

Implicit Resolvent Equation Problem in Hilbert Spaces

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Abstract: In this paper, we study an implicit resolvent equation problem with its corresponding implicit variational inclusion problem in Hilbert spaces. We establish that the implicit variational inclusion problem is equivalent to a fixed point problem and a relationship between implicit variational inclusion problem with an implicit resolvent equation problem is shown. This equivalence is used to suggest an iterative algorithm for solving implicit resolvent equation problem. Finally, an existence and convergence result is proved.

Keywords: Algorithm, Convergence, Implicit, Inclusion, Resolvent, Solution.

1 Introduction

Variational inequalities (inclusions) can be viewed as innovative and novel extension of the variational problems. The study of variational inequalities (inclusions) and their systems occupies a central and significant role in the interdisciplinary research between analysis, geometry, biology, elasticity, optimization, image processing, biomedical science and mathematical physics, etc., see e.g., [1, 2, 4–6, 8–11, 13–20]. The resolvent operator techniques for solving variational inequalities (inclusions) are important and of much importance for recent research going on. The resolvent operator technique is used to establish an equivalence between variational inclusions and resolvent equations. The resolvent operator technique is used to develop powerful and efficient numerical techniques for solving variational inclusions and other related problems.

In this paper, we study an implicit variational inclusion problem with its corresponding implicit resolvent equation problem. We define an iterative algorithm to approximate the solution of implicit resolvent equation problem. Finally, an existence and convergence result is proved for implicit resolvent equation problem in Hilbert space.

2 Formulation and Preliminaries

Throughout this paper, we suppose that X is a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle \cdot, \cdot \rangle$,

d is the metric induced by the norm $\|\cdot\|$, 2^X (respectively, $CB(X)$) is the family of all nonempty (respectively, closed and bounded) subsets of X , and $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$ defined by

$$D(P, Q) = \max \left\{ \sup_{x \in P} d(x, Q), \sup_{y \in Q} d(P, y) \right\},$$

where $d(x, Q) = \inf_{y \in Q} d(x, y)$ and $d(P, y) = \inf_{x \in P} d(x, y)$.

Let us recall the required definitions.

Definition 1. A mapping $g : X \rightarrow X$ is said to be

(i) *Lipschitz continuous* if, there exist a constant $\lambda_g > 0$ such that

$$\|g(x) - g(y)\| \leq \lambda_g \|x - y\|, \forall x, y \in X.$$

(ii) *monotone*, if

$$\langle g(x) - g(y), x - y \rangle \geq 0, \forall x, y \in X.$$

(iii) *strongly monotone* if, there exists a constant $\xi > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq \xi \|x - y\|^2, \forall x, y \in X.$$

(iv) *relaxed Lipschitz continuous* if, there exists a constant $r > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \leq -r \|x - y\|^2, \forall x, y \in X.$$

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Definition 2. A mapping $N : X \times X \times X \rightarrow X$ is said to be Lipschitz continuous with respect to first argument if, there exists a constant λ_{N_1} such that for all $x_1, y_1, x_2, x_3 \in X$,

$$\|N(x_1, x_2, x_3) - N(y_1, x_2, x_3)\| \leq \lambda_{N_1} \|x_1 - y_1\|.$$

Similarly, we can define the Lipschitz continuity of N in rest of the arguments.

Definition 3. A set-valued mapping $A : X \rightarrow CB(X)$ is said to be D -Lipschitz continuous if, there exists a constant δ_A such that

$$D(A(x), A(y)) \leq \delta_A \|x - y\|, \forall x, y \in X.$$

Definition 4. Let $H : X \rightarrow X$ be a mapping and $I : X \rightarrow X$ be an identity mapping. Then, a set-valued mapping $M : X \rightarrow 2^X$ is said to be $(I - H)$ -monotone if, M is monotone, H is relaxed Lipschitz continuous and

$$[(I - H) + \lambda M](X) = X,$$

where $\lambda > 0$ is a constant.

Definition 5. Let $H : X \rightarrow X$ be relaxed Lipschitz continuous mapping and $I : X \rightarrow X$ be an identity mapping. Suppose that $M : X \rightarrow 2^X$ is a set-valued, $(I - H)$ -monotone mapping. The relaxed resolvent operator $R_{\lambda, M}^{(I-H)} : X \rightarrow X$ associated with I, H and M is defined by

$$R_{\lambda, M}^{(I-H)}(x) = [(I - H) + \lambda M]^{-1}(x), \forall x \in X, \quad (1)$$

where $\lambda > 0$ is a constant.

Theorem 1. Let $H : X \rightarrow X$ be a r -relaxed Lipschitz continuous mapping, $I : X \rightarrow X$ be an identity mapping and $M : X \rightarrow 2^X$ be a set-valued $(I - H)$ -monotone mapping. Then the operator $[(I - H) + \lambda M]^{-1}$ is single-valued, where $\lambda > 0$ is a constant.

Proof. For any $z \in X$ and a constant $\lambda > 0$, let $x, y \in [(I - H) + \lambda M]^{-1}(z)$. Then,

$$\lambda^{-1}[z - (I - H)(x)] \in M(x);$$

$$\lambda^{-1}[z - (I - H)(y)] \in M(y).$$

Since M is monotone, we have

$$\begin{aligned} \langle -(I - H)(x) + z + (I - H)(y) - z, x - y \rangle &\geq 0; \\ -\langle (I - H)(x) - (I - H)(y), x - y \rangle &\geq 0; \\ -\langle x - H(x) - y + H(y), x - y \rangle &\geq 0; \\ \langle x - H(x) - y + H(y), x - y \rangle &\leq 0; \\ \langle x - H(x) - y + H(y), x - y \rangle &\leq 0; \\ \langle x - y, x - y \rangle - \langle H(x) - H(y), x - y \rangle &\leq 0. \end{aligned}$$

Since H is r -relaxed Lipschitz continuous, we have

$$\begin{aligned} 0 &\geq \langle x - y, x - y \rangle - \langle H(x) - H(y), x - y \rangle \\ &\geq \|x - y\|^2 + r\|x - y\|^2 \geq 0, \end{aligned}$$

it follows that $(1 + r)\|x - y\|^2 = 0$, which implies that $x = y$. Thus $[(I - H) + \lambda M]^{-1}$ is single-valued.

Theorem 2. Let $H : X \rightarrow X$ be a r -relaxed Lipschitz continuous mapping, $I : X \rightarrow X$ be an identity mapping and $M : X \rightarrow 2^X$ be a set-valued, $(I - H)$ -monotone mapping. Then the resolvent operator $R_{\lambda, M}^{I-H} : X \rightarrow X$ is $\frac{1}{[1+r]}$ -Lipschitz continuous. i.e.,

$$\|R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y)\| \leq \frac{1}{[1+r]} \|x - y\|, \forall x, y \in X.$$

Proof. Let any x and y be any given point in X . If follow from (1) that

$$\begin{aligned} R_{\lambda, M}^{I-H}(x) &= [(I - H) + \lambda M]^{-1}(x), \\ R_{\lambda, M}^{I-H}(y) &= [(I - H) + \lambda M]^{-1}(y). \end{aligned} \quad (2)$$

It follows that

$$\begin{aligned} \frac{1}{\lambda} [x - (I - H)(R_{\lambda, M}^{I-H}(x))] &\in M(R_{\lambda, M}^{I-H}(x)), \\ \frac{1}{\lambda} [y - (I - H)(R_{\lambda, M}^{I-H}(y))] &\in M(R_{\lambda, M}^{I-H}(y)). \end{aligned} \quad (3)$$

Since M is $(I - H)$ -monotone i.e., M is monotone, we have

$$\begin{aligned} 0 &\leq \frac{1}{\lambda} \langle x - (I - H)(R_{\lambda, M}^{I-H}(x)) - (y - (I - H)(R_{\lambda, M}^{I-H}(y))), \\ &\quad R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \rangle \\ &= \frac{1}{\lambda} \langle x - y - \{ (I - H)(R_{\lambda, M}^{I-H}(x)) - (I - H)(R_{\lambda, M}^{I-H}(y)) \}, \\ &\quad R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \rangle. \end{aligned} \quad (4)$$

It follows that

$$\begin{aligned} &\langle x - y, R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \rangle \\ &\geq \langle (I - H)(R_{\lambda, M}^{I-H}(x)) - (I - H)(R_{\lambda, M}^{I-H}(y)), \\ &\quad R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \rangle. \end{aligned} \quad (5)$$

By Cauchy-Schwartz inequality, (5) and r -relaxed Lipschitz continuity of H , we have

$$\begin{aligned} &\|x - y\| \|R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y)\| \\ &\geq \langle x - y, R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \rangle \\ &\geq \langle R_{\lambda, M}^{I-H}(x) - H(R_{\lambda, M}^{I-H}(x)) - R_{\lambda, M}^{I-H}(y) + H(R_{\lambda, M}^{I-H}(y)), \\ &\quad R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \rangle \\ &= \langle R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y), R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \rangle - \\ &\quad \langle H(R_{\lambda, M}^{I-H}(x)) - H(R_{\lambda, M}^{I-H}(y)), R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y) \rangle \\ &\geq \|R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y)\|^2 + r \|R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y)\|^2 \\ &= (1 + r) \|R_{\lambda, M}^{I-H}(x) - R_{\lambda, M}^{I-H}(y)\|^2. \end{aligned} \quad (6)$$

Thus, we have

$$\|R_{\lambda,M}^{I-H}(x) - R_{\lambda,M}^{I-H}(y)\| \leq \frac{1}{[1+r]} \|x - y\|,$$

i.e., the relaxed resolvent operator $R_{\lambda,M}^{I-H}$ is $\frac{1}{[1+r]}$ -Lipschitz continuous.

In support of Theorem 1 and Theorem 2, we have the following example.

Example 1. Let $X = \mathbb{R}^3$ with usual inner product. Let $H : X \rightarrow X$ be a mapping defined by

$$H(x) = (-x_1, -2x_2, -4x_3), \forall x = (x_1, x_2, x_3) \in X,$$

and the mapping $M : X \rightarrow 2^X$ is defined by

$$M(x) = (4x_1, 3x_2, x_3), \forall x = (x_1, x_2, x_3) \in X.$$

Then, it easy to check that H is 1-relaxed Lipschitz continuous and M is monotone. In addition, it is easy to verify that for $\lambda = 1$, $[(I - H) + \lambda M](X) = X$, which shows that M is $(I - H)$ -monotone mapping. Hence, the relaxed resolvent operator $R_{\lambda,M}^{I-H} : X \rightarrow X$ associated with I, H and M is of the form:

$$R_{\lambda,M}^{I-H}(x) = \left(\frac{x_1}{6}, \frac{x_2}{6}, \frac{x_3}{6}\right), \forall x = (x_1, x_2, x_3) \in X.$$

It is easy to see that the relaxed resolvent operator defined above is single-valued.

Now, we prove that $R_{\lambda,M}^{I-H}$ is Lipschitz continuous.

$$\begin{aligned} & \|R_{\lambda,M}^{I-H}(x) - R_{\lambda,M}^{I-H}(y)\| \\ &= \left\| \left(\frac{x_1}{6}, \frac{x_2}{6}, \frac{x_3}{6}\right) - \left(\frac{y_1}{6}, \frac{y_2}{6}, \frac{y_3}{6}\right) \right\| \\ &= \left\| \left(\frac{x_1 - y_1}{6}, \frac{x_2 - y_2}{6}, \frac{x_3 - y_3}{6}\right) \right\| \\ &= \left[\frac{(x_1 - y_1)^2}{36} + \frac{(x_2 - y_2)^2}{36} + \frac{(x_3 - y_3)^2}{36} \right]^{\frac{1}{2}} \\ &= \frac{1}{6} [(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2]^{\frac{1}{2}} \\ &\leq \frac{1}{2} \|x - y\|. \end{aligned}$$

Hence, the resolvent operator $R_{\lambda,M}^{I-H}$ is $\frac{1}{2}$ -Lipschitz continuous.

3 Resolvent equation problem and convergence result

In this section, we consider an implicit variational inclusion problem and an implicit resolvent equation problem. Based on fixed point formulation of implicit variational inequality problem, we establish an

equivalence relation between implicit variational inequality problem and implicit resolvent equation problem.

Let X be the real Hilbert space and $H, g : X \rightarrow X$, $N : X \times X \times X \rightarrow X$ be the single-valued mappings, $I : X \rightarrow X$ be an identity mapping. Suppose that $A, S, T : X \rightarrow CB(X)$ and $M : X \times X \rightarrow 2^X$ are the set-valued mappings such that M is $(I - H)$ -monotone. Then we consider the problem of finding $x \in X$, $u \in A(x)$, $v \in S(x)$, $w \in T(x)$ and $g(x) \cap \text{dom} M(\cdot, x) \neq \emptyset$ such that

$$0 \in N(u, v, w) + M(g(x), x). \quad (7)$$

Problem (7) is called implicit variational inclusion problem.

Below are some special cases of implicit variational inclusion problem (7).

- (i) If $T \equiv 0$, $g = I$, $N(u, v, w) = N(u, v)$ and $M(g(x), x) = M(x)$, then problem (7) reduces to the problem of finding $x \in X$, $u \in A(x)$, $v \in S(x)$ such that

$$0 \in N(u, v) + M(x). \quad (8)$$

A problem similar to (8) was considered and studied by Chang et al. [3] in the setting of Banach spaces.

- (ii) If $S, T \equiv 0$, $g = I$, A is a single-valued identity mapping and $N(x, \cdot, \cdot) = N(x)$, $M(g(x), x) = M(x)$ then problem (7) reduces to the problem of finding $x \in X$ such that

$$0 \in N(x) + M(x). \quad (9)$$

Problem (9) was considered and studied by Fang and Huag [7] in the setting of q -uniformly smooth Banach spaces with H -accretive operator M .

It is easy to see that implicit variational inclusion problem (7) includes many more known variational inclusions considered and studied in the literature in different settings.

In connection with the implicit variational inclusion problem (7), we consider the following implicit resolvent equation problem:

Find $z, x \in X$, $u \in A(x)$, $v \in S(x)$ and $w \in T(x)$ such that

$$N(u, v, w) + \lambda^{-1} J_{\lambda, M(\cdot, x)}^{I-H}(z) = 0, \quad (10)$$

where $\lambda > 0$ is a constant and $J_{\lambda, M(\cdot, x)}^{I-H}(z) = \left[I - (I - H) \left(R_{\lambda, M(\cdot, x)}^{I-H} \right) \right](z)$, where I is identity operator, $R_{\lambda, M(\cdot, x)}^{I-H}$ is the relaxed resolvent operator and $\left[I - (I - H) \left(R_{\lambda, M(\cdot, x)}^{I-H} \right) \right](z) = \left[z - (I - H) R_{\lambda, M(\cdot, x)}^{I-H}(z) \right]$.

We mention the following equivalence between implicit variational inclusion problem (7) and a fixed point problem which can be easily proved by using the definition of relaxed resolvent operator.

Lemma 1. Let $x \in X$, $u \in A(x)$, $v \in S(x)$ and $w \in T(x)$ is a solution of implicit variational inclusion problem (7) if and only if (x, u, v, w) satisfies the following equation:

$$g(x) = R_{\lambda, M(\cdot, x)}^{I-H} [(I-H)g(x) - \lambda N(u, v, w)], \quad (11)$$

where

$$R_{\lambda, M(\cdot, x)}^{I-H} = [(I-H) + \lambda M(\cdot, x)]^{-1},$$

and $\lambda > 0$ is a constant.

Now, we show that the implicit variational problem (7) is equivalent to implicit resolvent equation problem (10).

Lemma 2. The variational inclusion problem (7) has a solution (x, u, v, w) with $x \in X$, $u \in A(x)$, $v \in S(x)$ and $w \in T(x)$ if and only if the implicit resolvent equation problem (10) has a solution (z, x, u, v, w) with $z, x \in X$, $u \in A(x)$, $v \in S(x)$ and $w \in T(x)$, where

$$g(x) = R_{\lambda, M(\cdot, x)}^{I-H}(z), \quad (12)$$

and

$$z = [(I-H)g(x) - \lambda N(u, v, w)].$$

Proof. Let (x, u, v, w) be a solution of implicit variational inclusion problem (7). Then by Lemma 1, it satisfies the following equation:

$$g(x) = R_{\lambda, M(\cdot, x)}^{I-H} [(I-H)g(x) - \lambda N(u, v, w)].$$

Using the fact that $J_{\lambda, M(\cdot, x)}^{I-H}(z) = [I - (I-H)(R_{\lambda, M(\cdot, x)}^{I-H})](z)$ and above equation (11), we obtain

$$\begin{aligned} & J_{\lambda, M(\cdot, x)}^{I-H}(z) \\ &= J_{\lambda, M(\cdot, x)}^{I-H} [(I-H)g(x) - \lambda N(u, v, w)] \\ &= [I - (I-H)(R_{\lambda, M(\cdot, x)}^{I-H})] ((I-H)g(x) - \lambda N(u, v, w)) \\ &= ((I-H)g(x) - \lambda N(u, v, w)) - [(I-H)(R_{\lambda, M(\cdot, x)}^{I-H})] \\ &\quad ((I-H)g(x) - \lambda N(u, v, w)) \\ &= ((I-H)g(x) - \lambda N(u, v, w)) - \\ &\quad (I-H) [R_{\lambda, M(\cdot, x)}^{I-H} ((I-H)g(x) - \lambda N(u, v, w))] \\ &= ((I-H)g(x) - \lambda N(u, v, w)) - (I-H)g(x) \\ &= -\lambda N(u, v, w). \end{aligned}$$

It follows that

$$N(u, v, w) + \lambda^{-1} J_{\lambda, M(\cdot, x)}^{I-H}(z) = 0$$

with

$$z = [(I-H)g(x) - \lambda N(u, v, w)],$$

i.e., (z, x, u, v, w) is a solution of the implicit resolvent equation problem (10).

Conversely, let (z, x, u, v, w) is a solution of the implicit resolvent equation problem (10). i.e.,

$$N(u, v, w) + \lambda^{-1} J_{\lambda, M(\cdot, x)}^{I-H}(z) = 0.$$

Using the Definition of $J_{\lambda, M(\cdot, x)}^{I-H}$, we have

$$\begin{aligned} & [I - (I-H)(R_{\lambda, M(\cdot, x)}^{I-H})](z) = -\lambda N(u, v, w) \\ & z - (I-H)(R_{\lambda, M(\cdot, x)}^{I-H})(z) = -\lambda N(u, v, w) \\ & z - (I-H)(R_{\lambda, M(\cdot, x)}^{I-H}(z)) = -\lambda N(u, v, w) \end{aligned}$$

$$z = (I-H)(R_{\lambda, M(\cdot, x)}^{I-H}(z)) - \lambda N(u, v, w)$$

$$z = [(I-H)g(x) - \lambda N(u, v, w)]$$

$$R_{\lambda, M(\cdot, x)}^{I-H}(z) = R_{\lambda, M(\cdot, x)}^{I-H} [(I-H)g(x) - \lambda N(u, v, w)]$$

$$g(x) = R_{\lambda, M(\cdot, x)}^{I-H} [(I-H)g(x) - \lambda N(u, v, w)].$$

Thus, by Lemma 1, (x, u, v, w) is a solution of the implicit variational inclusion problem (7).

Alternative Proof. Let

$$z = [(I-H)g(x) - \lambda N(u, v, w)],$$

then from (12), we have

$$g(x) = R_{\lambda, M(\cdot, x)}^{I-H}(z).$$

Then,

$$z = [(I-H)(R_{\lambda, M(\cdot, x)}^{I-H}(z)) - \lambda N(u, v, w)]$$

$$z - (I-H)(R_{\lambda, M(\cdot, x)}^{I-H}(z)) = -\lambda N(u, v, w),$$

which implies that

$$[I - (I-H)(R_{\lambda, M(\cdot, x)}^{I-H})](z) = -\lambda N(u, v, w).$$

Using the fact that

$$J_{\lambda, M(\cdot, x)}^{I-H}(z) = [I - (I-H)(R_{\lambda, M(\cdot, x)}^{I-H})](z),$$

it follows that

$$J_{\lambda, M(\cdot, x)}^{I-H}(z) = -\lambda N(u, v, w),$$

and thus

$$N(u, v, w) + \lambda^{-1} J_{\lambda, M(\cdot, x)}^{I-H}(z) = 0,$$

the required implicit resolvent equation problem (10). $\square$

Based on above discussion, we now define the following iterative Algorithm for solving implicit resolvent equation problem (10).

Now, we prove an existence and convergence result for implicit resolvent equation problem (10).

Algorithm 1

For any initial points $z_0, x_0 \in X$, $u_0 \in A(x_0)$, $v_0 \in S(x_0)$ and $w_0 \in T(x_0)$, let

$$z_1 = [(I - H)g(x_0) - \lambda N(u_0, v_0, w_0)].$$

Take $z_1, x_1 \in X$ such that

$$g(x_1) = R_{\lambda, M(\cdot, x_1)}^{I-H}(z_1).$$

Since $u_0 \in A(x_0)$, $v_0 \in S(x_0)$ and $w_0 \in T(x_0)$, by Nadler's Theorem [12], there exists $u_1 \in A(x_1)$, $v_1 \in S(x_1)$ and $w_1 \in T(x_1)$ such that

$$\|u_0 - u_1\| \leq D(A(x_0), A(x_1)),$$

$$\|v_0 - v_1\| \leq D(S(x_0), S(x_1)),$$

$$\|w_0 - w_1\| \leq D(T(x_0), T(x_1)),$$

where $D(\cdot, \cdot)$ is the Hausdörf metric on $CB(X)$.

Let

$$z_2 = [(I - H)g(x_1) - \lambda N(u_1, v_1, w_1)]$$

and take any $z_2, x_2 \in X$ such that

$$g(x_2) = R_{\lambda, M(\cdot, x_2)}^{I-H}(z_2).$$

Continuing the above process inductively, we can compute the sequences $\{z_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ by the following iterative scheme:

$$g(x_n) = R_{\lambda, M(\cdot, x_n)}^{I-H}(z_n), \quad (13)$$

$$u_n \in A(x_n), \|u_n - u_{n+1}\| \leq D(A(x_n), A(x_{n+1})), \quad (14)$$

$$v_n \in B(x_n), \|v_n - v_{n+1}\| \leq D(S(x_n), S(x_{n+1})), \quad (15)$$

$$w_n \in C(x_n), \|w_n - w_{n+1}\| \leq D(T(x_n), T(x_{n+1})), \quad (16)$$

$$z_{n+1} = [(I - H)g(x_n) - \lambda N(u_n, v_n, w_n)], \quad (17)$$

where $\lambda > 0$ is a constant and $n = 0, 1, 2, \dots$.

Theorem 3. Let X be a real Hilbert space and $H, g : X \rightarrow X$ be the single-valued mappings such that g is ξ -strongly monotone, λ_g -Lipschitz continuous, H is r -relaxed Lipschitz continuous and λ_H -Lipschitz continuous. Suppose that $N : X \times X \times X \rightarrow X$ is a single-valued mapping such that N is Lipschitz continuous in all the three arguments with constants λ_{N_1} , λ_{N_2} and λ_{N_3} , respectively and $A, S, T : X \rightarrow CB(X)$ be the set-valued mappings such that A is δ_A -D-Lipschitz continuous, S is δ_S -D-Lipschitz continuous and T is δ_T -D-Lipschitz continuous. Suppose that set-valued mapping $M : X \times X \rightarrow 2^X$ is such that for a fixed $x \in X$, $M(\cdot, x)$ is $(I - H)$ -monotone mapping. Let $I : X \rightarrow X$ be an identity mapping and for each $x \in X$. Suppose that there exists constants $\lambda > 0$ and $h > 0$ such that the following conditions holds:

$$\|R_{\lambda, M(\cdot, x)}^{I-H}(z) - R_{\lambda, M(\cdot, y)}^{I-H}(z)\| \leq h\|x - y\|, \quad \forall x, y, z \in X, \quad (18)$$

and

$$0 \leq \lambda_g + \lambda_H \lambda_g + \lambda \lambda_{N_1} \delta_A + \lambda \lambda_{N_2} \delta_S + \lambda \lambda_{N_3} \delta_T < [1 + r][\xi - h], \quad \xi > h. \quad (19)$$

Then, there exist $z, x \in X$, $u \in A(x)$, $v \in S(x)$ and $w \in T(x)$ satisfying (10) and iterative sequences $\{z_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ generated by Algorithm 1 converge strongly to z, x, u, v and w , respectively.

Proof. Using Cauchy-Schwartz inequality and ξ -strong monotonicity of g , we have

$$\|g(x_n) - g(x_{n-1})\| \|x_n - x_{n-1}\| \geq \langle g(x_n) - g(x_{n-1}), x_n - x_{n-1} \rangle \geq \xi \|x_n - x_{n-1}\|^2,$$

which implies that

$$\|x_n - x_{n-1}\| \leq \frac{1}{\xi} \|g(x_n) - g(x_{n-1})\|. \quad (20)$$

As g is λ_g -Lipschitz continuous, H is λ_H -Lipschitz continuous, N is Lipschitz continuous in all three arguments with constants λ_{N_1} , λ_{N_2} , λ_{N_3} , respectively, A is δ_A -D-Lipschitz continuous, S is δ_S -D-Lipschitz continuous, T is δ_T -D-Lipschitz continuous, and using Algorithm 1, we obtain

$$\begin{aligned} & \|z_{n+1} - z_n\| \\ &= \|(I - H)g(x_n) - (I - H)g(x_{n-1}) - \lambda(N(u_n, v_n, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1}))\| \\ &\leq \|(I - H)g(x_n) - (I - H)g(x_{n-1})\| + \lambda \|N(u_n, v_n, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1})\| \\ &\leq \|g(x_n) - g(x_{n-1})\| + \|H(g(x_n)) - H(g(x_{n-1}))\| \\ &\quad + \lambda \|N(u_n, v_n, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1})\| \\ &\quad + \lambda \|N(u_{n-1}, v_n, w_n) - N(u_{n-1}, v_{n-1}, w_n)\| \\ &\quad + \lambda \|N(u_{n-1}, v_{n-1}, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1})\| \\ &\leq \|g(x_n) - g(x_{n-1})\| + \|H(g(x_n)) - H(g(x_{n-1}))\| \\ &\quad + \lambda \|N(u_n, v_n, w_n) - N(u_{n-1}, v_n, w_n)\| \\ &\quad + \lambda \|N(u_{n-1}, v_n, w_n) - N(u_{n-1}, v_{n-1}, w_n)\| \\ &\quad + \lambda \|N(u_{n-1}, v_{n-1}, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1})\| \\ &\leq \lambda_g \|x_n - x_{n-1}\| + \lambda_H \lambda_g \|x_n - x_{n-1}\| + \lambda \lambda_{N_1} \|u_n - u_{n-1}\| \\ &\quad + \lambda \lambda_{N_2} \|v_n - v_{n-1}\| + \lambda \lambda_{N_3} \|w_n - w_{n-1}\| \\ &\leq \lambda_g \|x_n - x_{n-1}\| + \lambda_H \lambda_g \|x_n - x_{n-1}\| + \lambda \lambda_{N_1} D(A(x_{n-1}), A(x_n)) \\ &\quad + \lambda \lambda_{N_2} D(S(x_{n-1}), S(x_n)) + \lambda \lambda_{N_3} D(T(x_{n-1}), T(x_n)) \\ &\leq \lambda_g \|x_n - x_{n-1}\| + \lambda_H \lambda_g \|x_n - x_{n-1}\| + \lambda \lambda_{N_1} \delta_A \|x_n - x_{n-1}\| \\ &\quad + \lambda \lambda_{N_2} \delta_S \|x_n - x_{n-1}\| + \lambda \lambda_{N_3} \delta_T \|x_n - x_{n-1}\| \\ &\leq [\lambda_g + \lambda_H \lambda_g + \lambda \lambda_{N_1} \delta_A + \lambda \lambda_{N_2} \delta_S + \lambda \lambda_{N_3} \delta_T] \|x_n - x_{n-1}\|. \quad (21) \end{aligned}$$

By using Theorem 2 and condition (18), we have

$$\begin{aligned}
 & \|g(x_n) - g(x_{n-1})\| \\
 &= \|R_{\lambda, M(\cdot, x_n)}^{I-H}(z_n) - R_{\lambda, M(\cdot, x_{n-1})}^{I-H}(z_{n-1})\| \\
 &= \|R_{\lambda, M(\cdot, x_n)}^{I-H}(z_n) - R_{\lambda, M(\cdot, x_n)}^{I-H}(z_{n-1}) + R_{\lambda, M(\cdot, x_n)}^{I-H}(z_{n-1}) \\
 &\quad - R_{\lambda, M(\cdot, x_{n-1})}^{I-H}(z_{n-1})\| \\
 &= \|R_{\lambda, M(\cdot, x_n)}^{I-H}(z_n) - R_{\lambda, M(\cdot, x_n)}^{I-H}(z_{n-1})\| + \|R_{\lambda, M(\cdot, x_n)}^{I-H}(z_{n-1}) \\
 &\quad - R_{\lambda, M(\cdot, x_{n-1})}^{I-H}(z_{n-1})\| \\
 &\leq \frac{1}{[1+r]} \|z_n - z_{n-1}\| + \|R_{\lambda, M(\cdot, x_n)}^{I-H}(z_{n-1}) \\
 &\quad - R_{\lambda, M(\cdot, x_{n-1})}^{I-H}(z_{n-1})\|.
 \end{aligned}$$

Using (18), we have

$$\|g(x_n) - g(x_{n-1})\| \leq \frac{1}{[1+r]} \|z_n - z_{n-1}\| + h \|x_n - x_{n-1}\|. \quad (22)$$

By (20) and (22), we have

$$\|x_n - x_{n-1}\| \leq \frac{1}{[1+r][\xi - h]} \|z_n - z_{n-1}\|. \quad (23)$$

Using (21), (22) and (23), we have

$$\begin{aligned}
 & \|z_{n+1} - z_n\| \\
 &\leq \left[\frac{\lambda_g + \lambda_H \lambda_g + \lambda \lambda_{N_1} \delta_A + \lambda \lambda_{N_2} \delta_S + \lambda \lambda_{N_3} \delta_T}{[1+r][\xi - h]} \right] \|z_n - z_{n-1}\|.
 \end{aligned}$$

i.e.,

$$\|z_{n+1} - z_n\| \leq \theta(*) \|z_n - z_{n-1}\|,$$

where

$$\theta(*) = \left[\frac{\lambda_g + \lambda_H \lambda_g + \lambda \lambda_{N_1} \delta_A + \lambda \lambda_{N_2} \delta_S + \lambda \lambda_{N_3} \delta_T}{[1+r][\xi - h]} \right].$$

From (19), we have $0 \leq \theta(*) < 1$ and consequently $\{z_n\}$ is a Cauchy sequence in X . Since X is a Hilbert space, there exist some $z \in X$ such that $z_n \rightarrow z$ as $n \rightarrow \infty$. From (23) we know that the sequence $\{x_n\}$ is also a Cauchy sequence in X . Therefore, there exists $x \in X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Since the mappings A , S and T are D -Lipschitz continuous. It follows from (14) – (16) that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are also Cauchy sequences, we can assume that $u_n \rightarrow u$, $v_n \rightarrow v$ and $w_n \rightarrow w$. By Algorithm 1, it follows that

$$\begin{aligned}
 z_{n+1} &= [(I-H)g(x_n) - \lambda N(u_n, v_n, w_n)] \rightarrow z = [(I-H)g(x) \\
 &\quad - \lambda N(u, v, w)], \text{ as } n \rightarrow \infty,
 \end{aligned} \quad (24)$$

and consequently

$$g(x_n) = R_{\lambda, M(\cdot, x_n)}^{I-H}(z_n) \rightarrow R_{\lambda, M(\cdot, x)}^{I-H}(z) = g(x), \text{ as } n \rightarrow \infty. \quad (25)$$

By (24), (25) and Lemma 2, we have

$$N(u, v, w) + \lambda^{-1} J_{\lambda, M(\cdot, x)}^{I-H}(z) = 0.$$

Finally, we prove that $u \in A(x)$, $v \in S(x)$ and $w \in T(x)$. In fact, since $\{u_n\} \in A(x_n)$ and

$$\begin{aligned}
 d(u_n, A(x)) &\leq \max \left\{ d(u_n, A(x)), \sup_{y_1 \in A(x)} d(A(x_n), y_1) \right\} \\
 &\leq \max \left\{ \sup_{y_2 \in A(x_n)} d(y_2, A(x)), \sup_{y_1 \in A(x)} d(A(x_n), y_1) \right\} \\
 &= D(A(x_n), A(x)),
 \end{aligned}$$

we have

$$\begin{aligned}
 d(u, A(x)) &\leq \|u - u_n\| + d(u_n, A(x)) \\
 &\leq \|u - u_n\| + D(A(x_n), A(x)) \\
 &\leq \|u - u_n\| + \delta_A \|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Which implies that $d(u, A(x)) = 0$. Since $A(x) \in CB(x)$, it follows that $u \in A(x)$. Similarly, we can prove that $v \in S(x)$ and $w \in T(x)$. This completes the proof.

4 Conclusion

Using the resolvent operator technique, one can prove that the variational inclusions are equivalent to the fixed point problems and resolvent equation problems. These comparable formulations have played a compelling aspect in establishing useful methods for solving the variational inclusions (inequalities) and related optimization problems. The aim of this work is to study an implicit resolvent equation problem with its equivalent implicit variational inclusion problem and corresponding fixed point problem by introducing a new type of relaxed resolvent operator. We suggest an iterative algorithm for solving implicit resolvent equation problem in Hilbert spaces. Note that the relaxed resolvent operator technique can be further applied for solving different kind of variational inclusions in diverse structure.

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