \frac{3n}{\alpha} - \frac{3n\alpha^2}{\alpha^3 + 6} + \sum_{i=1}^n \frac{2\alpha x_i}{\alpha^2 x_i + 3} \\ - \sum_{i=1}^n \frac{1}{x_i} = 0, \end{aligned} \quad (22)$$

which has no closed-form root; we solve it numerically and restart from several seeds to be sure of the global optimum. With $I(\hat{\alpha}) = -d^2\ell/d\alpha^2$ at $\hat{\alpha}$, the standard error is $I(\hat{\alpha})^{-1/2}$ and a 95% interval is $\hat{\alpha} \pm 1.96I(\hat{\alpha})^{-1/2}$.

5.2 Maximum product of spacings estimation of α

The maximum product of spacings(MPS) method, due to Cheng and Amin and independently to Ranneby, replaces the joint density by the gaps the fitted distribution leaves between consecutive order statistics, and tries to make those gaps as even as possible. For ordered data $x_{(1)} \leq \dots \leq x_{(n)}$ define the $n+1$ spacings

$$D_i(\alpha) = F(x_{(i)}; \alpha) - F(x_{(i-1)}; \alpha), \quad i = 1, \dots, n+1, \quad (23)$$

with $F(x_{(0)}; \alpha) = 0$ and $F(x_{(n+1)}; \alpha) = 1$, so that $\sum_i D_i(\alpha) = 1$. The MPS estimator maximises the geometric mean of the spacings, equivalently

$$G(\alpha) = \frac{1}{n+1} \sum_{i=1}^{n+1} \ln D_i(\alpha), \quad (24)$$

with estimating equation

$$\frac{dG}{d\alpha} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\alpha)} \frac{\partial D_i(\alpha)}{\partial \alpha} = 0. \quad (25)$$

This too is solved numerically; if a spacing collapses to zero because of tied observations we floor it at a tiny constant before taking logs. MPS is consistent and asymptotically as efficient as the MLE, and remains well behaved in some non-regular cases where the likelihood fails, so we build its interval from the observed information in the same way.

5.3 Bayesian estimation of α

For the Bayesian analysis we treat α as random and combine the likelihood with a prior. Because $\alpha > 0$, a natural and convenient choice is a gamma prior,

$$\pi(\alpha) = \frac{b_0^{a_0}}{\Gamma(a_0)} \alpha^{a_0-1} e^{-b_0\alpha}, \quad \alpha > 0, \quad (26)$$

with shape $a_0 > 0$ and rate $b_0 > 0$. We used a weak, data-dominated prior so the posterior is driven by the sample. By Bayes' theorem,

$$\begin{aligned} \pi(\alpha | \mathbf{x}) \propto & \alpha^{3n+a_0-1} (\alpha^3 + 6)^{-n} \\ & \times \exp\left(-b_0\alpha - \alpha \sum_{i=1}^n \frac{1}{x_i}\right) \prod_{i=1}^n (\alpha^2 x_i + 3), \end{aligned} \quad (27)$$

and under squared-error loss the Bayes estimator is the posterior mean $\hat{\alpha}_B = E(\alpha | \mathbf{x})$. This integral has no closed form, so we approximate the posterior by a random-walk Metropolis–Hastings sampler on the log scale: from $\alpha^{(k)}$ we propose $\alpha^* = \alpha^{(k)} e^\varepsilon$, $\varepsilon \sim N(0, \sigma^2)$, and accept with probability

$$\min\left\{1, \frac{\pi(\alpha^* | \mathbf{x})}{\pi(\alpha^{(k)} | \mathbf{x})} \frac{\alpha^*}{\alpha^{(k)}}\right\}, \quad (28)$$

the extra factor being the Jacobian of the log transform. After a burn-in, the posterior mean is the average of the retained draws and a 95% credible interval their empirical 2.5% and 97.5% quantiles. Each chain was seeded at the MLE to speed convergence without changing the stationary distribution.

5.4 Plug-in estimation of the entropy measures

Once a point estimate $\hat{\alpha}$ of the parameter is available—whether from MLE, the MPS, or the Bayesian estimators. The entropy measures are obtained by substitution, since every measure derived in Section 3 is a function of α alone. For the two closed-form measures this substitution is immediate:

$$\hat{\varepsilon}_1 = -\frac{3(2\hat{\alpha}^6 + 30\hat{\alpha}^3 + 135)}{16\hat{\alpha}(\hat{\alpha}^3 + 6)^2},$$

$$\hat{\varepsilon}_2 = -\frac{3(\hat{\alpha}^6 + 12\hat{\alpha}^3 + 45)}{16(\hat{\alpha}^3 + 6)^2},$$

which estimate the entropy and the weighted entropy, respectively. The four remaining measures depend on a reference time t , and their estimates follow in the same way by inserting $\hat{\alpha}$ into the incomplete-gamma expressions in (8), (10), (12), and (14).

Interval estimation requires no extra work, and is handled identically for the three estimators. Each measure can be written as $\varepsilon = g(\alpha)$ for a function g that is monotone in α over the relevant range, so a parameter interval (α_L, α_U) maps directly onto $(g(\alpha_L), g(\alpha_U))$ for the measure, with the endpoints ordered when g is decreasing. For the MLE and the MPS the parameter interval is the Wald interval built from the observed information; for the Bayesian estimator it is the equal-tailed 95% credible interval from the posterior draws. In every case the transformation step is the same, which keeps the three methods on a common footing in the comparison of Section 6.

6 Numerical Simulation

In this section we assess, by Monte Carlo simulation, how the three estimators of α —maximum likelihood (MLE), maximum product of spacings (MPS), and the Bayesian posterior mean obtained by MCMC—perform in finite samples, both for the parameter itself and for the plug-in estimates of the six entropy measures. The design is meant to mirror the sample sizes one actually meets in reliability and survival studies.

6.1 Design

Samples are generated from the IGD by the inverse-CDF method: we draw $u \sim \text{Uniform}(0, 1)$ and solve $F(x; \alpha) = u$ numerically for x , which is reliable because F is strictly increasing. We take three true values from the upper part of the band tabulated in Table 2,

$$\alpha \in \{11, 13, 15\},$$

the largest of which, $\alpha = 15$, coincides exactly with the last row of Table 2 and so lets the reader cross-check our true

entropy values against the published numbers. For each α we use the sample sizes $n \in \{25, 50, 100, 200, 500\}$, with $R = 500$ Monte Carlo replicates per scenario.

From each replicate we obtain the three estimates and their 95% intervals (Wald intervals from the observed information for MLE and MPS; the equal-tailed credible interval for the Bayesian estimator). Over the R replicates we report, for the parameter, the bias, the mean squared error (MSE), its root (RMSE), the average interval length (AL), and the coverage probability (CP). The six entropy functionals—EX, WEX, REX, WREX, PEX, WPEX—are evaluated by plugging $\hat{\alpha}$ into the closed-form expressions (EX, WEX) or the incomplete-gamma expressions (the four time-dependent ones), with the latter taken at the two reference times $t = 10$ and $t = 12$. Intervals for the entropy measures follow from the interval for α by the monotone-transformation method.

6.2 Results for the parameter

Table 3 collects the parameter results for all three estimators and the three true values. The behaviour is exactly what one hopes for and is consistent across α . All three estimators are consistent: the bias and the MSE shrink steadily as n grows, and the average interval length contracts at the expected rate. The MLE and the Bayesian posterior mean are almost indistinguishable throughout—the weak gamma prior lets the data dominate, so the posterior mean sits essentially on top of the MLE—and both carry a small *positive* bias.

The comparison gets distilled in Figure 5, which uses two panels—RMSE and coverage probability—taken at the representative value $\alpha = 15$. You can barely tell the three RMSE curves apart; they sit one on top of the other and taper off smoothly as n rises. Coverage behaves similarly, with all three lingering around the nominal 95% mark. The MLE follows the target most faithfully, while MPS and the Bayesian estimator slip a little beneath it once samples get very small.

Finally, Figure 6 presents the complete sampling distribution of $\hat{\alpha}$ through boxplots drawn separately for each method. As n grows, the spread for each one closes in symmetrically on the true value—no skew, no estimates running wild. The three panels are hard to tell apart at a glance, and that visual sameness is perhaps the strongest single piece of evidence that the estimators are in agreement.

6.3 Results for the six entropy measures

The six measurements are on very different numerical scales: WEX is around 10^{-13} while WPEX is close to 10^{-4} . Therefore, each column in the following tables includes its own power-of-ten factor, printed in the

Table 3: Performance of the MLE, MPS and Bayesian estimators of α for $\alpha = 11, 13, 15$.

α	Method	n	Bias	MSE	RMSE	AL	CP
11	MLE	25	0.19629	2.70771	1.64551	6.1653	0.952
		50	0.05129	1.02557	1.01271	4.3030	0.966
		100	0.06087	0.58279	0.76341	3.0455	0.958
		200	0.07110	0.31670	0.56276	2.1556	0.950
		500	-0.02020	0.11228	0.33508	1.3519	0.958
	MPS	25	-0.18543	2.52802	1.58998	5.9493	0.938
		50	-0.17846	1.01077	1.00537	4.2111	0.946
		100	-0.07618	0.57225	0.75647	3.0068	0.938
		200	-0.00692	0.30870	0.55561	2.1400	0.948
		500	-0.05697	0.11405	0.33771	1.3473	0.952
	BAYES	25	0.25200	2.61841	1.61815	5.9957	0.940
		50	0.08741	1.01684	1.00838	4.2585	0.964
		100	0.08006	0.58263	0.76330	3.0282	0.960
		200	0.07803	0.31398	0.56034	2.1471	0.948
		500	-0.01786	0.11221	0.33498	1.3461	0.960
13	MLE	25	0.16503	3.35738	1.83231	7.2681	0.952
		50	0.11222	1.85884	1.36339	5.1190	0.940
		100	0.09253	0.88412	0.94028	3.6144	0.946
		200	0.03472	0.41610	0.64506	2.5444	0.944
		500	0.00202	0.16960	0.41183	1.6052	0.946
	MPS	25	-0.28842	3.18648	1.78507	7.0135	0.942
		50	-0.15984	1.79253	1.33885	5.0110	0.924
		100	-0.07017	0.86053	0.92764	3.5687	0.938
		200	-0.05757	0.41295	0.64261	2.5261	0.944
		500	-0.04147	0.17013	0.41247	1.5997	0.944
	BAYES	25	0.16289	3.13130	1.76955	7.0520	0.962
		50	0.11171	1.78095	1.33452	5.0376	0.940
		100	0.09425	0.86665	0.93094	3.5772	0.936
		200	0.03502	0.41281	0.64250	2.5346	0.948
		500	0.00202	0.16874	0.41078	1.5966	0.948
15	MLE	25	0.23518	5.40076	2.32395	8.4231	0.950
		50	0.15018	2.17809	1.47584	5.9231	0.950
		100	-0.00618	1.12679	1.06151	4.1448	0.946
		200	0.02636	0.57852	0.76061	2.9373	0.956
		500	0.03563	0.22819	0.47769	1.8589	0.954
	MPS	25	-0.27868	5.09199	2.25654	8.1358	0.940
		50	-0.16328	2.09860	1.44865	5.7992	0.948
		100	-0.19450	1.13503	1.06538	4.0922	0.946
		200	-0.08180	0.57735	0.75984	2.9159	0.948
		500	-0.01515	0.22586	0.47524	1.8525	0.950
	BAYES	25	0.12744	4.80801	2.19272	8.1087	0.938
		50	0.09307	2.04198	1.42898	5.8091	0.958
		100	-0.02591	1.09713	1.04744	4.0972	0.946
		200	0.01408	0.56832	0.75387	2.9133	0.944
		500	0.03000	0.22705	0.47650	1.8499	0.948

header, allowing each entry to remain within a readable range. Instead of separating the two reference times, we group $t = 10$ and $t = 12$ into a single table by estimation method, identified by the first column, allowing the reader to consult both cases simultaneously.

The MSE and bias for the three methods appear in Tables 4–9. Once the column scales are taken into account, the same story repeats itself: the bias is never significant, the mean squared error (MSE) progressively

decreases as n increases, and the values remain consistent across all three actual values of α and at both reference times. WPEX, the weighted measure of the past, shows the highest values; this is not surprising, since it puts its weight in the upper tail, where sampling variability is greatest. EX and WEX, by contrast, have closed forms and turn out to be the steadiest of the bunch. As for the Bayesian MSE (Table 6), it meets or beats the MLE

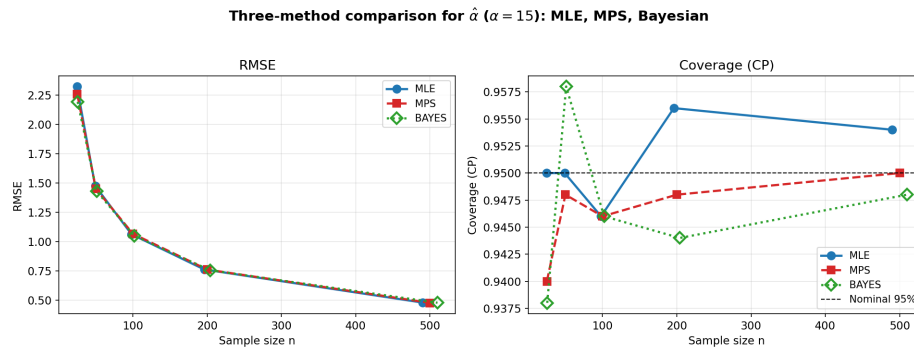


Fig. 5: Three-method comparison for $\hat{\alpha}$ at $\alpha = 15$: RMSE (left) and coverage probability (right) versus n , for MLE, MPS and the Bayesian posterior mean.

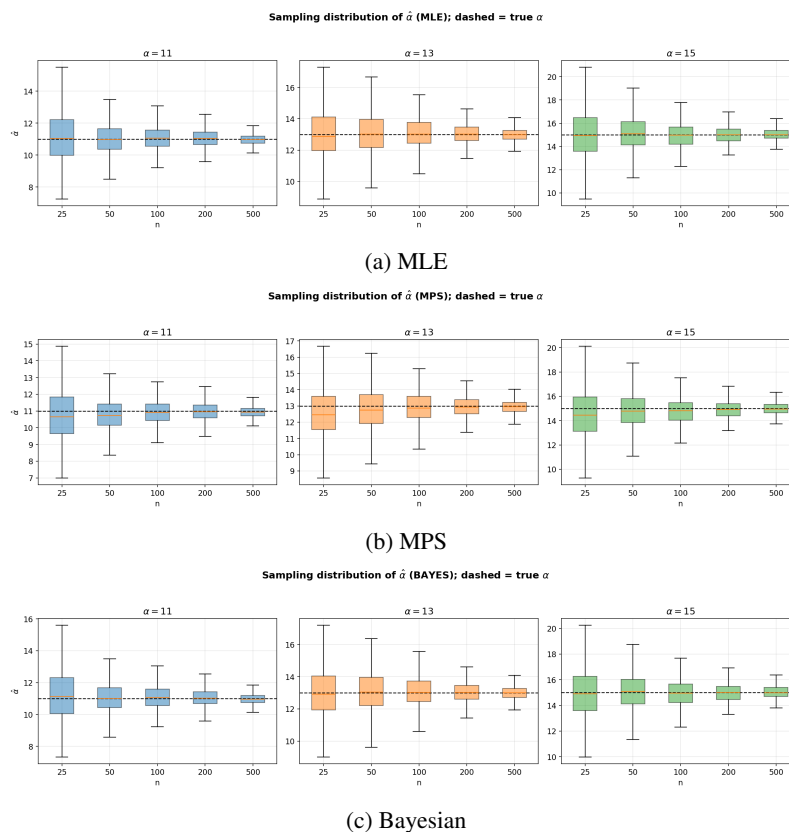


Fig. 6: Sampling distribution of $\hat{\alpha}$ across sample sizes, for the three estimators and the three true values $\alpha = 11, 13, 15$.

figures (Table 4) everywhere, so the Bayesian estimator holds its own throughout and never falls behind.

A direct, head-to-head comparison of the three methods lives in Figure 7, which plots the RMSE of the six core measures at $\alpha = 15$. The curves practically overlap, with the Bayesian one resting on or just under the MLE. Figure 8, lastly, lays out the MCMC diagnostics underpinning the Bayesian estimator for one

representative sample. The chain mixes nicely, with an acceptance rate close to 0.56; the running mean settles down fast; the posterior clusters tightly around the true value while the prior stays nearly flat across its support; and the autocorrelation falls inside the confidence band after just a few lags.

Table 4: MSE of the six core extropy estimators (MLE).

α	n	t -independent		$t = 10$				$t = 12$			
		EX ($\times 10^{-6}$)	WEX ($\times 10^{-13}$)	REX ($\times 10^{-7}$)	WREX ($\times 10^{-5}$)	PEX ($\times 10^{-7}$)	WPEX ($\times 10^{-4}$)	REX ($\times 10^{-7}$)	WREX ($\times 10^{-5}$)	PEX ($\times 10^{-7}$)	WPEX ($\times 10^{-4}$)
11	25	24.426	16.533	25.009	23.867	18.630	9.034	13.667	18.989	50.541	5.446
	50	9.820	4.057	9.756	9.334	5.629	3.339	5.302	7.380	19.807	2.003
	100	5.718	2.114	5.600	5.363	3.010	1.880	3.038	4.231	11.527	1.126
	200	2.956	0.819	3.027	2.898	1.156	1.028	1.644	2.289	5.487	0.616
	500	1.111	0.325	1.087	1.042	0.495	0.360	0.589	0.821	2.189	0.215
13	25	15.408	1.845	26.818	25.244	13.532	12.495	15.090	20.760	12.463	7.666
	50	8.858	0.752	15.205	14.346	4.846	6.823	8.517	11.737	6.034	4.174
	100	4.315	0.289	7.357	6.953	1.343	3.213	4.108	5.668	2.503	1.961
	200	2.057	0.119	3.490	3.301	0.418	1.504	1.946	2.686	1.107	0.917
	500	0.844	0.046	1.428	1.351	0.128	0.612	0.796	1.099	0.438	0.373
15	25	13.610	0.490	36.522	33.778	55.638	21.974	21.261	28.888	10.660	13.698
	50	6.023	0.125	15.600	14.516	16.332	8.646	8.983	12.260	1.677	5.361
	100	3.182	0.054	8.181	7.624	7.624	4.445	4.699	6.420	0.496	2.753
	200	1.583	0.022	4.185	3.898	3.985	2.288	2.406	3.286	0.156	1.417
	500	0.629	0.008	1.660	1.547	1.500	0.900	0.953	1.303	0.025	0.557

Table 5: MSE of the six extropy estimators (MPS).

α	n	t -independent		$t = 10$				$t = 12$			
		EX ($\times 10^{-6}$)	WEX ($\times 10^{-13}$)	REX ($\times 10^{-7}$)	WREX ($\times 10^{-5}$)	PEX ($\times 10^{-7}$)	WPEX ($\times 10^{-4}$)	REX ($\times 10^{-7}$)	WREX ($\times 10^{-5}$)	PEX ($\times 10^{-7}$)	WPEX ($\times 10^{-4}$)
11	25	28.028	27.207	24.301	23.272	28.715	8.128	13.176	18.353	70.334	4.873
	50	10.944	5.720	9.843	9.437	8.174	3.216	5.325	7.423	25.046	1.923
	100	6.044	2.608	5.575	5.345	3.854	1.821	3.016	4.204	13.196	1.089
	200	3.007	0.924	2.974	2.848	1.373	0.994	1.612	2.246	5.887	0.595
	500	1.151	0.352	1.108	1.063	0.548	0.364	0.600	0.836	2.321	0.218
13	25	18.022	3.121	26.756	25.310	9.956	11.469	14.902	20.574	19.775	6.997
	50	9.671	1.049	15.089	14.277	3.656	6.451	8.403	11.603	8.416	3.933
	100	4.531	0.359	7.285	6.896	0.997	3.089	4.053	5.599	3.191	1.882
	200	2.129	0.136	3.497	3.310	0.327	1.483	1.946	2.688	1.295	0.903
	500	0.863	0.049	1.439	1.362	0.110	0.612	0.801	1.106	0.477	0.373
15	25	15.677	0.779	36.466	33.942	39.984	20.154	20.976	28.628	9.789	12.501
	50	6.562	0.173	15.548	14.520	12.453	8.184	8.890	12.164	1.571	5.059
	100	3.454	0.068	8.413	7.857	6.570	4.429	4.811	6.583	0.462	2.737
	200	1.650	0.025	4.226	3.942	3.651	2.269	2.424	3.313	0.136	1.404
	500	0.635	0.008	1.652	1.541	1.426	0.888	0.948	1.296	0.021	0.550

6.4 Summary

The simulation study supports a clean conclusion. All three estimators of α are consistent and nearly unbiased from $n = 50$ onward, and even at $n = 25$ the bias is modest. They differ only in the small-sample bias—positive and tiny for the MLE and the Bayesian posterior mean, mildly negative for the MPS—and they are interchangeable in MSE and RMSE, exactly as the asymptotic theory predicts. The plug-in estimators of all six extropy measures inherit these properties at every α and at both reference times, and the transformation-based

intervals reach the correct coverage with no additional approximation. The Bayesian estimator, far from being inferior, matches or slightly improves on the MLE here because of the weak prior, while the MPS is a sound, theoretically justified alternative whose handling of the tail can be useful for heavy-tailed data. For the sample sizes typical of reliability and survival work, all three procedures are reliable.

Table 6: MSE of the six entropy estimators (**Bayesian**).

α	n	t -independent		$t = 10$				$t = 12$			
		EX ($\times 10^{-6}$)	WEX ($\times 10^{-13}$)	REX ($\times 10^{-7}$)	WREX ($\times 10^{-5}$)	PEX ($\times 10^{-7}$)	WPEX ($\times 10^{-4}$)	REX ($\times 10^{-7}$)	WREX ($\times 10^{-5}$)	PEX ($\times 10^{-7}$)	WPEX ($\times 10^{-4}$)
11	25	22.872	13.668	24.073	22.964	16.016	8.773	13.168	18.289	45.295	5.291
	50	9.541	3.754	9.638	9.219	5.168	3.322	5.242	7.295	18.781	1.994
	100	5.658	2.027	5.589	5.351	2.881	1.883	3.032	4.224	11.262	1.128
	200	2.916	0.797	2.999	2.870	1.117	1.020	1.629	2.268	5.379	0.611
	500	1.108	0.322	1.086	1.041	0.490	0.360	0.588	0.820	2.177	0.215
13	25	14.376	1.638	25.062	23.596	12.134	11.642	14.097	19.396	11.304	7.140
	50	8.509	0.714	14.585	13.763	4.530	6.533	8.168	11.257	5.767	3.996
	100	4.218	0.279	7.208	6.811	1.322	3.150	4.025	5.553	2.424	1.923
	200	2.041	0.118	3.463	3.275	0.413	1.492	1.930	2.665	1.099	0.910
	500	0.840	0.046	1.421	1.344	0.126	0.609	0.792	1.093	0.437	0.371
15	25	12.698	0.468	33.067	30.640	45.964	19.417	19.184	26.101	8.742	12.087
	50	5.765	0.122	14.719	13.706	14.689	8.080	8.465	11.559	1.490	5.007
	100	3.121	0.053	7.983	7.441	7.308	4.323	4.583	6.262	0.468	2.677
	200	1.564	0.022	4.118	3.836	3.872	2.245	2.366	3.233	0.149	1.391
	500	0.627	0.008	1.653	1.541	1.487	0.895	0.949	1.297	0.024	0.554

Table 7: Bias of the six entropy estimators (**MLE**).

α	n	t -independent		$t = 10$				$t = 12$			
		EX ($\times 10^{-4}$)	WEX ($\times 10^{-8}$)	REX ($\times 10^{-5}$)	WREX ($\times 10^{-4}$)	PEX ($\times 10^{-5}$)	WPEX ($\times 10^{-4}$)	REX ($\times 10^{-5}$)	WREX ($\times 10^{-4}$)	PEX ($\times 10^{-5}$)	WPEX ($\times 10^{-4}$)
11	25	-1.127	-34.186	14.834	14.140	-58.472	-42.941	11.563	13.405	-43.972	-34.145
	50	-1.281	-14.344	3.342	3.127	-24.891	-12.227	2.706	3.102	-21.118	-9.815
	100	0.235	-6.654	5.018	4.830	-12.433	-12.649	3.834	4.473	-7.932	-9.988
	200	1.335	-1.254	6.467	6.286	-4.180	-13.675	4.836	5.681	1.093	-10.689
	500	-0.957	-2.895	-2.175	-2.144	-4.437	3.279	-1.573	-1.868	-5.944	2.497
13	25	-1.735	-11.815	9.836	9.063	-57.021	-38.899	8.146	9.249	-44.695	-31.371
	50	-0.590	-6.227	7.348	6.868	-32.994	-25.586	5.934	6.794	-24.623	-20.536
	100	0.561	-2.331	7.087	6.760	-16.957	-19.633	5.504	6.383	-10.067	-15.597
	200	0.060	-1.238	2.524	2.392	-7.839	-7.573	1.985	2.293	-5.252	-6.039
	500	-0.248	-0.699	-0.086	-0.109	-2.912	-0.785	-0.023	-0.043	-2.760	-0.665
15	25	-1.615	-6.022	11.902	10.631	-73.152	-56.223	10.300	11.489	-55.440	-45.620
	50	0.117	-2.222	9.498	8.822	-35.410	-33.838	7.725	8.808	-24.149	-27.195
	100	-1.363	-1.649	-2.267	-2.370	-11.876	-0.898	-1.440	-1.800	-12.293	-1.008
	200	-0.198	-0.628	1.364	1.224	-8.359	-6.313	1.175	1.314	-6.375	-5.120
	500	0.342	-0.099	2.696	2.567	-5.310	-7.495	2.098	2.429	-2.650	-5.956

7 Real data analysis

To illustrate the practical usefulness of the fitted distributions, we analyze a real dataset representing infection times (in months) of kidney dialysis patients. This dataset was previously studied by [18, 36] and contains $n = 28$ observations. The observed values are the following:

2.5, 2.5, 3.5, 3.5, 3.5, 4.5, 5.5, 6.5, 6.5, 7.5, 7.5, 7.5, 7.5, 8.5, 9.5, 10.5, 11.5, 12.5, 12.5, 13.5, 14.5, 14.5, 21.5, 21.5, 22.5, 22.5, 25.5, 27.5.

The descriptive statistics are as follows:

From Table 10, the positive skewness indicates moderate right-skewness, while the kurtosis value suggests a distribution slightly flatter than the normal distribution.

The proposed IGD distribution is compared with four established competing models, namely the inverse Gompertz (IGoD) [26], inverse Lindley (ILD) [52], inverse XLindley (IXLD) [19], and inverse X-gamma (IXgD) [54] distributions. All models were fitted to the data using the maximum likelihood estimation (MLE) method.

Table 8: Bias of the six extropy estimators (MPS).

α	n	t -independent		$t = 10$				$t = 12$			
		EX ($\times 10^{-4}$)	WEX ($\times 10^{-8}$)	REX ($\times 10^{-5}$)	WREX ($\times 10^{-4}$)	PEX ($\times 10^{-5}$)	WPEX ($\times 10^{-4}$)	REX ($\times 10^{-5}$)	WREX ($\times 10^{-4}$)	PEX ($\times 10^{-5}$)	WPEX ($\times 10^{-4}$)
11	25	-13.336	-63.718	-22.459	-22.339	-85.121	25.589	-15.921	-19.023	-99.034	18.914
	50	-8.609	-28.787	-19.244	-18.980	-40.739	28.880	-13.920	-16.523	-54.057	21.984
	100	-4.083	-14.478	-8.449	-8.351	-21.369	11.899	-6.080	-7.229	-27.060	9.003
	200	-1.101	-5.405	-1.195	-1.213	-9.052	0.313	-0.805	-0.978	-9.580	0.134
	500	-2.115	-4.858	-5.795	-5.688	-6.833	9.858	-4.237	-5.013	-11.074	7.585
13	25	-12.157	-22.705	-31.585	-31.205	-48.851	47.161	-22.794	-27.095	-70.247	35.842
	50	-6.743	-11.677	-17.548	-17.340	-27.132	26.075	-12.657	-15.048	-38.986	19.806
	100	-3.086	-5.226	-7.822	-7.739	-13.135	11.265	-5.627	-6.695	-18.336	8.529
	200	-2.009	-2.804	-5.949	-5.849	-5.673	9.943	-4.338	-5.138	-9.944	7.635
	500	-1.222	-1.417	-4.081	-3.995	-1.885	7.466	-3.004	-3.546	-4.964	5.776
15	25	-10.453	-10.748	-31.667	-31.398	-34.784	45.710	-22.746	-27.117	-55.343	34.605
	50	-5.187	-4.488	-17.216	-16.963	-11.176	28.335	-12.521	-14.855	-23.251	21.721
	100	-4.571	-2.939	-18.397	-17.949	2.467	36.386	-13.653	-16.079	-11.967	28.314
	200	-2.020	-1.310	-7.890	-7.713	0.081	15.126	-5.834	-6.880	-5.985	11.742
	500	-0.507	-0.403	-1.645	-1.625	-1.289	2.576	-1.190	-1.415	-2.408	1.965

Table 9: Bias of the six extropy estimators (Bayesian).

α	n	t -independent		$t = 10$				$t = 12$			
		EX ($\times 10^{-4}$)	WEX ($\times 10^{-8}$)	REX ($\times 10^{-5}$)	WREX ($\times 10^{-4}$)	PEX ($\times 10^{-5}$)	WPEX ($\times 10^{-4}$)	REX ($\times 10^{-5}$)	WREX ($\times 10^{-4}$)	PEX ($\times 10^{-5}$)	WPEX ($\times 10^{-4}$)
11	25	0.956	-27.834	20.467	19.667	-51.651	-52.619	15.687	18.281	-33.001	-41.599
	50	-0.099	-11.897	6.910	6.622	-22.103	-18.657	5.330	6.201	-15.644	-14.785
	100	0.844	-5.530	6.906	6.679	-11.138	-16.082	5.224	6.114	-5.210	-12.643
	200	1.562	-0.825	7.154	6.959	-3.649	-14.909	5.341	6.277	2.140	-11.642
	500	-0.883	-2.768	-1.944	-1.919	-4.284	2.860	-1.404	-1.667	-5.617	2.173
13	25	-1.427	-10.802	9.996	9.252	-53.486	-37.994	8.212	9.348	-41.323	-30.599
	50	-0.477	-5.910	7.425	6.955	-31.766	-25.315	5.973	6.847	-23.433	-20.302
	100	0.631	-2.213	7.273	6.944	-16.691	-19.918	5.639	6.543	-9.673	-15.814
	200	0.072	-1.220	2.557	2.425	-7.792	-7.622	2.009	2.321	-5.182	-6.077
	500	-0.247	-0.696	-0.085	-0.108	-2.899	-0.783	-0.022	-0.042	-2.746	-0.663
15	25	-2.937	-6.209	3.564	2.672	-59.883	-33.936	3.852	4.010	-50.216	-27.946
	50	-0.713	-2.440	4.818	4.324	-29.658	-22.282	4.148	4.640	-22.654	-18.071
	100	-1.664	-1.737	-3.909	-3.952	-10.033	3.065	-2.691	-3.259	-11.920	2.117
	200	-0.394	-0.692	0.328	0.225	-7.293	-3.859	0.388	0.395	-6.224	-3.188
	500	0.249	-0.131	2.216	2.104	-4.855	-6.377	1.734	2.004	-2.613	-5.077

Table 10: Descriptive statistics for the kidney dialysis infection-time dataset.

n	Mean	Median	Variance	SD	Skewness	Kurtosis	Min	Max	Range
28	11.321	9.000	54.967	7.414	0.765	2.422	2.50	27.50	25.00

The estimation and model comparison results are summarized in Tables 11 and 12. Table 11 reports the likelihood-based model selection measures, including the log-likelihood (LogLik), information criteria (AIC, BIC, CAIC, and HQIC), the absolute standardised average error (ASAE), and extropy-based measures (Ext and

WExt). Table 12 presents the goodness-of-fit diagnostics, including the Kolmogorov–Smirnov (KS) statistic and its associated p-value, the Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics with their corresponding p-values, together with the maximum likelihood parameter estimates and their standard errors.

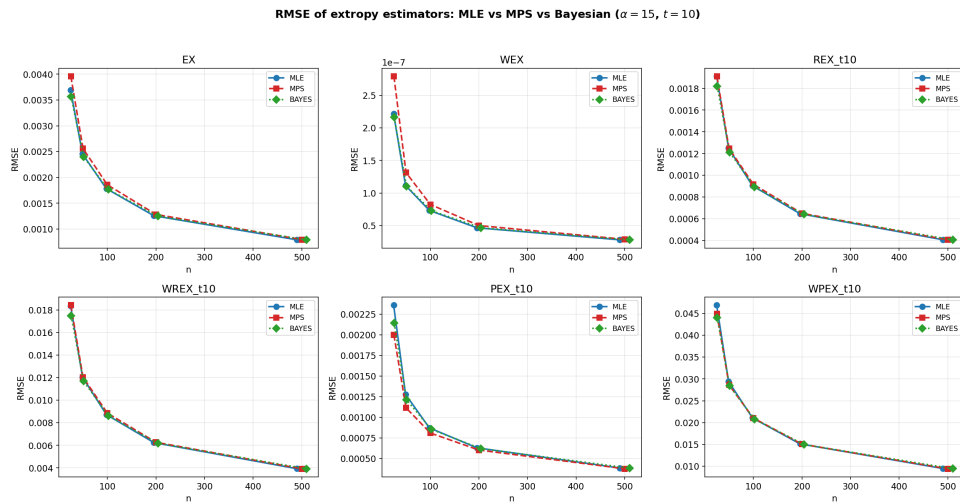


Fig. 7: RMSE of the six core entropy estimators under MLE, MPS and Bayesian at $\alpha = 15, t = 10$, as a function of n .

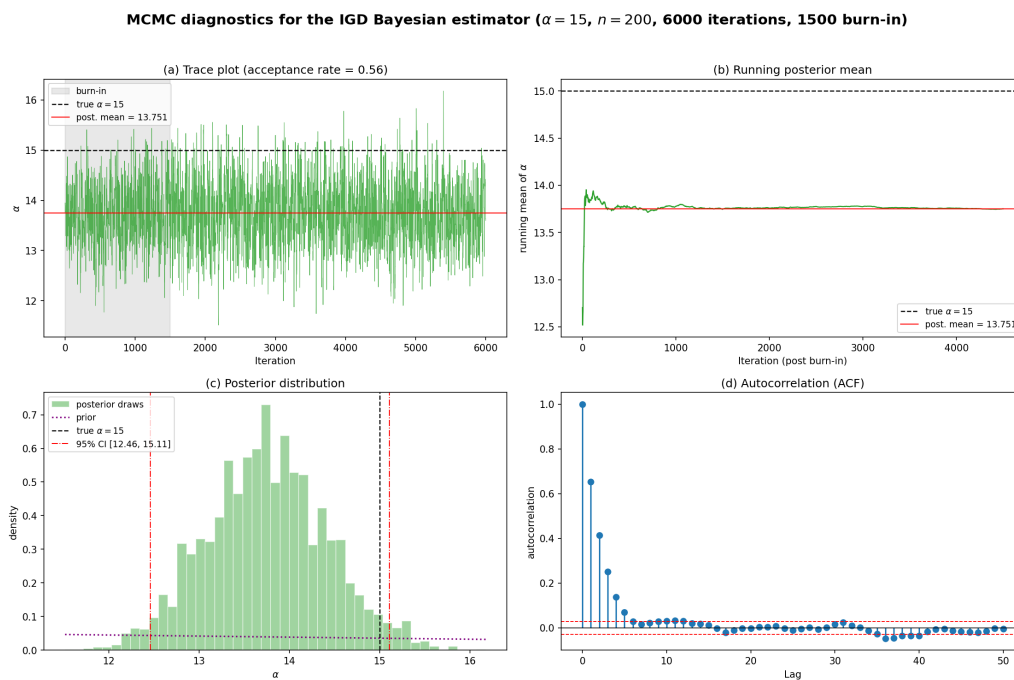


Fig. 8: MCMC diagnostics for the Bayesian estimator at $\alpha = 15$ ($n = 200$): (a) trace plot with burn-in shaded, (b) running posterior mean, (c) posterior density with the prior and the 95% credible interval, and (d) the autocorrelation function.

Table 11: Model selection criteria and entropy measures for the fitted distributions.

Model	LogLik	AIC	BIC	CAIC	HQIC	ASAE	Ext	WExt
IGD	92.4404	186.8808	188.2130	187.0346	187.2880	0.0808	-0.02637	-0.18750
IGoD	94.8951	193.7903	196.4547	194.2703	194.6048	0.3026	-0.02517	-0.15849
ILD	96.4008	194.8016	196.1338	194.9554	195.2089	0.5410	-0.01778	-0.12579
IXLD	96.8926	195.7851	197.1173	195.9390	196.1924	0.5603	-0.01666	-0.12146
IXgD	96.9952	195.9904	197.3226	196.1443	196.3977	0.6093	-0.01675	-0.12034

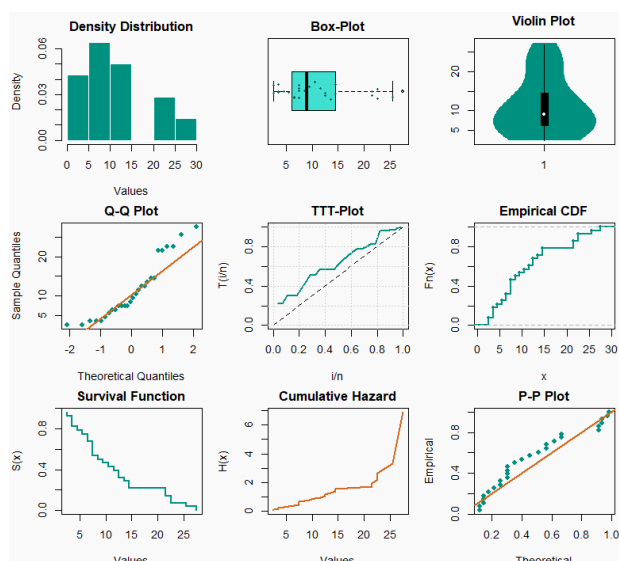


Fig. 9: Exploratory data analysis for the kidney dialysis infection-time dataset.

Table 12: Goodness-of-fit statistics, parameter estimates, and standard errors.

Model	KS	W^*	A^*	$\hat{\alpha}$	$\hat{\lambda}$
	KS_{pv}	W^*_{pv}	A^*_{pv}	$SE(\hat{\alpha})$	$SE(\hat{\lambda})$
IGD	0.1134	0.0651	0.4963	14.2375	—
	0.8645	0.7858	0.7492	1.8967	—
IGoD	0.1678	0.1279	0.9166	4.7588	3.2600
	0.4094	0.4670	0.4032	1.4768	1.7509
ILD	0.2257	0.2409	1.5282	7.9094	—
	0.1153	0.2007	0.1700	1.3578	—
IXLD	0.2299	0.2567	1.6007	7.2906	—
	0.1036	0.1801	0.1544	1.3343	—
IXgD	0.2394	0.2921	1.7791	8.5406	—
	0.0807	0.1421	0.1223	1.4607	—

Table 11 displays clear separations between the fits. Against all candidate distributions, the IGD model attains lowest AIC (186.88), BIC (188.21), CAIC (187.03) and HQIC (187.29) implying best goodness-of-fit with parsimonious modeling. In addition, IGD also results in the lowest ASAE value (0.0808) suggesting better predictive performance and less average fitting error against the competitors.

The extropy-based metrics also give additional insights into model behavior. While Ext and WExt are negative for all fitted distributions by definition, the IGD model has the lowest extropy (Ext = -0.0264 , WExt = -0.1875) which means it has higher concentration and more stable uncertainty profile than the other models.

The goodness-of-fit results in Table 12 also confirm the IGD model's dominance. In particular, IGD has minimum ($KS = 0.1134$) and maximum ($KS_{pv} = 0.8645$) which means no rejection of the fitted distribution. By comparison, the other models produce larger KS statistics and smaller p-values.

We observe the same trend with W^* and A^* , where IGD distribution has lowest values of $W^* = 0.0651$, $A^* = 0.4963$ and highest corresponding p-values (0.7858 and 0.7492). These findings suggest that IGD best fits the observed distribution both in the centre and the tails.

The graphical diagnostics of Figure 10 back up the numerical results. These fitted CDF and survival curves emphasize that the IGD model tracks the empirical behavior much more faithfully through most of the observation range. And once again, both the histogram with fitted densities and the P-P plot indicate more agreement between observed and fitted probabilities for the IGD specification.

Overall, however, the results indicate that the IGD distribution is the best fit for the kidney dialysis infection-time dataset out of the remaining candidates. Taken collectively, the increased likelihood-based criteria, lower prediction error, positive goodness-of-fit diagnostics, and graphical concordance highlight the versatility and general applicability of the IGD model for modeling positively skewed medical survival data.

8 Conclusion

This paper proposed the inverse Garhy distribution (IGD), a lifetime model of one-parameter constructed by inverting the Garhy distribution. Six various extropy measures-of-uncertainty are calculated and analyzed. We obtained closed-form expressions for extropy, weighted extropy, residual extropy, weighted residual extropy, past extropy and weighted past extropy. These results add to the growing body of work applying extropy theory in reliability and survival contexts, and they give a clearer picture of how uncertainty is distributed across the lifetime of a unit. The usual distributional properties were obtained: density, cdf, survival and hazard rate function, moments, incomplete and inverse moments, quantile function, stress-strength reliability. The model shows positive skewness, heavy tails, and flexible hazard shapes, making it a reasonable choice for lifetime and reliability data where extreme observations are present. Maximum likelihood was used to estimate the unknown parameters and extropy measures, with confidence intervals constructed from the observed Fisher information matrix. The IGD was then fitted to kidney dialysis infection times and contrasted with several competing inverse distributions. Outperformed each alternative on standard goodness-of-fit and information criteria. The results are promising, but the study was limited to full samples, and the model under censoring, truncation, or ranked set sampling was not studied here. Classical estimation was

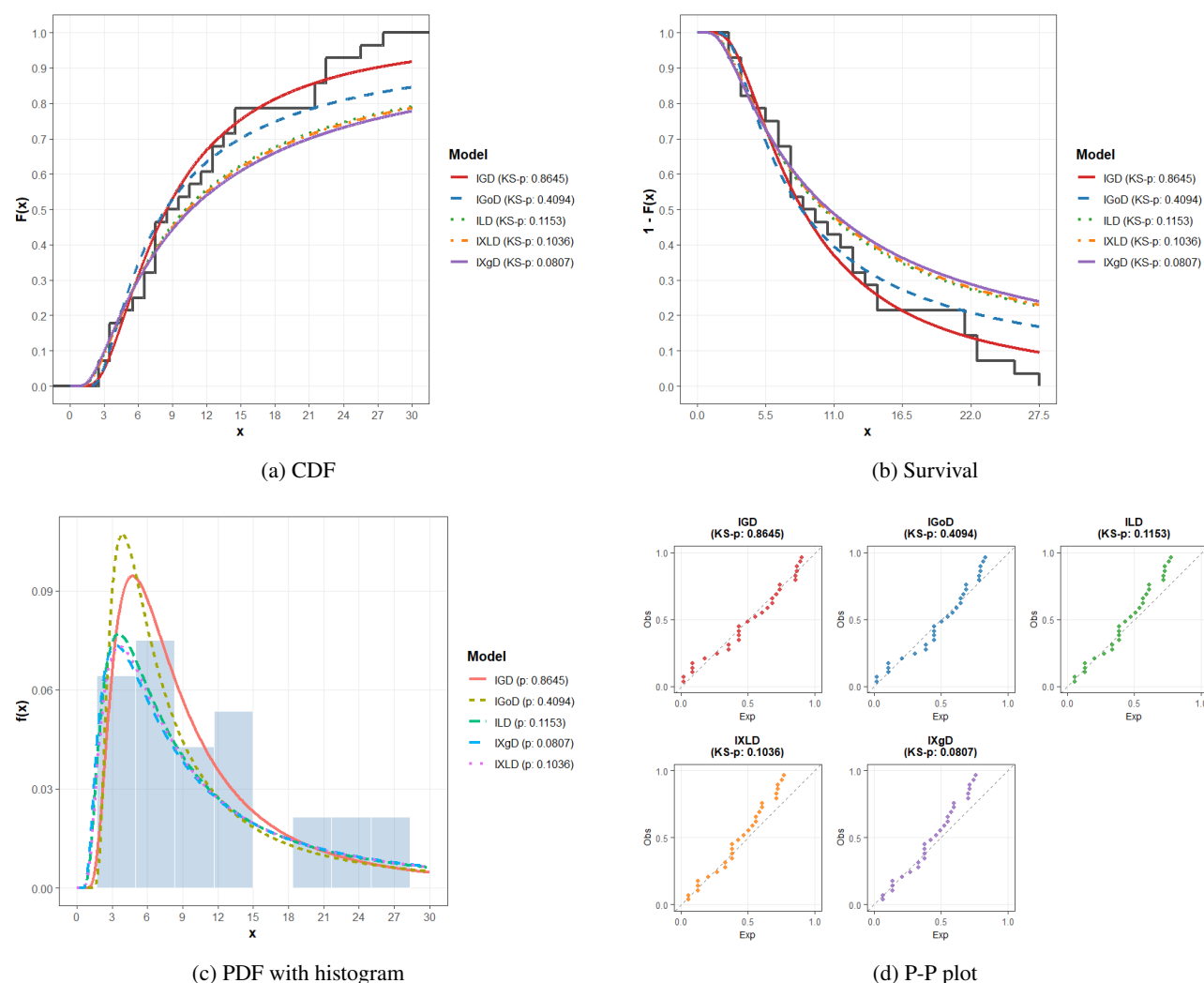


Fig. 10: Combined diagnostic plots: (a) CDF, (b) survival function (CCDF), (c) PDF with histogram, and (d) P-P plot. The IGD model (solid red line) closely follows the empirical estimates.

the sole inferential method contemplated. These gaps organically indicate the next steps. Bayesian and classical estimation approaches for the IGD deserve attention, as well as their behavior under various censoring schemes and ranked set sampling designs. Regression models built on the IGD would broaden its practical reach, and applications to engineering, biomedical, and environmental reliability datasets beyond the one investigated here would help to eliminate where the model functions well and where it does not.

Data Availability Statement

The data that support the findings of this study are available on request from the corresponding author. The

data are not publicly available due to privacy or ethical restrictions.

Conflicts of Interest

The authors declare no conflict of interest.

Acknowledgment

The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number: 26UQU4340323GSSR03.

Funding Statement

This research work was funded by Umm Al-Qura University, Saudi Arabia under grant number: 26UQU4340323GSSR03.

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