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Statistical Inferences of Reduced Modified Weibull Distribution under Progressive Type-II Censoring Data

Saad J. Almalki[1,*], *Gamal. A. Abd-Elmougod*[2] *and Maie J. Al-Malki*[1]

[1]Mathematics Department, Faculty of Science Taif University, P. o. Box 11099 Taif 21944, Saudi Arabia
[2]Mathematics Department, Faculty of Science, Damanhour University, Damanhour, Egypt

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Abstract: In reliability engineering and lifetime analysis many applications required a bathtub shaped or increasing hazard rate function. In this paper, we considered a reduced form of the new modified Weibull (NMW) distribution when the data are presented in the form of progressive type-II censoring data. The parameter estimators under maximum likelihood estimation (MLE) for point and approximate confidence interval are presented. Also, Bayesian approach under simulate posterior distribution with the help of MCMC the method is adopted. The developed results discussed with data analysis and assessed with Moto Carlo simulation study.

Keywords: Modified Weibull distribution, Progressive type-II censoring, Bathtub shaped hazard rate function, Maximum likelihood estimation, Bayesian estimation, MCMC.

1 Introduction

For the problem of accommodate bathtub shaped hazard rates, different researchers proposed a generalizations and modifications of the Weibull distribution. Different modifications of Weibull distribution are discussed and presented by different authors see, [1], [2], [3] and [4]. The modifications of Weibull distribution that present models with two or three parameters are discussed by [[5]-[4] and [6]], Also, the modifications with four parameter [7] and for the modifications with five parameter, [8]. Then, the large sets of distribution are generalized with generalized linear exponential distribution as bathtub shaped hazard rate presented by [9]. The unimodal, decreasing, increasing or bathtub shaped hazard rate beta-Weibull distribution is proposed by [10]. The four parameter generalized modified Weibull distribution proposed by [11].

The new modified Weibull distribution presented by [12] has a cumulative distribution function (CDF) given by

$$F(x) = 1 - \exp\left\{-\alpha x^{\theta} - \beta x^{\gamma}\exp\{\lambda x\}\right\},\ x > 0,\ \alpha,\beta,\gamma,\theta,\lambda > 0, \tag{1}$$

where the parameters θ and γ are called shape parameters, α and β are called scale parameters and λ is called accelerated parameter. The reduced version of the new modified Weibull distribution proposed by [13] with $\theta = \gamma = \frac{1}{2}$, has a cumulative distribution function (CDF) given by

$$F(t) = 1 - \exp\left\{-\sqrt{x}\left(\alpha + \beta\exp\{\lambda x\}\right)\right\}, \tag{2}$$

where α and β are called scale parameters and λ is called accelerated parameter. The reduced modified Weibull (RMW) distribution has a bathtub shaped hazard rate (FRF) function.

The data obtained from a real life testing, may be complete or censored data. The word complete data are used when we observe the lifetime of all units under the test. But, censored data is used when the informations about the lifetime for some but not all units under test. When, we talk about censored data, type-I and type-II censoring schemes are appeared as the oldest censoring scheme. In the mechanism of type-I censoring scheme, the test time is prefixed but, the number of failure units is randomly. Also, in the mechanism of type-II censoring scheme, the number of failure units is prefixed

* Corresponding author e-mail: saad8582@hotmail.com

and the test time is randomly. Each of these two types of censoring schemes, units is removed from the test only at the final point. Hence, we can say that the two types of censoring scheme do not allows to remove units from the test other than the final point. Then, to satisfies the property of removed units at any step of the test, progressive censoring scheme can serve this property for more details about progressive censoring scheme [14] has presented extensive review of this scheme. Also, for more generlization of progressive censoring scheme see [15] and [16].

For the progressive type-II censoring scheme, suppose n independent uints are randomly selected from the life product to put under a life test. Also, the pre-fixed integer number m satisfies that $m \leq n$ as well as the censoring scheme $\mathbf{R} = (R_1, R_2, ..., R_m)$ are proposed . When the first failure $T^{\mathbf{R}}_{1;m,n}$ is observed R_1 units is randomly selected to removed from the test. When the second failure $T^{\mathbf{R}}_{2;m,n}$ is observed R_2 units is randomly selected to removed from the test. The test is continual until the m-th failure $T^{\mathbf{R}}_{m;m,n}$ is observed then, the $R_{\mathbf{m}}$ srvival units are removed from the test. Hence, the set of random variables $\underline{T}=(T^{\mathbf{R}}_{1;m,n}, T^{\mathbf{R}}_{2;m,n}, ..., T^{\mathbf{R}}_{m;m,n})$ is called a progressively type-II censoring sample collected with the progressive censored scheme $\mathbf{R}$ and the prior integers $(m, n, \mathbf{R})$ satisfies that, $n = m + \sum_{i=1}^{m} R_i$. If the failure time $T^{\mathbf{R}}_{i;m,n}$, $i = 1, 2, ..., m$ distributed with PDF and CDF given respecitvely by $f(t)$ and $F(t)$, then the joint likelihood function of observed progressive type-II censoring sample $\underline{t}=(t^{\mathbf{R}}_{1;m,n}, t^{\mathbf{R}}_{2;m,n}, ..., t^{\mathbf{R}}_{m;m,n})$ is formulated by

$$L(\underline{t}|\underline{\theta}) = Q\prod_{i=1}^{m} f(t_i)[\overline{F}(t_i)]^{R_i}, \tag{3}$$

where $\overline{F}(.) = 1 - F(.)$, $t_i = t^{\mathbf{R}}_{1;m,n}$, $0 < t_1 < t_2 < ... < t_m < \infty$ and

$$Q = \prod_{i=1}^{m}(n - \sum_{j=1}^{i-1} R_i - i + 1). \tag{4}$$

The type-II censoring scheme and complete sample cases can be obtained as a special case of progressive type-II censoring scheme.

Models with bathtub shaped or increasing hazard rate functions is more required for several applications in reliability and lifetime analysis. So, our objective in this paper is focus on new form of RMW distribution with three parameters as the bathtub shape FRF with the general censoring scheme called progressive type-II censoring scheme. Therefore, we present the NRMW distribution and mainly focus for developing the point and interval estimate of model parameters with classical MLE and Bayes methods with the help of MCMC methods. The developed theoretical results are discussed and assessed with the two, real data analysis and Moto Carlo simulation studies.

The paper has planed as the following. The model and its properties given in Section 2. The point and corresponding approximate confidence interval estimate With ML methods is discussed in Section 3. Also, Bayesian approach under MCMC method for point and credible interval estimate is discussed in Section 4. The model has been building for the real data set in Section 5. Monte Carlo simulation results are presented in Section 6. Finally, a numerical brief comments in Section 7.

2 Model Description

Now, we define the new reduce modified Weibull (NRMW) distribution obtained from (1) for given the value $\theta = \gamma = 2$ with CDF given by

$$F(x) = 1 - e^{-\alpha x^2 - \beta x^2 e^{\lambda x}}, \; x > 0, \; \alpha, \beta, \lambda > 0. \tag{5}$$

Also, the parametrs α and β are scale parameters and λ is called accelerated parameter. Then , the probability density function (PDF) of NRMW distribution is given by

$$f(x) = (2\alpha x + \beta(2 + \lambda x)x\exp\{\lambda x\})\exp\left\{-x^2(\alpha + \beta\exp\{\lambda x\})\right\}. \tag{6}$$

Different shapes of probability and hazard rate functions for different parameter values has presented in figure 1 and 2 . It's clear that the hazard rate function is increasing with different parameter values and the PDF is more flexible. This distribution includes some important models which are used widely in survival analysis such as Rayleigh distribution when β=λ=0.

Let X denote a random variable having the NRMW distribution then.

1.The $r-$th moment of X is given by

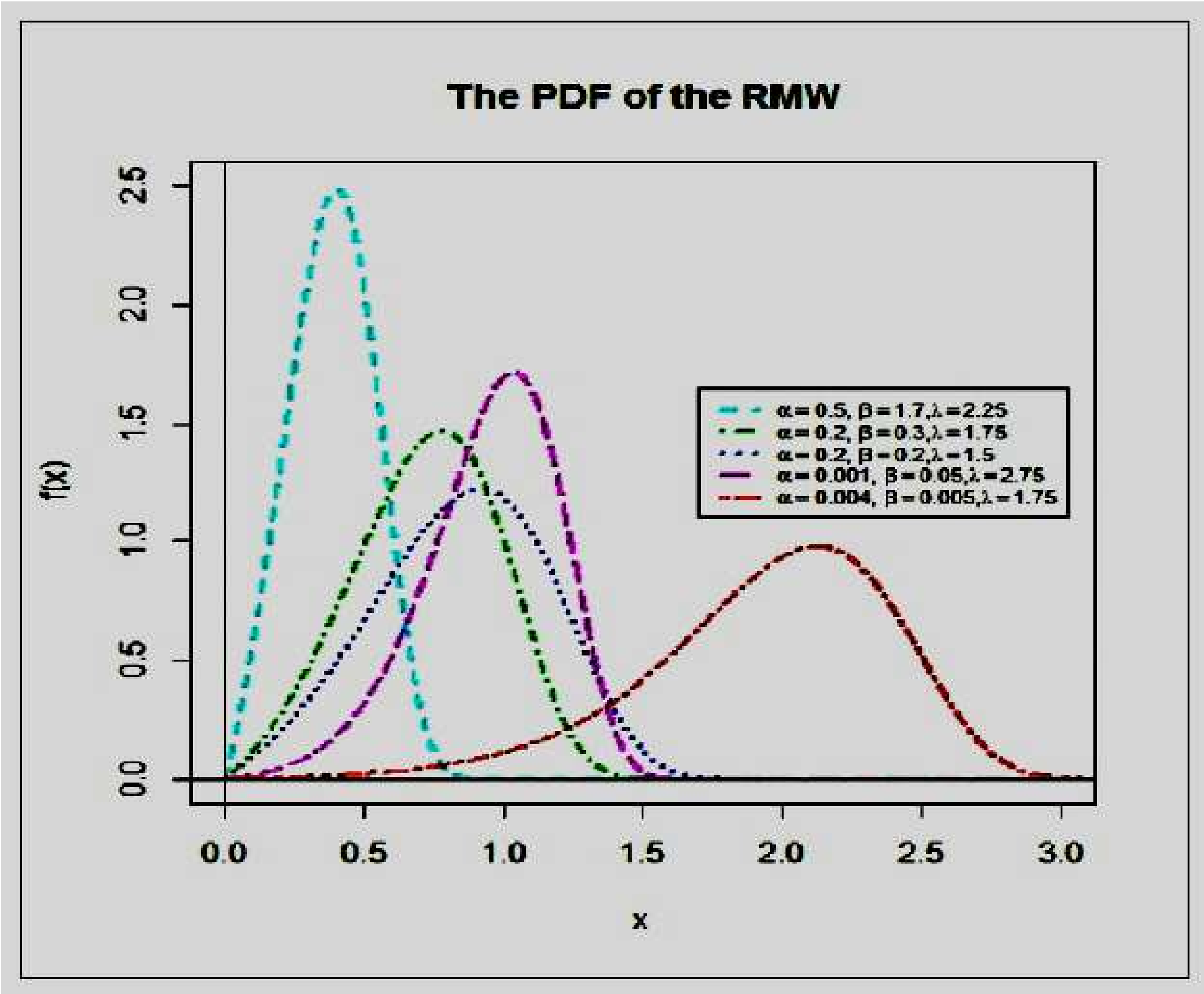

Fig. 1: The probability functions of the NRMW distribution.

$$\mu_r = E(x^r) = \frac{r}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-\beta)^i(\lambda i)^j}{i!j!}\alpha^{-\frac{2i+j+r}{2}}\Gamma(\frac{2i+j+r}{2}),\ r=1,2,...,\tag{7}$$

2.The moment generating function of X is given by

$$M_x(t) = 1+\frac{1}{2}\sum_{n,m,i=0}^{\infty}\frac{(-\beta)^n(\lambda n)^m t^{i+1}}{n!m!i\,!}\left[\alpha^{-\frac{2n+m+i+1}{2}}\Gamma(\frac{2n+m+i+1}{2})\right]\tag{8}$$

Where $\Gamma(.)$ is the gamma function.

3 Maximum Likelihood Estimation

In this section, we built the point and interval estimate of NRMW parameters for given progressive type-II censoring sample

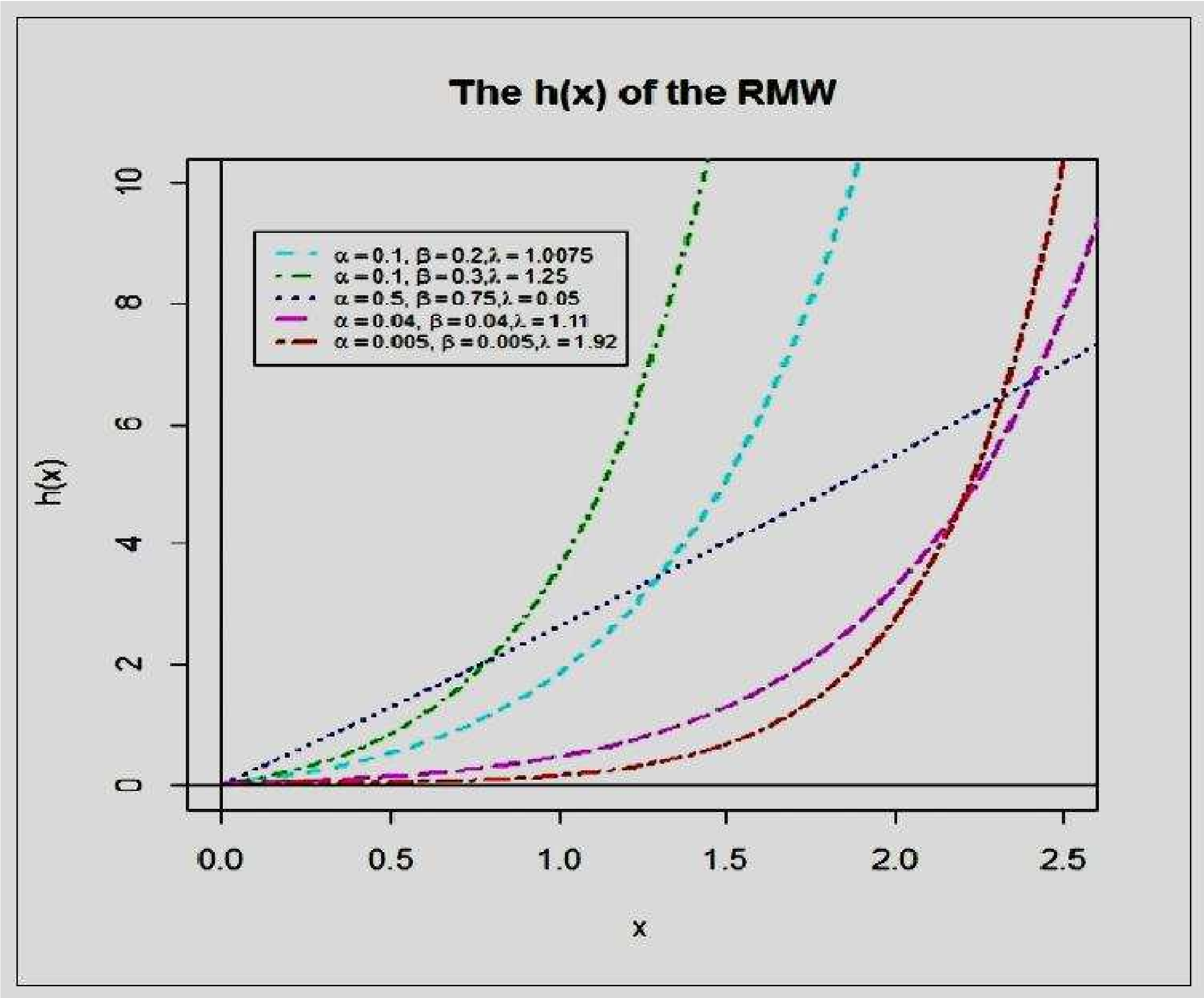

Fig. 2: The hazard functions of the NRMW distribution.

3.1 Point MLE

Under observed progressive type-II censoring sample $\underline{t}$=($t^{\mathbf{R}}_{1;m,n}$, $t^{\mathbf{R}}_{2;m,n}$, ..., $t^{\mathbf{R}}_{m;m,n}$) and distribution given by (3) and (4) the likelihood function (3) is reduced to

$$L(\alpha,\beta,\lambda|\underline{t}) = Q\left(\prod_{i=1}^{m} t_i\right)\exp\left\{\sum_{i=1}^{m}\log\left(2\alpha+\beta(2+\lambda t_i)\exp\{\lambda t_i\}\right)-\sum_{i=1}^{m}(R_i+1)t_i^2\left(\alpha+\beta\exp\{\lambda t_i\}\right)\right\}. \tag{9}$$

Then, without normalized constant the logarithms of (7) is reduced to

$$\ell(\alpha,\beta,\lambda|\underline{t}) = \sum_{i=1}^{m}\log\left(2\alpha+\beta(2+\lambda t_i)\exp\{\lambda t_i\}\right)-\sum_{i=1}^{m}(R_i+1)t_i^2\left(\alpha+\beta\exp\{\lambda t_i\}\right). \tag{10}$$

The likelihood equations are obtained from (8) after taken the first partial derivatives with respect to parameters vector $\underline{\eta}= (\alpha,\beta,\lambda)$ as follows

$$\sum_{i=1}^{m}\frac{2}{2\alpha+\beta(2+\lambda t_i)\exp\{\lambda t_i\}}-\sum_{i=1}^{m}(R_i+1)t_i^2=0, \tag{11}$$

$$\sum_{i=1}^{m}\frac{(2+\lambda t_i)\exp\{\lambda t_i\}}{2\alpha+\beta(2+\lambda t_i)\exp\{\lambda t_i\}}-\sum_{i=1}^{m}(R_i+1)t_i^2\exp\{\lambda t_i\}=0, \tag{12}$$

and

$$\sum_{i=1}^{m}\frac{(3+\lambda t_i)t_i\exp\{\lambda t_i\}}{2\alpha+\beta(2+\lambda t_i)\exp\{\lambda t_i\}}-\sum_{i=1}^{m}(R_i+1)t_i^3\exp\{\lambda t_i\}=0. \tag{13}$$

Then, the likelihood equations are reduced to three nonlinear equations (9), (10) and (11) solve numerical with any itteration method such as Newton Raphson to present the mximum values of $\underline{\eta}=(\alpha,\beta,\lambda)$ as to obtain $\underline{\hat{\eta}}=(\hat{\alpha},\hat{\beta},\hat{\lambda})$.

3.2 Interval estimation

From the loglikelihood function (6), the second derivitive for respect $\underline{\eta}=(\alpha,\beta,\lambda)$ is given by

$$\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\alpha^2}=\sum_{i=1}^{m}\frac{-4}{[2\alpha+\beta(2+\lambda t_i)\exp\{\lambda t_i\}]^2}, \tag{14}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\beta^2}=\sum_{i=1}^{m}\frac{-(2+\lambda t_i)^2\exp\{2\lambda t_i\}}{[2\alpha+\beta(2+\lambda t_i)\exp\{\lambda t_i\}]^2} \tag{15}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\lambda^2}=\sum_{i=1}^{m}\frac{\partial}{\partial\lambda}\left[\frac{\beta(3+\lambda t_i)t_i\exp\{\lambda t_i\}}{[2\alpha+\beta(2+\lambda t_i)\exp\{\lambda t_i\}]^2}\right]-\beta\sum_{i=1}^{m}(R_i+1)t_i^4\exp\{\lambda t_i\}, \tag{16}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\alpha\partial\beta}=\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\beta\partial\alpha}=-\sum_{i=1}^{m}\frac{2(2+\lambda t_i)\exp\{\lambda t_i\}}{[2\alpha+\beta(2+\lambda t_i)\exp\{\lambda t_i\}]^2}, \tag{17}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\alpha\partial\lambda}=\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\lambda\partial\alpha}=-\sum_{i=1}^{m}\frac{2\beta(3+\lambda t_i)t_i\exp\{\lambda t_i\}}{[2\alpha+\beta(2+\lambda t_i)\exp\{\lambda t_i\}]^2}, \tag{18}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\beta\partial\lambda}=\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\lambda\partial\beta}=\sum_{i=1}^{m}\frac{\partial}{\partial\lambda}\left[\frac{(2+\lambda t_i)\exp\{\lambda t_i\}}{2\alpha+\beta(2+\lambda t_i)\exp\{\lambda t_i\}}\right]-\sum_{i=1}^{m}(R_i+1)t_i^3\exp\{\lambda t_i\}. \tag{19}$$

Then, the Fisher information matrix (FIM) is built with minus expectation of derivatives (14) to (19) as follows

$$\mathrm{FIM}=-\mathrm{E}\left(\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\eta_i\partial\eta_j}\right), i,j=1,\ 2,\ 3. \tag{20}$$

The FIM needing to compute the minus expectation of second derivatives which in general is more difficult. Practice, we replace it with approximate Fisher information matrix defined by

$$\mathrm{FIM}_0=-\left(\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\eta_i\partial\eta_j}\right)_{\hat{\eta}=(\hat{\alpha},\hat{\beta},\hat{\lambda})}, i,j=1,\ 2,\ 3. \tag{21}$$

The approximate form (21) from FIM with the normality distribution of MLE of model parameters $\underline{\hat{\eta}}=(\hat{\alpha},\hat{\beta},\hat{\lambda})$ are used to built the approximate confidence intervals of model parameters $\underline{\eta}=(\alpha,\beta,\lambda)$. The estimators are distributed with $N\left(\underline{\eta},\mathrm{FIM}_0^{-1}(\underline{\hat{\eta}})\right)$ with the $\mathrm{FIM}_0^{-1}(\underline{\hat{\eta}})$ denoted to the inverse of approximate FIM. Hence, the approximate 100(1-ζ)% confidence intervals of $\underline{\eta}=(\alpha,\beta,\lambda)$ is formulated as

$$\hat{\alpha}\mp y_{\frac{\zeta}{2}}\sqrt{v_{11}},\ \hat{\beta}\mp y_{\frac{\zeta}{2}}\sqrt{v_{22}},\ \hat{\lambda}\mp y_{\frac{\zeta}{2}}\sqrt{v_{33}} \tag{22}$$

where the value $y_{\frac{\zeta}{2}}$ is denoted to the percentile of the standard normal distribution with right-tail probability ζ and the diagonal values of $\mathrm{FIM}_0^{-1}(\underline{\hat{\eta}})$ has the values v_{11}, v_{22} and v_{33}.

4 Bayesian Approach under MCMC Method

The available informations about the model parameters are exposed by the past experience which is presented by prior distribution. Also, informations exposed by the data which is presented by the likelihood function. Then, the Bayesian approach need to formulate the parameters posterior distribution. So, we adopt independent gamma priors for all parameters values $\underline{\eta} = (\alpha, \beta, \lambda)$ as follow

$$h^*(\underline{\eta}) \propto \prod_{i=1}^{3} \eta_i^{a_i-1} \exp(-b_i\eta_i),\ \eta_i > 0,\ a_i,\ b_i > 0,\ i = 1,2,3. \tag{23}$$

Hence, in generally the posterior distribution can be described by

$$h(\underline{\eta}) \propto h^*(\underline{\eta}) L\left(\underline{\eta}|\underline{t}\right). \tag{24}$$

Then, the Bayes estimation for the parameters or any function of parameters dependent on the mathematical formula of $h(\underline{\eta})$ and the corresponding loss function. Hence, without loss the generality the Bayes estimate of function $\Psi(\underline{\eta})$ under squared error loss (SEL) function is given by

$$\hat{\Psi}_B = \int_{\underline{\eta}} \Psi(\underline{\eta}) h(\underline{\eta}) d\eta_1 d\eta_2 d\eta_3. \tag{25}$$

The mathematical form of posterior distribution (24) as well as estimation in (25) in general need to complex integration which approximate with some numerical methods such as numerical integration. The more suitable method which can serve this problem specially under high dimensional cases called MCMC method which is adopted in this paper as follows

4.1 MCMC approach

From the likelihood function (7) and prior distribution (21) the posterior distribution can be described by

$$h(\alpha,\beta,\lambda|\underline{t}) \propto$$

$$\alpha^{a_1-1}\beta^{a_2-1}\lambda^{a_3-1}\exp\left\{-b_1\alpha - b_2\beta - b_3\lambda + \sum_{i=1}^{m}\log\left(2\alpha + \beta(2+\lambda t_i)\exp\{\lambda t_i\}\right) - \sum_{i=1}^{m}(R_i+1)t_i^2\left(\alpha + \beta\exp\{\lambda t_i\}\right)\right\} \tag{26}$$

Hence, the full conditional PDF of posterior distribution for the vector $\underline{\eta} = (\alpha, \beta, \lambda)$ is given by

$$h_1(\alpha|\beta,\lambda,\underline{t}) \propto \Omega \times \text{Gamma}(a_1,\ b_1 + \sum_{i=1}^{m}(R_i+1)t_i^2), \tag{27}$$

$$h_2(\beta|\alpha,\lambda,\underline{t}) \propto \Omega \times \text{Gamma}(a_2,\ b_2 + \sum_{i=1}^{m}(R_i+1)t_i^2\exp\{\lambda t_i\}), \tag{28}$$

and

$$h_3(\lambda|\alpha,\beta,\underline{t}) \propto \Omega\lambda^{a_3-1}\exp\left\{-b_3\lambda - \beta\sum_{i=1}^{m}(R_i+1)t_i^2\exp\{\lambda t_i\}\right\}, \tag{29}$$

where

$$\Omega = \exp\left\{\sum_{i=1}^{m}\log\left(2\alpha + \beta(2+\lambda t_i)\exp\{\lambda t_i\}\right)\right\} \tag{30}$$

The function (27) to (29) has shown that, the conditional distributions of parameters vector $\underline{\eta}$. Then the problem of generation the data from posterior distribution to present the empirical form of posterior distribution need to more information about the Metropolis Hasting (MH) [17] under Gibbs algorithms as for the recently review of MH under Gibbs see [18] and [19] follows

4.2 MH under Gibbs algorithms

Step 1:Begin with intial, $s=1$ and $\underline{\eta}^{(0)}=(\hat{\alpha},\hat{\beta},\hat{\lambda})$.

Step 2:Under non-symetric proposal distribution generate $\alpha^{(s)}$ from gamma distribution Gamma$(a_1, b_1+\sum_{i=1}^{m}(R_i+1)t_i^2)$ given in (25).

Step 3:Under non-symetric proposal distribution generate $\beta^{(s)}$ from gamma distribution Gamma$(a_2, b_2+\sum_{i=1}^{m}(R_i+1)t_i^2\exp\{\lambda t_i\})$ given in (28).

Step 4:Under a symetric proposal normal distribution generate $\lambda^{(s)}$ from (27) with proposal $N\left(\lambda^{(s-1)},\ \sqrt{v_{33}}\right)$.

Step 5:Record the vector $\underline{\eta}^{(s)}=(\alpha^{(s)},\beta^{(s)},\lambda^{(s)})$.

Step 6:Put $s=s+1$.

Step 7:Repet the steps from step2 to step6, M times.

Step 8:The Bayes MCMC estimate of function $\Psi(\underline{\eta})$ is gevien by

$$\hat{\Psi}_{\mathrm{B}}=\frac{1}{M-M^*}\sum_{i=M^*+1}^{M}\Psi^{(i)}, \tag{31}$$

where M^* is the number of MCMC iteration needing to achieved the stationary distribution (burn-in) and the corsponding posterior variance of Ψ is given by

$$\widehat{V}(\Psi|\underline{t})=\frac{1}{M-M^*}\sum_{i=M^*+1}^{M}\left(\Psi^{(i)}-\hat{\Psi}_{\mathrm{B}}\right)^2. \tag{32}$$

Step 10:The credible interval estimators with 100(1-ζ)% of Ψ is given by

$$\left(\Psi_{\frac{\zeta}{2}(M-M^*)},\Psi_{(1-\frac{\zeta}{2})(M-M^*)}\right). \tag{33}$$

5 Data Analysis

In this section, we check the efficiency of the theoretical results of NRMW distribution for analysis the data sets. Then, we generate a progressive type-II sample of NRMW with parameters $\underline{\eta}=(0.5,0.1,1.5)$ and the censoring parameters (n,m,R)=(50,30,{2, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 0, 2, 0, 2, 0, 0, 0, 0, 2, 0, 0, 2, 0, 2, 0, 2, 0, 0, 2}). The prior information corresponding with true parameters values are selected to be (a_i,b_i)={(2,4), (1,5), (3,2)}. Then, the results of MLE and Bayes etimate are given in Table (2). for Bayesian approach, we run the chan 11000 with the first 1000 value as brun-in and the point and interval estimate are presented in Table 2. The figures from (1) to (8) describe the MCMC sample.

Table 1: The simulate data set.

0.1502	0.1761	0.2817	0.3321	0.3696	0.3846	0.4637	0.4840	0.4974	0.5153
0.5957	0.6033	0.6206	0.6379	0.6619	0.6809	0.6847	0.6930	0.7324	0.7507
0.8395	0.9142	0.9618	0.9654	0.9936	0.9965	1.0164	1.0833	1.0854	1.1532

Table 2: MLEs, Bays estimate and 95% approximate intervales estimate

Pa.s	$(.)_{\mathrm{ML}}$	$(.)_{\mathrm{MCMC}}$	95% ACI	95% CI
$\alpha=0.5$	0.40281	0.3126	(0.1242, 2.1543)	(0.1649, 0.5387)
$\beta=0.1$	0.16842	0.0590	(0.0024, 0.5125)	(0.0619, 0.1405)
$\lambda=1.5$	1.48876	2.1557	(1.1212, 4.2142)	(1.1443, 3.8239)

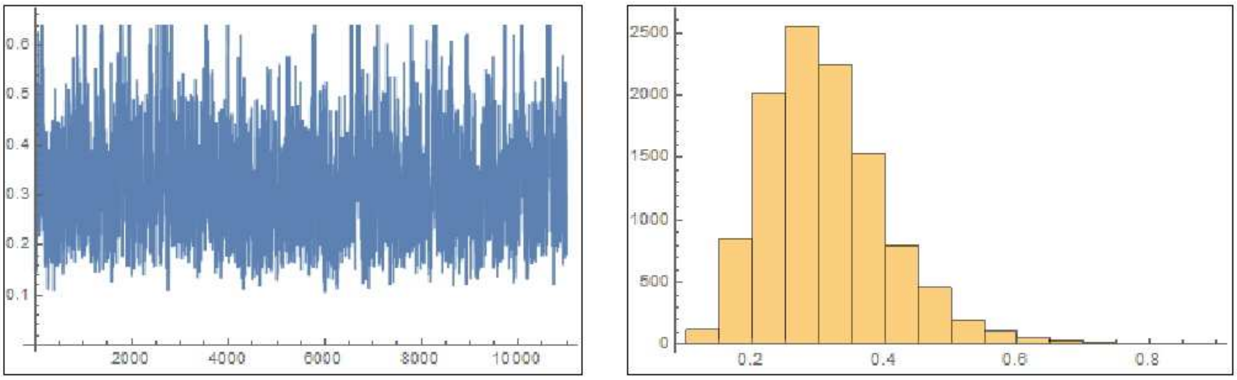

Fig. 3: The simulation number and the corresponding histogram of α generated by MCMC.

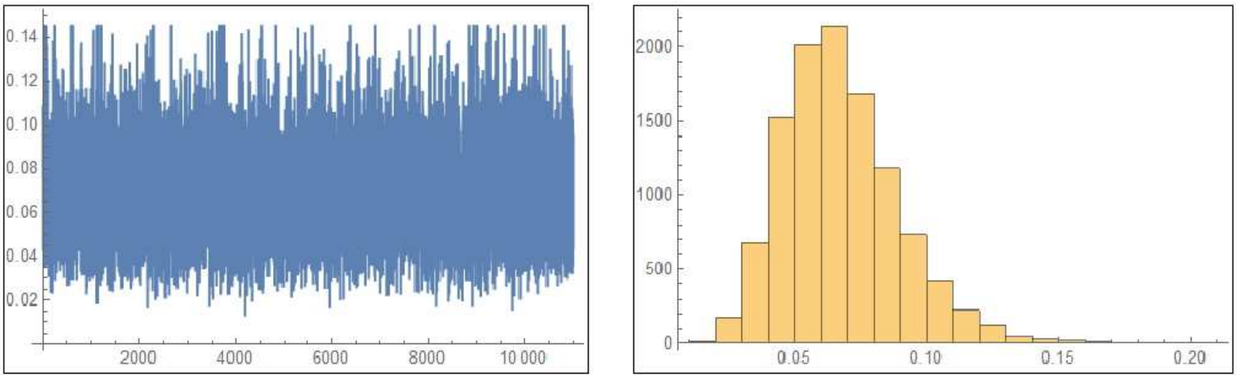

Fig. 4: The simulation number and the corresponding histogram of β generated by MCMC.

6 Simulation Studies

The estimation results developed in this article under ML and Bayes methods are compared and assessed with Monto Carlo simulation study. Also, we measured the effect each of sample size n and affect sample size m for different censoring schemes. The simulation study are done for two sets of parameters vector $\underline{\eta}$ and the corresponding prior information are selected to satisfies $\mathrm{E}(\eta_i) \simeq \frac{a_i}{b_i}$. Different tolls are adopted to assess the estimate, mean (ME) and mean squared error (MSE) are used to assess the point estimate but, the average interval length (AEL) and probability coverage (PC), where ME and MSE are given by

$$\begin{cases} \mathrm{ME} = \bar{\eta}_i = \sum_{i=1}^{N} \hat{\eta}_i^{(i)} \\ \mathrm{MSE} = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\eta}_i^{(i)} - \bar{\eta}_i \right)^2 \end{cases}, \tag{32}$$

where $\underline{\eta} = (\alpha, \beta, \lambda)$. The parameters vector is selected to be

Case I:$\underline{\eta}$=(0.1, 0.5, 1.0) and the corresponding prior information is described by, non-informative prior $\mathrm{P}_0 = \{(a_i, b_i) = (0.00001, 0.00001)\}$, $i = 1,2,3,4$ as well as informative prior $\mathrm{P}_1 = \{(1, 5), (2, 3), (3, 2)\}$

Case II$\underline{\eta}$=(0.5, 0.01, 0.01) and the corresponding prior information is described by P_0 as well as informative prior $\mathrm{P}_2 = \{(1, 5), (2, 3), (3, 2)\}$.

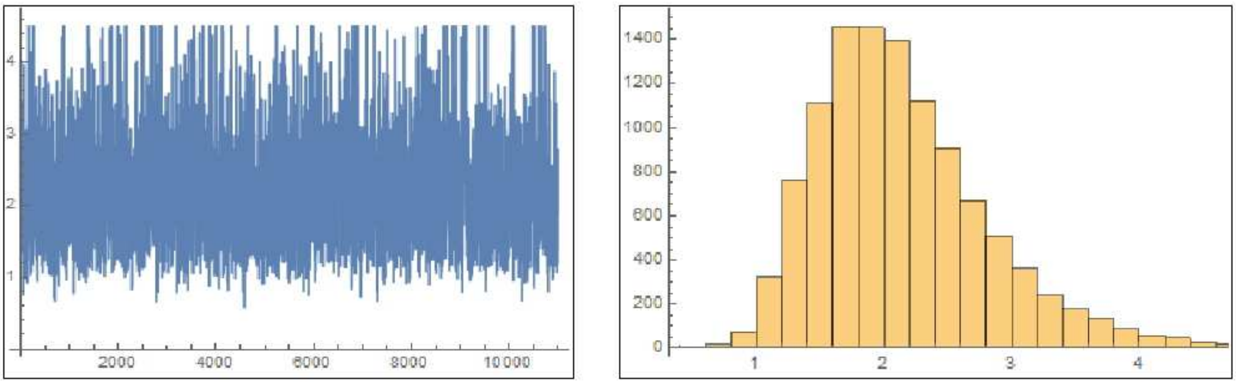

Fig. 5: The simulation number and the corresponding histogram of λ generated by MCMC..

The simulation results are formulated under generate 1000 different sample NRMW distribution. The Bayesian approach with MCMC method have chan with 10500 iteration taken the first 500 iteration as the burn-in. Progressive censoring scheme (PCS) that is used in our simulation can be described as

$$\text{PCS}=\begin{cases} I:\ R_i=\begin{cases} 0,\ i<m \\ n-m,\ i=m \end{cases} \\ II:\ R_i=\begin{cases} 0,\ i>1 \\ n-m,\ i=1 \end{cases} \\ III:\ R_i=\begin{cases} 0,\ i\neq \frac{m+1}{2} \text{ and } m \text{ odd or } i\neq \frac{m}{2} \text{ and } m \text{ even} \\ n-m,\ i=\frac{m+1}{2} \text{ and } m \text{ odd or } i=\frac{m}{2} \text{ and } m \text{ even} \end{cases} \\ V:\ R_i=\begin{cases} 0,\ i>n-m \\ 1,\ i\leq n-m \end{cases} \end{cases}$$

The simulation results are presented in Tables (3) to (6) as follows.

Table 3: The MEs and MSEs (Case I).

(n,m)	PCS	tolls	$(.)_{ML}$			$(.)_{BMCMC-P_0}$			$(.)_{BMCMC-P_1}$		
			α	β	λ	α	β	λ	α	β	λ
(30,20)	*I*	MEs	0.214	0.642	1.235	0.217	0.635	1.222	0.177	0.600	1.201
		MSEs	0.074	0.124	0.475	0.065	0.111	0.462	0.044	0.098	0.411
	II	MEs	0.182	0.605	1.201	0.188	0.542	1.200	0.145	0.584	1.211
		MSEs	0.050	0.111	0.445	0.049	0.110	0.441	0.032	0.065	0.380
	III	MEs	0.210	0.635	1.228	0.211	0.629	1.217	0.171	0.598	1.203
		MSEs	0.066	0.118	0.467	0.058	0.108	0.454	0.039	0.091	0.406
	V	MEs	0.212	0.618	1.221	0.213	0.624	1.212	0.166	0.589	1.211
		MSEs	0.062	0.115	0.464	0.052	0.102	0.450	0.035	0.088	0.400
(50,30)	*I*	MEs	0.185	0.613	1.207	0.188	0.605	1.190	0.150	0.572	1.170
		MSEs	0.055	0.103	0.456	0.057	0.090	0.443	0.025	0.079	0.390
	II	MEs	0.152	0.575	1.170	0.160	0.513	1.171	0.114	0.555	1.184
		MSEs	0.030	0.090	0.426	0.031	0.092	0.422	0.013	0.047	0.362
	III	MEs	0.180	0.605	1.201	0.185	0.603	1.180	0.149	0.570	1.174
		MSEs	0.047	0.101	0.427	0.039	0.088	0.438	0.018	0.072	0.384
	V	MEs	0.184	0.587	1.201	0.187	0.601	1.183	0.141	0.558	1.187
		MSEs	0.044	0.100	0.444	0.035	0.091	0.431	0.017	0.061	0.381
(50,40)	*I*	MEs	0.171	0.600	1.198	0.172	0.587	1.175	0.137	0.561	1.158
		MSEs	0.041	0.092	0.442	0.042	0.076	0.431	0.011	0.064	0.376
	II	MEs	0.139	0.562	1.155	0.147	0.500	1.159	0.100	0.543	1.171
		MSEs	0.016	0.057	0.412	0.016	0.078	0.409	0.001	0.033	0.349
	III	MEs	0.166	0.601	1.190	0.172	0.587	1.166	0.138	0.557	1.161
		MSEs	0.038	0.091	0.414	0.025	0.076	0.425	0.009	0.066	0.374
	V	MEs	0.171	0.572	1.193	0.173	0.591	1.169	0.130	0.541	1.177
		MSEs	0.031	0.092	0.432	0.024	0.088	0.420	0.005	0.049	0.369
(70,50)	*I*	MEs	0.160	0.592	1.190	0.162	0.574	1.162	0.122	0.549	1.150
		MSEs	0.030	0.081	0.433	0.032	0.064	0.422	0.002	0.051	0.366
	II	MEs	0.130	0.550	1.147	0.140	0.491	1.148	0.098	0.532	1.160
		MSEs	0.008	0.050	0.401	0.010	0.071	0.402	0.000	0.025	0.341
	III	MEs	0.154	0.594	1.181	0.163	0.578	1.160	0.130	0.548	1.152
		MSEs	0.030	0.082	0.405	0.014	0.066	0.417	0.001	0.060	0.366
	V	MEs	0.132	0.566	1.189	0.165	0.588	1.160	0.122	0.533	1.168
		MSEs	0.024	0.088	0.427	0.018	0.079	0.411	0.001	0.040	0.360

Table 4: The MELs and CPs (Case I).

(n,m)	PCS	tolls	$(.)_{ML}$			$(.)_{BMCMC-P_0}$			$(.)_{BMCMC-P_1}$		
			α	β	λ	α	β	λ	α	β	λ
(30,20)	*I*	MELs	0.321	1.812	3.895	0.302	1.799	3.887	0.277	1.701	3.823
		CPs	0.88	0.87	0.89	0.89	0.90	0.89	0.90	0.91	0.90
	II	MELs	0.290	1.782	3.865	0.275	1.771	3.870	0.255	1.675	3.801
		CPs	0.89	0.90	0.90	0.90	0.91	0.89	0.91	0.91	0.92
	III	MELs	0.310	1.803	3.882	0.291	1.788	3.879	0.268	1.689	3.812
		CPs	0.87	0.89	0.90	0.89	0.90	0.90	0.90	0.90	0.91
	V	MELs	0.308	1.800	3.881	0.294	1.783	3.883	0.265	1.680	3.810
		CPs	0.89	0.89	0.89	0.90	0.90	0.91	0.92	0.91	0.91
(50,30)	*I*	MELs	0.291	1.782	3.866	0.275	1.770	3.861	0.249	1.673	3.800
		CPs	0.90	0.91	0.90	0.92	0.92	0.91	0.92	0.93	0.92
	II	MELs	0.262	1.754	3.837	0.249	1.741	3.842	0.230	1.646	3.771
		CPs	0.92	0.96	0.92	0.94	0.93	0.90	0.92	0.93	0.97
	III	MELs	0.283	1.775	3.854	0.264	1.757	3.848	0.240	1.655	3.783
		CPs	0.90	0.94	0.92	0.91	0.92	0.93	0.94	0.91	0.91
	V	MELs	0.281	1.773	3.851	0.269	1.754	3.853	0.238	1.649	3.779
		CPs	0.90	0.91	0.92	0.92	0.93	0.94	0.92	0.95	0.92
(50,40)	*I*	MELs	0.279	1.769	3.850	0.261	1.756	3.848	0.234	1.660	3.786
		CPs	0.91	0.91	0.92	0.92	0.92	0.90	0.92	0.93	0.92
	II	MELs	0.248	1.741	3.822	0.235	1.728	3.828	0.214	1.635	3.756
		CPs	0.95	0.96	0.92	0.93	0.93	0.91	0.92	0.95	0.94
	III	MELs	0.270	1.761	3.841	0.252	1.743	3.833	0.225	1.641	3.772
		CPs	0.93	0.94	0.92	0.93	0.92	0.93	0.94	0.92	0.91
	V	MELs	0.268	1.761	3.838	0.254	1.741	3.839	0.224	1.636	3.764
		CPs	0.95	0.92	0.92	0.92	0.90	0.94	0.94	0.92	0.91
(70,50)	*I*	MELs	0.258	1.750	3.832	0.242	1.740	3.830	0.221	1.642	3.763
		CPs	0.95	0.91	0.92	0.92	0.95	0.92	0.92	0.93	0.92
	II	MELs	0.229	1.722	3.805	0.217	1.709	3.804	0.200	1.615	3.737
		CPs	0.89	0.94	0.92	0.92	0.93	0.91	0.92	0.95	0.93
	III	MELs	0.255	1.748	3.829	0.237	1.731	3.814	0.208	1.623	3.754
		CPs	0.94	0.94	0.93	0.93	0.92	0.93	0.94	0.92	0.92
	V	MELs	0.250	1.745	3.819	0.235	1.727	3.814	0.205	1.618	3.747
		CPs	0.92	0.90	0.92	0.92	0.90	0.94	0.92	0.92	0.94

Table 5: The MEs and MSEs (Case 2).

(n,m)	PCS	tolls	$(.)_{ML}$			$(.)_{BMCMC-P_0}$			$(.)_{BMCMC-P_1}$		
			α	β	λ	α	β	λ	α	β	λ
(30,20)	*I*	MEs	0.6324	0.0136	0.0135	0.6143	0.0125	0.0130	0.5804	0.0122	0.0128
		MSEs	0.1323	0.0087	0.0077	0.1312	0.0081	0.0071	0.1220	0.0045	0.0077
	II	MEs	0.6112	0.0125	0.0124	0.6022	0.0114	0.0119	0.5790	0.0113	0.0119
		MSEs	0.1307	0.0071	0.0066	0.1300	0.0069	0.0068	0.1207	0.0032	0.0063
	III	MEs	0.6212	0.0128	0.0127	0.6074	0.0119	0.0126	0.5801	0.0118	0.0124
		MSEs	0.1316	0.0078	0.0069	0.1303	0.0065	0.0065	0.1211	0.0039	0.0068
	V	MEs	0.6201	0.0127	0.0124	0.6069	0.0118	0.0127	0.5811	0.0115	0.0127
		MSEs	0.1314	0.0073	0.0064	0.1300	0.0061	0.0059	0.1202	0.0034	0.0063
(50,30)	*I*	MEs	0.6301	0.0131	0.0130	0.6138	0.0121	0.0127	0.5798	0.0118	0.0124
		MSEs	0.1290	0.0069	0.0059	0.1285	0.0065	0.0053	0.1201	0.0028	0.0060
	II	MEs	0.6101	0.0122	0.0121	0.6017	0.0109	0.0114	0.5788	0.0108	0.0114
		MSEs	0.1256	0.0054	0.0049	0.1271	0.0048	0.0050	0.1280	0.0018	0.0047
	III	MEs	0.6207	0.0123	0.0122	0.6070	0.0117	0.0122	0.5789	0.0112	0.0120
		MSEs	0.1265	0.0061	0.0051	0.1273	0.0048	0.0047	0.1185	0.0021	0.0051
	V	MEs	0.6209	0.0124	0.0125	0.6068	0.0119	0.0124	0.5807	0.0109	0.0126
		MSEs	0.1256	0.0057	0.0049	0.1281	0.0049	0.0041	0.1180	0.0018	0.0046
(50,40)	*I*	MEs	0.6250	0.0127	0.0124	0.6103	0.0116	0.0122	0.5769	0.0113	0.0119
		MSEs	0.1269	0.0056	0.0051	0.1257	0.0056	0.0045	0.1170	0.0019	0.0049
	II	MEs	0.6101	0.0118	0.0117	0.6000	0.0103	0.0109	0.5751	0.0104	0.0109
		MSEs	0.1224	0.0042	0.0038	0.1242	0.0039	0.0041	0.1251	0.0011	0.0038
	III	MEs	0.6170	0.0118	0.0119	0.6041	0.0118	0.0121	0.5761	0.01058	0.0117
		MSEs	0.1239	0.0052	0.0043	0.1245	0.0039	0.0038	0.1161	0.0014	0.0041
	V	MEs	0.6181	0.0120	0.0121	0.6035	0.0114	0.0120	0.5763	0.0102	0.0124
		MSEs	0.1237	0.0048	0.0050	0.1252	0.0038	0.0032	0.1152	0.0009	0.0038
(70,50)	*I*	MEs	0.6231	0.0118	0.0114	0.6057	0.0107	0.0111	0.5725	0.0102	0.0108
		MSEs	0.1250	0.0047	0.0042	0.1239	0.0044	0.0037	0.1152	0.0008	0.0038
	II	MEs	0.6072	0.0108	0.0108	0.6001	0.0091	0.0101	0.5722	0.0095	0.0102
		MSEs	0.1205	0.0033	0.0029	0.1219	0.0027	0.0030	0.1232	0.0001	0.0029
	III	MEs	0.6142	0.0107	0.0109	0.6013	0.0111	0.0114	0.5730	0.01047	0.0110
		MSEs	0.1221	0.0041	0.0025	0.1224	0.0030	0.0028	0.1143	0.0004	0.0033
	V	MEs	0.6149	0.0111	0.01201	0.6008	0.0102	0.0111	0.5737	0.0095	0.0114
		MSEs	0.1218	0.0027	0.0041	0.1235	0.0026	0.0022	0.1137	0.0002	0.0029

Table 6: The MEs and MSEs (Case I).

(n,m)	PCS	tolls	$(.)_{ML}$			$(.)_{BMCMC-P_0}$			$(.)_{BMCMC-P_1}$		
			α	β	λ	α	β	λ	α	β	λ
(30,20)	*I*	MELs	1.992	0.152	0.184	1.887	0.144	0.179	1.742	0.114	0.125
		CPs	0.88	0.89	0.90	0.90	0.91	0.89	0.90	0.91	0.92
	II	MELs	1.972	0.135	0.167	1.865	0.126	0.151	1.720	0.100	0.108
		CPs	0.89	0.90	0.90	0.91	0.90	0.90	0.91	0.91	0.91
	III	MELs	1.984	0.143	0.175	1.879	0.135	0.168	1.735	0.105	0.114
		CPs	0.89	0.89	0.89	0.90	0.90	0.90	0.91	0.91	0.91
	V	MELs	1.980	0.141	0.171	1.873	0.132	0.163	1.731	0.101	0.110
		CPs	0.89	0.90	0.90	0.90	0.91	0.90	0.91	0.91	0.90
(50,30)	*I*	MELs	1.965	0.135	0.165	1.868	0.125	0.160	1.724	0.100	0.107
		CPs	0.90	0.89	0.91	0.90	0.92	0.89	0.91	0.91	0.92
	II	MELs	1.951	0.117	0.148	1.846	0.107	0.132	1.704	0.85	0.100
		CPs	0.91	0.90	0.91	0.91	0.90	0.92	0.91	0.91	0.93
	III	MELs	1.965	0.122	0.157	1.860	0.117	0.145	1.718	0.090	0.100
		CPs	0.90	0.91	0.90	0.91	0.92	0.90	0.91	0.93	0.91
	V	MELs	1.959	0.119	0.154	1.855	0.114	0.141	1.714	0.082	0.098
		CPs	0.92	0.90	0.91	0.90	0.91	0.90	0.96	0.91	0.93
(50,40)	*I*	MELs	1.944	0.121	0.151	1.845	0.114	0.147	1.712	0.089	0.098
		CPs	0.91	0.91	0.92	0.90	0.93	0.90	0.92	0.92	0.92
	II	MELs	1.939	0.105	0.135	1.831	0.094	0.119	1.690	0.838	0.085
		CPs	0.92	0.91	0.91	0.91	0.93	0.92	0.93	0.91	0.96
	III	MELs	1.952	0.111	0.143	1.845	0.108	0.131	1.707	0.078	0.086
		CPs	0.92	0.91	0.94	0.93	0.92	0.94	0.91	0.95	0.92
	V	MELs	1.945	0.105	0.141	1.842	0.104	0.130	1.701	0.069	0.081
		CPs	0.92	0.91	0.93	0.90	0.93	0.92	0.96	0.93	0.93
(70,50)	*I*	MELs	1.929	0.110	0.138	1.840	0.102	0.135	1.701	0.080	0.090
		CPs	0.93	0.91	0.95	0.92	0.93	0.94	0.92	0.93	0.97
	II	MELs	1.924	0.092	0.122	1.819	0.090	0.111	1.678	0.824	0.076
		CPs	0.92	0.92	0.91	0.93	0.93	0.92	0.95	0.90	0.94
	III	MELs	1.941	0.102	0.131	1.832	0.099	0.118	1.700	0.065	0.071
		CPs	0.91	0.91	0.93	0.93	0.92	0.94	0.92	0.95	0.93
	V	MELs	1.933	0.101	0.129	1.830	0.097	0.118	1.689	0.054	0.068
		CPs	0.90	0.92	0.93	0.90	0.91	0.92	0.96	0.92	0.94

7 Comments

The models with bathtub shaped or increasing hazard rate function have modeled a several real lifetime data. Also, the quality of the life products is measured based on the information that obtaining from a life testing experiment. So, in this paper, we are considered the reduced form of modified Wiebull distribution as a life distribution under general censoring scheme called progressive type-II censoring scheme. The estimation results with ML and Bayes is developed and assessed with numerical computation. The numerical results has shown that

1: From the numerical results the proposed model under progressive type-II censoring scheme can serve different life products.
2: The Bayes estimation under non-informative prior are more closed to maximum likelihood estimation.
3: The informative priors P_1 and P_2 serve better than maximum likelihood estimation.
4: The results is more better for increasing sample size m.
5: The scheme that has withdraw in the first step perform better than other scheme.
6: The different choose of model parameters has satisfies the validity of the results.

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Conflicts of Interests
The authors declare that they have no conflicts of interests

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