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Optimal Control of an SIR Epidemic Model with a Saturated Treatment

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Abstract: In this paper, we formulate an optimal control problem for an SIR epidemic model with saturated incidence and saturated treatment. Two main efforts, namely treatment and vaccination are considered to limit the disease transmission. The impacts of vaccination and treatment on the disease transmission are discussed through the basic reproduction number. Then to achieve control of the disease, a control problem is formulated and the existence of the control is shown. Two control functions are used, one for vaccinating the susceptible and the other for treatment efforts for infectious individuals. Appropriate optimal control methods are used to characterize the optimal levels of the two controls. The effectiveness of the proposed control solution is shown by comparing the behavior of controlled and uncontrolled systems. Numerical results show the impacts of two controls in decreasing both susceptible and infectious members of the population.

Keywords: Epidemic model; Vaccination; Optimal control; Numerical methods

1 Introduction

Recently, mathematical models describing the dynamics of infectious diseases have played an important role in the disease control in epidemiology. Most of the models are interesting in the formulation of the mode of disease transmission [1,2]. Several authors have suggested many nonlinear incidence rates to model the disease transmission process [3,4]. With these rates, many interesting and problematic transmission dynamics of disease such as periodic orbits, Hopf bifurcations and multiple equilibria have been shown, which state a comprehensive qualitative illustration of the disease dynamics and give better implications for the control or prediction of diseases [5].

The epidemic spread causes deaths of millions of people as well as expenditure of vast amount of money in health care and medical management. It is, therefore, essential that adequate attention must be paid to stop the spread of such diseases. Several studies in the literature have been carried out to investigate the role of treatment and vaccination on the spread of diseases (see [6,7] and the references therein). A discrete-time epidemic model with vaccination for measles is derived in Linda [8]. The effect of vaccination on the spread of periodic diseases, using discrete-time model, was studied by Mickens [9]. The effectiveness of constant and pulse vaccination policies using *SIR* model were compared in [10]. Their theoretical results showed that under constant vaccination, the dynamic behavior of the disease model is similar to with no vaccination.

A number of studies have used the applications of optimal control theory in epidemiological models [11,12, 13]. Some of these studies focused on the effect of vaccination on the dynamics of the disease [14]. Gumel and Moghadas [15] investigated a disease transmission model by considering the impact of a protective vaccine and found the optimal vaccine coverage threshold required for disease control and elimination. Kar and Batabyal [16] used optimal control to study a nonlinear *SIR* epidemic model with a vaccination program. Various modeling studies have been made to study the role of optimal control using *SIR* epidemic model ([17,19,20, 21]. In [18], Gul et al. considered an *SIR* epidemic model using vaccination as control. Makinde and Okosun [22] applied optimal control to study the impact of chemo-therapy on malaria disease with infective immigrants, while Hattaf et al. [23] used optimal control strategies associated with preventing exogenous

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reinfection based on a exogenous reinfection tuberculosis model. The authors in Lashari et al. [24], investigated the fundamental role of three type of controls, personal protection, treatment, and mosquito reduction strategies in controlling malaria. In [25], a mathematical model of a vector-borne disease that incorporates both direct and indirect transmission was formulated. Analysis of the model revealed that the model exhibits the phenomenon of backward bifurcation with standard incidence. Then the model was further extended taking into account the density-dependent demographic parameters and control functions to asses the impact of some control measures by using optimal control techniques. Our aim is to analyze the effects of vaccinating the susceptible individuals and giving treatment to infectious individuals in a general *SIR* epidemic model. These analysis reveal the possibilities to develop strategies that manipulate the level of vaccination and treatment efforts. It is important to mention here that our work is different from some of the other related works cited in this paper because the model uses nonlinear incidence with two control variables (vaccination and treatment). Note that in this paper, we shall deal with the optimal control of the disease and we refer the interested reader to [26] for the mathematical analysis of the model.

The purpose of this paper is to consider a general *SIR* epidemic model [26] to incorporate optimal control strategies in the form of treatment and vaccination to decrease the number of susceptible and infectious individuals with minimum investment in disease control. The main feature of the present paper is not to consider a special disease but to present a method of how to treat this class of optimization problems. The problem is formulated as an optimal control problem with two control variables (that represent vaccination and treatment strategies). To do this, we use a time dependent percentage of susceptible and infected populations as control in the *SIR* model. Thus, the optimal control (vaccination and treatment) strategy is to minimize the susceptible and infected individuals as well as the cost of implementing the two controls. The model will then be used to determine cost-effective strategies for combatting the spread of an infectious disease in a given population. We illustrate how the optimal control theory and the percentage of the vaccination u_1 and treatment u_2 can be applied to minimize the susceptible and infected individuals. Then, we derive the optimality system for the *SIR* model with the percentage of vaccinated and treated individuals.

The organization of the paper is as follows. The mathematical model with controls is developed in Section 2. Analysis of the model with constant controls is presented in Section 3. The necessary conditions for an optimal control pair and the corresponding states are derived using Pontryagin's Maximum Principle in Section 4. The numerical simulations of the optimal control model are given in Section 5. Lastly, the conclusions are given in Section 6.

2 The epidemic model with controls

We will study an *SIR* epidemic model with saturated incidence rate and saturated treatment function. The saturated incidence rate can interpret the psychological effect or the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals [27]. The inclusion of saturated treatment function describe the effect of the infected individuals being delayed for treatment [26]. Furthermore, it is assumed that the cure rate of infected increases at a rate proportional to $u_2(t)$, where u_2 is treatment of infective and $r_0 > 0$ is a rate constant. We will divide the total population at time t, denoted by $N(t)$ into three subgroups: susceptible (S), infective (I) and recovered (R), individuals. The susceptible individuals become infected at a saturated incidence rate $\frac{\beta SI}{(1+kI)}$, where β is transmission rate and k is nonnegative that measure the inhibitory effect. By treatment, the infected individuals recover at a saturated treatment function $g(I) = \frac{(r+r_0u_2)I}{(1+\alpha I)}$, where r is cure rate, r_0 is the rate constant and α is positive that quantify the extent of the effect of the infected being delayed for treatment. Further, $\frac{1}{1+\alpha I}$ defines the opposite effect of the infected being delayed for treatment and $\frac{1}{1+kI}$ describes the psychological effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. Parameter definitions and assumptions lead to the following system of ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{\beta SI}{1+kI} - \mu S - u_1 S,\\
\frac{dI}{dt} &= \frac{\beta SI}{1+kI} - \frac{(r+r_0u_2)I}{1+\alpha I} - (\gamma+\delta+\mu)I, \qquad (1)\\
\frac{dR}{dt} &= \gamma I + \frac{(r+r_0u_2)I}{1+\alpha I} - \mu R + u_1 S,
\end{aligned}$$

with initial conditions given at $t = 0$. Λ, μ, γ, δ are the recruitment rate of the population, the natural mortality rate of the population, the natural recovery rate of the infective individuals and the disease induced death rate, respectively. The control variables u_1 is vaccination coverage of susceptible. The objective functional $\mathscr{F}$ formulates the optimization problem of interest, particularly, that of determining the efficient control strategies. The objective is to minimize the number of susceptible and infected individuals at a minimal cost over $[0,T]$ (a finite time interval).

The functional $\mathscr{F}$ is given by

$$\mathscr{F}(u_1,u_2) = \int_0^T \left(A_1S_h + A_2I_h + \frac{1}{2}(B_1u_1^2 + B_2u_2^2)\right) dt. \quad (2)$$

We adopt to model the control efforts by linear combination of $u_i^2(t)$ $(i=1,2)$. The constants A_i and B_i $(i=1,2)$ are weights constants which help to balance each term in the integrand so that none of the terms dominates. The terms $B_1u_1^2$ and $B_2u_2^2$ represent the costs associated with vaccination of susceptible and treatment of infected, respectively. The cost associated with the first control could come from the cost of antimicrobial drugs. Whereas the cost associated with the second control could arise from medical treatment of the infected people, cost associated with treating patients with other health complications or cost of drug. The problem is to find optimal functions $(u_1^*(t),u_2^*(t))$ such that

$$\mathscr{F}(u_1^*,u_2^*)=\min\{\mathscr{F}(u_1,u_2),\ (u_1,u_2)\in U\}$$

where the control set is defined as

$$\begin{aligned} U=\{(u_1,u_2)|&u_i(t) \text{ is Lebesgue measurable}\\ &\text{on } [0,1],\ 0\le u_1(t),u_2(t)\le 1,\ t\in[0,T]\}, \end{aligned} \tag{3}$$

subject to the system (1) and suitable initial conditions [17]. We use Pontryagin's Maximum Principle to solve this optimal control problem. Before deriving the optimality system and proving the existence of an optimal control for system (1), first, we analyze the model by considering constant controls.

3 Analysis of the model with constant controls

In this section, we determine the steady state solutions and their stability, the bifurcation behavior as well as the basic reproductive number of system (1) by assuming that the control parameters are constant. The first, two equations in system (1) are independent of the third equation, and therefore third equation can be excluded without loss of generality. Thus, we can rewrite system (1) as

$$\begin{aligned} \frac{dS}{dt}&=\Lambda-\frac{\beta SI}{1+kI}-\mu S-u_1S,\\ \frac{dI}{dt}&=\frac{\beta SI}{1+kI}-\frac{(r+r_0u_2)I}{1+\alpha I}-(\gamma+\delta+\mu)I. \end{aligned} \tag{4}$$

The disease free equilibrium of system (4) is given by $E_0=(\frac{\Lambda}{\mu+u_1},0)$. The basic reproduction number, denoted by R_0, associated to the system (4), is given by

$$R_0=\frac{\beta\Lambda}{(r+\gamma_0u_2+\gamma+\delta+\mu)(\mu+u_1)}.$$

The threshold R_0 is called the basic reproduction number, which is defined as the average number of secondary infections produced by an infected individual in a completely susceptible population. The vaccination and treatment in our system can have a great effect on R_0. To see the effect of u_1 and u_2 on R_0, straightforward computation gives

$$\frac{\partial R_0}{\partial u_1}=\frac{-\beta\Lambda}{(r+r_0u_2+\gamma+\delta+\mu)(\mu+u_1)^2},$$

$$\frac{\partial R_0}{\partial u_2}=\frac{-r_0\beta\Lambda}{(r+r_0u_2+\gamma+\delta+\mu)^2(\mu+u_1)},$$

thus $\frac{\partial R_0}{\partial u_1}<0$, and $\frac{\partial R_0}{\partial u_2}<0$.

From this analysis, we see that a higher vaccination u_1 of susceptible and higher treatment to infected u_2 both decreases R_0. This aspect can be a very useful control strategy and will be further explored in Section 5 through numerical simulation.

The variational matrix M_0 corresponding to E_0 is given by,

$$\begin{pmatrix} -\mu-u_1 & \frac{-\beta\Lambda}{\mu+u_1}\\ 0 & (R_0-1)(r+r_0u_2+\gamma+\delta+\mu)\end{pmatrix}.$$

Therefore, the disease free equilibrium E_0 is locally asymptotically stable if $R_0<1$. The endemic equilibrium is given by $E_1=(S^*,I^*)$, where

$$S^*=\frac{\Lambda(1+kI^*)}{\beta I^*+(\mu+u_1)(1+kI^*)},$$

and I^* satisfy the following quadratic equation

$$AI^2+BI+C=0, \tag{5}$$

where

$$\begin{aligned} A&=\alpha(\gamma+\delta+\mu)(\beta+k(\mu+u_1)),\\ B&=\alpha(\gamma+\delta+\mu)(\mu+u_1)\\ &\quad+(r+r_0u_2+\gamma+\delta+\mu)(\beta+k(\mu+u_1))-\alpha\beta\Lambda,\\ C&=(r+r_0u_2+\gamma+\delta+\mu)(\mu+u_1)(1-R_0). \end{aligned} \tag{6}$$

Note that, the coefficient A in (5) is always positive and C is positive if $R_0<1$, C is negative if $R_0>1$, and $C=0$ if $R_0=1$. Thus, we have the following result (see [26] for more detail).

Theorem 3.1. System (2) has a backward bifurcation at $R_0=1$, if and only if the coefficient B in (5) is less than 0.

The existence of backward bifurcation at $R_0=1$ ($C=0$), so that $\beta\Lambda=(r+\gamma_0u_2+\gamma+\delta+\mu)(\mu+u_1)$, can also be expressed explicitly in terms of α. The condition $b<0$, with $\beta\Lambda=(r+\gamma_0u_2+\gamma+\delta+\mu)(\mu+u_1)$, is equivalent to

$$\alpha>\frac{\beta\Lambda(\beta+k(\mu+u_1))}{(\mu+u_1)^2(r+r_0u_2)}. \tag{7}$$

So, backward bifurcation occurs at $R_0=1$ if and only if α satisfy the relation (7). From this, it can be easily seen that the parameter α, which lead to the backward

bifurcation, decrease as the control variables u_1 and u_2 increases. The occurrence of backward bifurcation suggests that disease eradication is achievable only when the effect of the infected being delayed for the treatment can not exceed some level. Moreover, for the disease not to become endemic again, treatment and vaccination controls must be maintained at this level for all time.

Theorem 3.2. If $R_0 > 1$, then the endemic equilibrium is locally asymptotically stable if $\alpha < min\{\frac{\beta\Lambda(\beta+k(\mu+u_1))}{(\mu+u_1)^2(r+r_0u_2)}, \frac{\beta+k(\mu+u_1)+k(\delta+\gamma+\mu+r+r_0u_2)}{r+r_0u_2}\}$.

Proof. The proof is worked out for a similar case in ([26], Theorem 4.2).

4 Characterization of the optimal control

In this section, first we prove the existence of an optimal control for system (1).

Theorem 4.1. There exist optimal controls $(u_1^*(t), u_2^*(t))$ and corresponding solutions, S^*, I^* and R^* that minimizes $\mathscr{F}(u_1,u_2)$ over U.

Proof. The integrand of the objective functional $\mathscr{F}$ given by (2) is a convex function of (u_1,u_2) and the state system (1) satisfies the Lipshitz property with respect to the state variables since state solutions are bounded. The existence of optimal controls follows [28].

In order to find an optimal solution , we need to find the Lagrangian and Hamiltonian for the problem $(1)-(2)$. The Lagrangian of the control problem is given by

$$L = A_1S + A_2I + 1/2(B_1u_1^2 + B_2u_2^2).$$

We need the minimal value of the Lagrangian. For this, the Hamiltonian H for the control problem, where λ_i, $i = 1,2,3$ are the adjoint variables, is given by

$$\begin{aligned} H &= L(S,I,u_1,u_2) + \lambda_1\left[\Lambda - \frac{\beta SI}{1+kI} - \mu S - u_1S\right] \\ &+ \lambda_2\left[\frac{\beta SI}{1+kI} - \frac{(r+r_0u_2)I}{1+\alpha I} - (\gamma+\delta+\mu)I\right] \\ &+ \lambda_3\left[\gamma I + \frac{(r+r_0u_2)I}{1+\alpha I} - \mu_h R + u_1S\right]. \end{aligned} \tag{8}$$

We now derive the necessary conditions, using Pontryagin's maximum principle [29], that optimal control functions and corresponding states must satisfy.

Theorem 4.2. Given an optimal control pair (u_1^*,u_2^*) and a solution (S^*,I^*,R^*) of the corresponding state system (1)-(2), there exists adjoint variables λ_i, $i = 1,2,3$ satisfying

$$\begin{aligned} \frac{d\lambda_1(t)}{dt} &= \frac{\beta(\lambda_1-\lambda_2)I}{1+kI} + (\lambda_1-\lambda_3)u_1 + \mu\lambda_1 - A_1, \\ \frac{d\lambda_2(t)}{dt} &= \frac{\beta(\lambda_1-\lambda_2)S}{(1+\alpha I)^2} + \frac{(r+r_0u_2)(\lambda_2-\lambda_3)}{(1+\alpha I)^2} \\ &\quad +(\gamma+\delta+\mu)\lambda_2 - \gamma\lambda_3 - A_2, \\ \frac{d\lambda_3(t)}{dt} &= \mu\lambda_3, \end{aligned} \tag{9}$$

with transversality conditions

$$\lambda_i(T) = 0, \qquad i = 1,2,3. \tag{10}$$

Furthermore, the control functions u_1^* and u_2^* are given by

$$u_1^* = \max\{\min\{\frac{(\lambda_1-\lambda_3)S^*}{B_1}, 1\}, 0\}, \tag{11}$$

$$u_2^* = \max\{\min\{\frac{(\lambda_2-\lambda_3)r_0I^*}{B_2(1+\alpha I^*)}, 1\}, 0\}. \tag{12}$$

Proof. In order to determine the transversality conditions and the adjoint equations, we use the Hamiltonian (8). The adjoint system results from Pontryagin's Maximum Principle [29].

$$\frac{d\lambda_1(t)}{dt} = -\frac{\partial H}{\partial S}, \quad \frac{d\lambda_2(t)}{dt} = -\frac{\partial H}{\partial I}, \quad \frac{d\lambda_3(t)}{dt} = -\frac{\partial H}{\partial R},$$

with $\lambda_i(T) = 0$, $i = 1,2,3$.

In order to obtain the characterization of the control given by (11)-(12), solving the equations,

$$\frac{\partial H}{\partial u_1} = 0, \quad \frac{\partial H}{\partial u_2} = 0, \tag{13}$$

on the interior of the control set and using the property of the control space U, we can obtain the desired characterization (11) and (12). □

Here we call formulas (11) and (12) for (u_1^*,u_2^*) the characterization of the optimal control. The optimal control and the state are found by solving the optimality system, which consists of the state system (1), the adjoint system (9), initial conditions at $t = 0$, boundary conditions (10), and the characterization of the optimal control (11)-(12). To solve the optimality system we use the transversality and initial conditions together with the characterization of the optimal control (u_1^*,u_2^*) given by (11)-(12). In addition, the second derivative of the Lagrangian with respect to u_1 and u_2, respectively, is positive, which shows that the optimal problem is minimum at controls u_1^* and u_2^*. Therefore, taking the

state system together with the adjoint system, the optimal control, and the transversality conditions, we have the following optimality system:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{\beta S(t)I(t)}{1+kI} - \mu(t)S \\
&\quad - \max\{\min\{\tfrac{(\lambda_1-\lambda_3)S}{B_1},1\},0\}S, \\
\frac{dI}{dt} &= \frac{\beta S(t)I(t)}{1+kI} \\
&\quad -(r+r_0\max\{\min\{\tfrac{(\lambda_2-\lambda_3)r_0 I}{B_2(1+\alpha I)},1\},0\})\tfrac{I(t)}{1+\alpha I} \\
&\quad -(\gamma+\delta+\mu)I(t), \\
\frac{dR}{dt} &= \gamma I + (r+r_0\max\{\min\{\frac{(\lambda_2-\lambda_3)r_0 I}{B_2(1+\alpha I)},1\},0\})\frac{I(t)}{1+\alpha I} \\
&\quad -\mu_h R_h(t) + \max\{\min\{\frac{(\lambda_1-\lambda_3)S}{B_1},1\},0\}S,
\end{aligned} \tag{14}$$

with H^* at $(t,S^*,I^*,R^*,u_1^*,u_2^*,\lambda_1,\lambda_2,\lambda_3)$:

$$\begin{aligned}
H^* &= A_1S^* + A_2I^* \\
&\quad + \tfrac{1}{2}\big(B_1(\max\{\min\{\frac{(\lambda_1-\lambda_3)S^*}{B_1},1\},0\})^2 \\
&\quad + B_2(\max\{\min\{\frac{(\lambda_2-\lambda_3)r_0I^*}{B_2(1+\alpha I^*)},1\},0\})^2 \\
&\quad + \lambda_1\tfrac{dS^*}{dt} + \lambda_2\tfrac{dI^*}{dt} + \lambda_3\tfrac{dR^*}{dt},
\end{aligned} \tag{15}$$

$$u_1^* = \max\{\min\{\frac{(\lambda_1-\lambda_3)S^*}{B_1},1\},0\},$$

$$u_2^* = \max\{\min\{\frac{(\lambda_2-\lambda_3)r_0I^*}{B_2(1+\alpha I)},1\},0\},$$

$$\lambda_i(T) = 0.$$

The problem described above is a two point boundary value problem, with specified initial conditions for the state system and terminal boundary conditions for adjoint equations. To find out the optimal control and state, we will numerically solve the above systems (14) and (15).

5 Numerical results and discussion

In this section, the optimality system is solved using Runge-Kutta fourth order scheme. The optimal strategy is achieved by solving the adjoint and state systems and the transversality conditions. We note that this is a two-point boundary-value problem, with separated boundary conditions at times $t = 0$ and $t = T$. It is our aim to solve this problem for the value $T = 50$. This value was chosen to represent the time (in days) at which vaccination and treatment is stopped. In our numerical simulation, first we start to solve the state equations (1) using Runge-Kutta fourth order forward in time with a guess for the controls

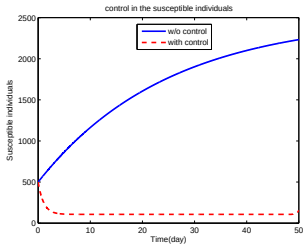

Fig. 1: The plot represents population of susceptible individuals with and without control.

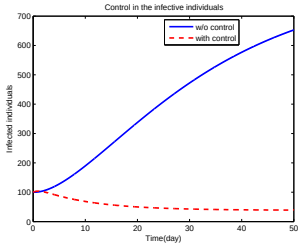

Fig. 2: The plot represents population of infective individuals with and without control.

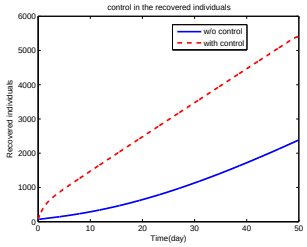

Fig. 3: The plot represents population of recovered individuals with and without control.

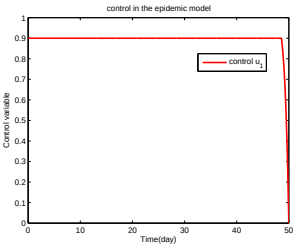

Fig. 4: Optimal control u_1 given by (11).

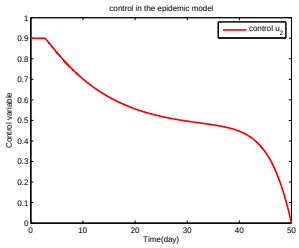

Fig. 5: Optimal control u_2 given by (12).

over the simulated time. Then, using the current iteration of the state equations, the adjoint equations in the system (9) are solved by a backward method with the transversality conditions (10). We update the controls by using a convex combination of the controls in the previous iteration and the value from the characterizations of the system (11)-(12). Repeat this process and stop iterations if the values of unknowns at the previous iteration are very close to the ones at the present iteration. We may refer the reader to see [30,31] such iterative algorithms for more detail.

The susceptible, infected, and recovered individuals with and without control are plotted using the parameters values as: $\Lambda = 100$, $\beta = 0.02$, $\mu = 0.000039$, $\alpha = 0.01$, $\gamma = 0.08$, $\delta = 0.02$, $k = 0.5$, $r = 0.2$, $r_0 = 0.2$. The values for weight constants are $A_1 = 0.09$, $A_2 = 0.02$, $B_1 = 10$ and $B_2 = 10$. When viewing the graphs, remember that each of the individuals without control is marked by un-dashed lines and individuals with control are marked by dashed lines. The graphs from simulating the model, given in Fig. 1-Fig. 3, help to compare the population of susceptible (S), the infected (I) and the recovered individuals (R) both with controls and without controls.

In Fig. 1, we have plotted susceptible individual with and without controls. We see that the population of the susceptible individuals sharply decreases in first two or three days after that it begins to increase very slowly and goes to its stable state.

In Fig. 2, we see that if there are no controls infected individuals without controls continually increases, but if there are controls the population of infected individuals begins to decrease from the very beginning day of vaccination and treatment and gradually decreases as time goes on.

In Fig. 3, the population of recovered individuals increases rapidly with controls. As expected, the population of the susceptible group decreases with time while that of the recovered group gradually increases for the inclusion of vaccinated susceptible group and treated infectious group.

Fig. 4 and Fig. 5 represent the optimal controls u_1^* and u_2^*. The control vanishes in day 50 and there remains very small number of susceptible and infected individuals.

6 Conclusion

In this paper, we analyzed an optimal control strategy in the SIR epidemic model with saturated incidence and saturated treatment. In case of constant control, it is found that the model exhibits backward bifurcation. The epidemiological implication of backward bifurcation is that for effective eradication and control of the disease, R_0 should be less than a critical values less than one. Moreover, achieving this may be too costly, because it means that for constant controls, one needs to keep vaccinating and treating for infinite time. Therefore, we considered time dependent controls as a way out, to ensure the eradication of the disease in a finite time. In this light, we set up an optimal control problem in the form of treatment and vaccination to minimize the number of infective and susceptible populations. A comparison between optimal control and without control was also presented. It is easy to see that the optimal control is much more effective for reducing the number of infected individuals. The number of susceptible, infected and recovered individuals with the optimal control and without control are shown in figures to illustrate the overall picture of the epidemic. Moreover, it is observed that a higher vaccination and treatment rate decreases the basic reproduction number. So, vaccination and treatment has positive impact in controlling the transmission of the disease.

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