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Alpha Power Transformed Fréchet Distribution

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Abstract: The Fréchet distribution has several applications in different fields of study and is most commonly used for modeling extreme events. In recent time, modifications of the Fréchet distribution have been proposed to improve its fit when used for modeling lifetime data. In this paper, a new modification called the alpha power transformed Fréchet distribution is proposed and studied. The parameters of the model are estimated using maximum-likelihood estimation and simulation studies are performed to investigate the properties of the estimators for the parameters. Applications of the model are demonstrated using two-real data sets. Finally, bivariate and multivariate extensions of the model are proposed using copulas.

Keywords: Fréchet, alpha power transformed, bivariate, multivariate, copulas.

1 Introduction

Following the early works of Pearson on the development of statistical distributions using system of differential equations, barrage of methodologies have been proposed for generating new statistical distributions [1]. Some of the proposed techniques in literature includes: translation [2] and quantile [3,4] methods. However, the methods of differential equations [1,5], translation and quantile techniques were developed prior to the 1980.
From the 1980 up, researchers shifted attention from these early methods to methods of adding parameters to existing distributions or combining existing distributions. This new approach of generating statistical distributions that were proposed since the 1980s includes: beta-generated method [6], transformed-transformer method [7], exponentiated transformed-transformer method [8], exponentiated generalized transformed-transformer method [9] and exponentiated generalized method [10]. These methods have been employed to modify existing distributions with the goal of making them more flexible in modeling data with different kinds of failure rates such as upside down bathtub, bathtub and non-monotonically increasing or decreasing failure rates among others.
Recently, the Fréchet distribution which was developed for modeling extreme events such as one-day rainfall and river discharge have been generalized to make it provide a more reasonable parametric fit to data arising from all fields of study [11] . Some of the modifications are: beta-exponential Fréchet [12], transmuted Fréchet [13], transmuted exponentiated Fréchet [14] and Kumaraswamy Fréchet [15].
Thus, the goal of this study is to develop another generalization of the Fréchet distribution called the Alpha Power Transformed (APT) Fréchet (APFT) distribution using the idea of [16]. For an arbitrary baseline Cumulative Distribution Function (CDF), the CDF of the APT family of distributions as:

$$F_{\mathrm{APT}}(x)=\begin{cases}\frac{\alpha^{F(x)}-1}{\alpha-1}, \text{if } \alpha>0,\ \alpha\neq 1, x\in\mathbb{R}\\ F(x), \text{if } \alpha>0,\ \alpha=1, x\in\mathbb{R}\end{cases}, \quad (1)$$

and the corresponding Probability Density Function (PDF) as:

$$f_{\mathrm{APT}}(x)=\begin{cases}\frac{\log\alpha}{\alpha-1}f(x)\alpha^{F(x)}, \text{if } \alpha>0,\ \alpha\neq 1, x\in\mathbb{R}\\ f(x), \text{if } \alpha>0,\ \alpha=1, x\in\mathbb{R}\end{cases}, \quad (2)$$

where, $F(x)$ is an absolute continuous distribution function with PDF $f(x)$ [16].

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The rest of the paper is organized as follows: In section 2, the CDF, PDF, survival and hazard rate functions are defined. In section 3, a representation mixture of the APTF model is given. In section 4, statistical properties of the model are derived. In section 5, estimators for the parameters of the model are developed using maximum-likelihood estimation technique. In section 6, Monte Carlo simulations are performed to investigate the finite sample properties of the estimators. In section 7, applications of the APTF distribution are demonstrated using real-data sets. In section 8, bivariate and multivariate extensions of the APTF distribution are proposed using copulas. The concluding remarks of the study are finally given in section 9.

2 APTF Distribution

Suppose X is a Fréchet random variable with parameters $a,b>0$. Then the CDF and PDF associated to X are respectively given by:

$$F(x)=e^{-\left(\frac{a}{x}\right)^b},\ x>0,\ \text{and}\ f(x)=ba^bx^{-b-1}e^{-\left(\frac{a}{x}\right)^b},\ x>0. \tag{3}$$

Hence, a random variable X is said to have the APTF distribution if its CDF is of the form:

$$F_{\text{APTF}}(x)=\begin{cases}\frac{\alpha^{e^{-\left(\frac{a}{x}\right)^b}}-1}{\alpha-1},\ \alpha>0,\ \alpha\neq1,\ x>0\\ e^{-\left(\frac{a}{x}\right)^b},\ \alpha>0,\ x>0\end{cases}, \tag{4}$$

where $\alpha,b>0$ are shape parameters and $a>0$ is a scale parameter. The CDF in equation (4) is obtained by substituting the CDF in equation (3) into (1). The corresponding PDF of the APTF distribution is:

$$f_{\text{APTF}}(x)=\begin{cases}\frac{\log\alpha}{\alpha-1}ba^bx^{-b-1}e^{-\left(\frac{a}{x}\right)^b}\alpha^{e^{-\left(\frac{a}{x}\right)^b}},\ \alpha\neq1,\ x>0\\ ba^bx^{-b-1}e^{-\left(\frac{a}{x}\right)^b},\ \alpha=1,\ x>0\end{cases}. \tag{5}$$

The APTF distribution houses a number of sub-models such a Fréchet, Inverse Exponential (IE), APT Inverse Exponential (APIE), Inverse Rayleigh (IR), APT Inverse Rayleigh (APTIR), one-parameter Fréchet and one-parameter APTF (OAPTF) distributions. Table 1 displays the special cases of the APTF distribution. It is important to note that where the values of α are not stated as $\alpha=1$ in Table 1, then $\alpha\neq1$.
Figure 1 displays the shapes of the density function of the APTF distribution for some selected parameter values. It can be seen that the PDF is unimodal and right skewed with different degrees of kurtosis.

The survival and the hazard rate functions of the APTF distribution are respectively given by:

$$S_{\text{APTF}}(x)=\begin{cases}\frac{\alpha}{\alpha-1}\left(1-\alpha^{e^{-\left(\frac{a}{x}\right)^b}-1}\right),\ \alpha\neq1,\ x>0\\ 1-e^{-\left(\frac{a}{x}\right)^b},\ \alpha=1,\ x>0\end{cases}, \tag{6}$$

and

$$h_{\text{APTF}}(x)=\begin{cases}\frac{\log\alpha}{\left(1-\alpha^{e^{-\left(\frac{a}{x}\right)^b}-1}\right)}ba^bx^{-b-1}e^{-\left(\frac{a}{x}\right)^b}\alpha^{e^{-\left(\frac{a}{x}\right)^b}-1},\ \alpha\neq1,\ x>0\\ \frac{ba^bx^{-b-1}e^{-\left(\frac{a}{x}\right)^b}}{1-e^{-\left(\frac{a}{x}\right)^b}},\ \alpha=1,\ x>0\end{cases}. \tag{7}$$

Figure 2 shows the plot of the hazard rate function of the APTF distribution for some selected parameter values. The hazard rate function exhibit decreasing and upside down bathtub failure rate for the selected parameters values.

3 Mixture Representation

The mixture representation of the density is very useful when deriving the statistical properties of generalized distributions. In this section, the mixture representation of the APTF density function is derived. Employing the series representation

$$\alpha^u=\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}u^i,$$

and $\alpha\neq1$, the density of the APTF distribution can be written as:

$$f_{\text{APTF}}(x)=\frac{ba^bx^{-b-1}}{\alpha-1}\sum_{i=0}^{\infty}\frac{(\log\alpha)^{i+1}}{i!}e^{-(i+1)\left(\frac{a}{x}\right)^b}. \tag{8}$$

4 Statistical Properties

It is imperative to derive the statistical properties when a new distribution is developed. In this section, the statistical properties of the APTF distribution are derived for the case of $\alpha\neq1$, since for the case of $\alpha=1$ it is simply the properties of the Fréchet distribution.

4.1 Quantile Function

The quantile function plays a useful role when simulating random variates from a statistical distribution. The quantile function of the APTF distribution, say $x=Q(p)$ is given by:

$$Q(p)=\left\{\frac{1}{\alpha}\left[\log\left(\frac{\log\alpha}{\log(1+p(\alpha-1))}\right)\right]^{\frac{1}{b}}\right\}^{-1},\quad 0<p<1. \tag{9}$$

Table 1: Sub-models of APTF distribution

Sub-model	α	a	b
Fréchet	1	a	b
IE	1	a	1
APTIE	α	a	1
IR	1	a	2
APTIR	α	a	2
One parameter Fréchet	1	1	b
OAPTF	α	1	b

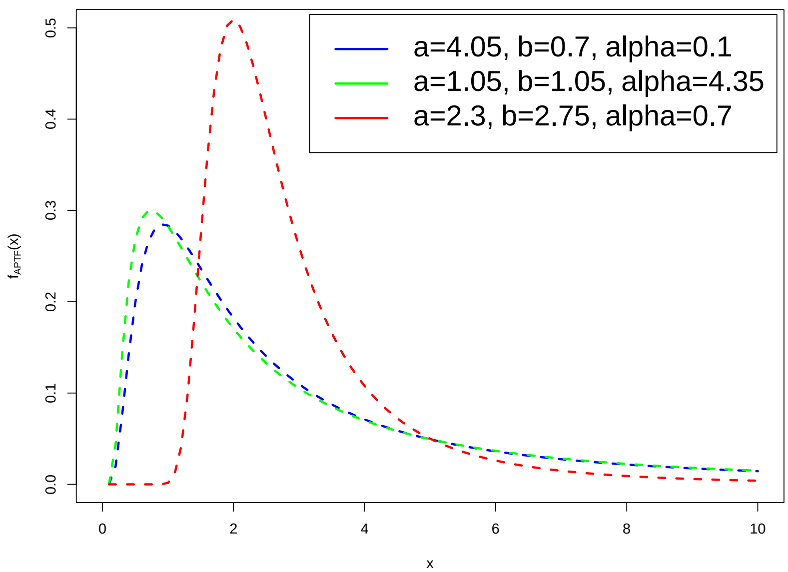

Fig. 1: Plot of the APTF distribution density function

The median is obtained by substituting $p = 0.5$ into the quantile function. Hence, the median is:

$$Q(0.5) = \left\{ \frac{1}{\alpha} \left[\log \left(\frac{\log \alpha}{\log(1 + 0.5(\alpha - 1))} \right) \right]^{\frac{1}{b}} \right\}^{-1}. \tag{10}$$

For several heavy tailed distributions, the classical measures of skewness and kurtosis cannot be computed due to nonexistence of higher moments. In such situations, the quantile can be employed to estimate such measures. The Bowley's coefficient of skewness which is based on quartiles can be used to estimate the coefficient of skewness. It is defined as:

$$B = \frac{Q(0.75) - 2Q(0.5) + Q(0.25)}{Q(0.75) - Q(0.25)},$$

[17]. Similarly, the coefficient of kurtosis can be estimated using the Moors' coefficient of kurtosis which is defined based on the octiles as:

$$M = \frac{Q(0.875) - Q(0.625) - Q(0.375) + Q(0.125)}{Q(0.75) - Q(0.25)},$$

[18]. Figure 3 shows the Bowley's coefficient of skewness and Moors' coefficient of kurtosis for some selected parameter values. It can be seen that for smaller values of α both measures increase whereas for larger values they

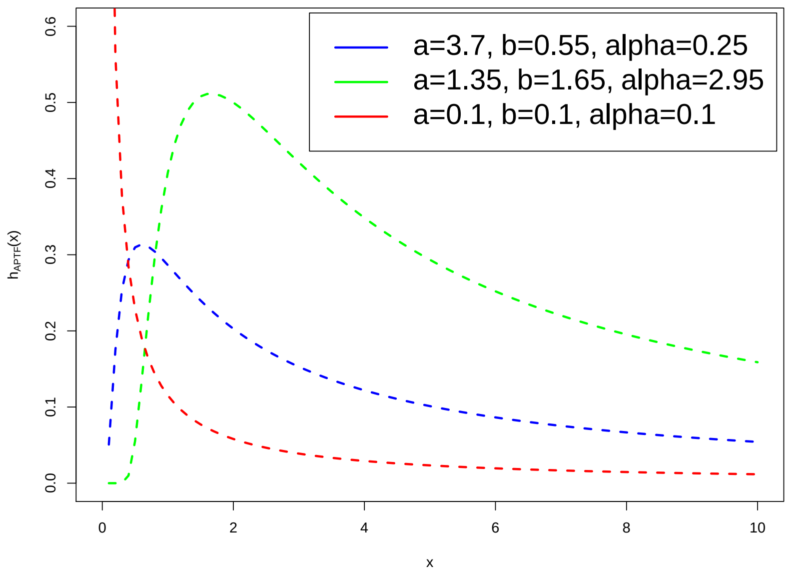

Fig. 2: Plot of the hazard rate function of APTF distribution

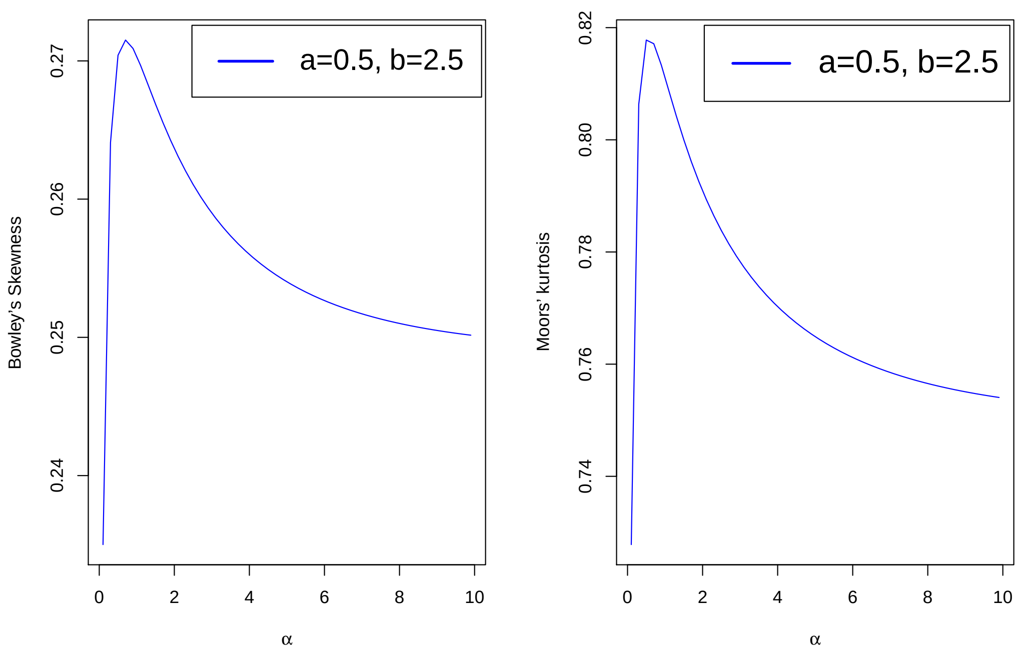

Fig. 3: Plot of Bowley's Skewness and Moors' kurtosis

are decreasing.

4.2 Moments

The moment of a random variable plays a useful role when computing measures of central tendencies, dispersions and shapes. The r^{th} non-central moment of the APTF random variable is:

$$\begin{aligned}\mu_r' &= \int_0^\infty x^r dF_{\text{APTF}}(x) \\ &= \int_0^\infty x^r \frac{ba^b x^{-b-1}}{\alpha-1}\sum_{i=0}^\infty \frac{(\log\alpha)^{i+1}}{i!} e^{-(i+1)(\frac{a}{x})^b} dx \\ &= \frac{ba^b}{\alpha-1}\sum_{i=0}^\infty \frac{(\log\alpha)^{i+1}}{i!}\int_0^\infty x^{r-b-1} e^{-(i+1)(\frac{a}{x})^b} dx \\ &= \frac{a^r}{\alpha-1}\sum_{i=0}^\infty \frac{(\log\alpha)^{i+1}}{(i+1)!}(i+1)^{\frac{r}{b}}\Gamma\left(1-\frac{r}{b}\right), r<b, \quad (11)\end{aligned}$$

for $r = 1,2,\ldots$, where $\Gamma(\cdot)$ is the gamma function. Table 2 displays the first six moments, Standard Deviations (SD), Coefficient of Variation (CV), Coefficient of Skewness (CS) and Coefficient of Kurtosis (CK). The values for SD, CV, CS and CK are respectively given by:

$$SD = \sqrt{\mu_2' - \mu^2},$$

$$CV = \frac{\sigma}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1},$$

$$CS = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{\frac{3}{2}}}$$

and

$$CK = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2}.$$

4.3 Moment-Generating Function

The moment-generating function of random variable X that follows the APTF distribution, if it exist, is given by:

$$\begin{aligned}M_X(t) &= E(e^{tX}) \\ &= \sum_{r=0}^\infty \frac{t^r}{r!}\mu_r' \\ &= \sum_{r=0}^\infty \frac{t^r}{r!}\frac{a^r}{\alpha-1}\sum_{i=0}^\infty \frac{(\log\alpha)^{i+1}}{(i+1)!}(i+1)^{\frac{r}{b}}\Gamma\left(1-\frac{r}{b}\right), r<b.\end{aligned} \quad (12)$$

4.4 Incomplete Moment

The incomplete moment has important applications in different fields of study. The first incomplete moment is used in estimation of the Bonferroni and Lorenz curves which are useful in economics, reliability, demography, medicine and insurance. The r^{th} incomplete moment of the APTF random variable is:

$$\begin{aligned}\varphi_r(t) &= \int_0^t x^r dF_{\text{APTF}}(x) \\ &= \frac{ba^b}{\alpha-1}\sum_{i=0}^\infty \frac{(\log\alpha)^{i+1}}{i!}\int_0^t x^{r-b-1} e^{-(i+1)(\frac{a}{x})^b} dx.\end{aligned} \quad (13)$$

Using the complementary incomplete gamma function, this yields:

$$\varphi_r(t) = \frac{a^r}{\alpha-1}\sum_{i=0}^\infty \frac{(\log\alpha)^{i+1}}{(i+1)!}(i+1)^{\frac{r}{b}}\Gamma\left(1-\frac{r}{b},(i+1)(\frac{a}{t})^b\right), \quad (14)$$

$r < b$, where $\Gamma(q,z) = \int_z^\infty w^{q-1}e^{-w}dw$ is the complementary incomplete gamma function.

4.5 Mean Residual Life and Mean Inactivity Time

The Mean Residual Life (MRL) or the life expectancy at age t is the expected additional life length for a unit, which is alive at age t. The MRL has several important applications in life insurance, maintenance and product quality control, and also demography and economics are among others. The MRL is given by:

$$m_X(t) = E(X-t|X>t), t>0.$$

It can therefore be expressed as:

$$m_X(t) = \frac{(\mu - \varphi_1(t))}{S(t)} - t,$$

where $\mu = \mu_1'$, $\varphi_1(t)$ is the first incomplete moment and $S(t)$ is the survival function. Thus, the MRL of the APTF distribution is:

$$m_X(t) = \frac{\left[\mu - \frac{a}{\alpha-1}\sum_{i=0}^\infty \frac{(\log\alpha)^{i+1}}{(i+1)!}(i+1)^{\frac{1}{b}}\Gamma\left(1-\frac{1}{b},(i+1)(\frac{a}{t})^b\right)\right]}{S_{\text{APTF}}(t)} - t, b>1. \quad (15)$$

The Mean Inactivity Time (MIT) is the waiting time elapsed since the failure of an item on condition that the failure had occurred in $(0,t)$. The MIT of the APTF random variable X is defined for $t>0$ as:

$$\psi_X(t) = E(t-X|X\le t).$$

Table 2: First six moments, SD, CV, CS and CK

$\mu_r^{'}$	$a=0.5, b=10.5, \alpha=0.5$	$a=2.5, b=10.5, \alpha=1.5$
$\mu_1^{'}$	0.5202	2.6994
$\mu_2^{'}$	0.2749	7.4206
$\mu_3^{'}$	0.1480	20.8427
$\mu_4^{'}$	0.0815	60.0851
$\mu_5^{'}$	0.0462	178.9574
$\mu_6^{'}$	0.0274	556.4143
SD	0.0655	0.3658
CV	0.1259	0.1355
SK	1.8908	1.8168
CK	10.6600	9.8328

This can further be expressed as:

$$\psi_X(t) = t - \frac{\varphi_1(t)}{F(t)}.$$

Substituting the first incomplete moment and the CDF of the APTF random variable yields its MIT as:

$$\psi_X(t) = t - \frac{\alpha}{(\alpha-1)F_{\text{APTF}}(t)} \sum_{i=0}^{\infty} \frac{(\log\alpha)^{i+1}}{(i+1)!}(i+1)^{\frac{1}{b}}\Gamma\left(1-\frac{1}{b}, (i+1)(\frac{a}{t})^b\right).$$

4.6 Entropy

Entropy has been used in the engineering sciences and information theory as measures of variation of uncertainty. The Rényi entropy of a random variable X having the APTF distribution is given as:

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\int_0^\infty f_{\text{APTF}}^{\delta}(x)dx\right], \delta > 0 \text{ and } \delta \neq 1.$$

From equation (5), we can write

$$f_{\text{APTF}}^{\delta}(x) = \left(\frac{\log\alpha}{\alpha-1}\right)^{\delta} a^{\delta b} b^{\delta} \times \sum_{i=0}^{\infty} \frac{(\delta\log\alpha)^i}{i!} x^{-\delta(b+1)} e^{-(\delta+i)(\frac{a}{x})^b}, \quad (16)$$

using the approach employed to expand the density in equation (8). Thus, the Rényi entropy is given by:

$$I_R(\delta) = \frac{1}{1-\delta} \times \log\left[\left(\frac{\log\alpha}{\alpha-1}\right)^{\delta} a^{\delta b} b^{\delta} \sum_{i=0}^{\infty} \frac{(\delta\log\alpha)^i}{i!} \int_0^\infty x^{-\delta(b+1)} e^{-(\delta+i)(\frac{a}{x})^b} dx\right]$$
$$= \frac{1}{1-\delta} \times \log\left[A\sum_{i=0}^{\infty} \frac{(\delta\log\alpha)^i}{i!}(\delta+i)^{(1+\frac{1}{b})(1-\delta)}\Gamma\left(\delta+\frac{(\delta-1)}{b}\right)\right], \quad (17)$$

$\delta > 0, \delta \neq 1, A = \left(\frac{\log\alpha}{\alpha-1}\right)^{\delta} a^{1-\delta} b^{\delta-1}$. The Rényi entropy converges to the Shannon entropy as δ approaches 1. The δ-entropy, say $H(\delta)$ of the APTF random variable is defined by:

$$H(\delta) = \frac{1}{\delta-1}\log\left[1 - I_{\delta}(x)\right],$$

where

$$I_{\delta}(x) = \int_0^\infty f_{\text{APTF}}^{\delta}(x)dx, \delta > 0 \text{ and } \delta \neq 1.$$

Hence,

$$H(\delta) = \frac{1}{\delta-1} \times \log\left[1 - B\sum_{i=0}^{\infty} \frac{(\delta\log\alpha)^i}{i!}(\delta+i)^{(1+\frac{1}{b})(1-\delta)}\Gamma\left(\delta+\frac{(\delta-1)}{b}\right)\right],$$

$B = \left(\frac{\log\alpha}{\alpha-1}\right)^{\delta} a^{1-\delta} b^{\delta-1}, \delta > 0, \text{ and } \delta \neq 1.$

4.7 Stochastic Ordering

Stochastic ordering is the commonest way of describing ordering mechanism in lifetime distributions. Let $X_1 \sim APTF(a,b,\alpha_1)$ and $X_2 \sim APTF(a,b,\alpha_2)$. The random variable X_2 is stochastically greater than X_1 in the:

–stochastic order $(X_1 \leq_{st} X_2)$ if the associated CDFs satisfy: $F_{X_1}(x) \leq F_{X_1}(x)$ for all x.
–hazard rate order $(X_1 \leq_{hr} X_2)$ if the associated hazard rate function satisfies: $h_{X_1}(x) \leq h_{X_1}(x)$ for all x.
–likelihood ratio order $(X_1 \leq_{lr} X_2)$ if $\frac{f_{X_1}(x)}{f_{X_2}(x)}$ is a decreasing function of x.

Given the PDFs of X_1 and X_2,

$$\frac{f_{X_1}(x)}{f_{X_2}(x)} = \left(\frac{\alpha_2-1}{\alpha_1-1}\right)\left(\frac{\log\alpha_1}{\log\alpha_2}\right)\left(\frac{\alpha_1}{\alpha_2}\right)^{e^{-(\frac{a}{x})^b}}. \tag{18}$$

Taking the logarithm and differentiating the ratio of the densities yield:

$$\frac{d}{dx}\log\frac{f_{X_1}(x)}{f_{X_2}(x)} = a^b b x^{-b-1} e^{-(\frac{a}{x})^b}\log\left(\frac{\alpha_1}{\alpha_2}\right) < 0, \tag{19}$$

if $\alpha_1 < \alpha_2 \forall x > 0$. Thus, for $\alpha_1 < \alpha_2$,$X_1 \leq_{lr} X_2 \ \forall x$. It follows from the implications of stochastic ordering that:

$$X_1 \leq_{lr} X_2 \Longrightarrow X_1 \leq_{hr} X_2 \Longrightarrow X_1 \leq_{st} X_2.$$

4.8 Order Statistics

Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ represents order statistics obtained from the APTF distribution. Then the PDF, $f_{p:n}(x)$, of the p^{th} order statistic $X_{r:n}$ is:

$$f_{p:n}(x) = \frac{1}{B(p,n-p+1)}[F(x)]^{p-1}[1-F(x)]^{n-p} f(x), \tag{20}$$

where $F(x)$ and $f(x)$ are the CDF and PDF of the APTF distribution respectively, and $B(\cdot,\cdot)$ is the beta function. Substituting the PDF and the CDF of the APTF distribution gives:

$$f_{p:n}(x) = \frac{n! a^b b x^{-b-1}\log\alpha}{(\alpha-1)^n (p-1)!(n-p)!}\left(1-\alpha^{e^{-(\frac{a}{x})^b}}\right)^{n-p} \times \left(\alpha^{e^{-(\frac{a}{x})^b}}-1\right)^{p-1} \alpha^{e^{-(\frac{a}{x})^b}+n-p}. \tag{21}$$

Hence, the PDFs of the smallest and the largest order statistics are respectively given by:

$$f_{X_{(1)}}(x) = \frac{n a^b b x^{-b-1}\log\alpha}{(\alpha-1)^n}\left(1-\alpha^{e^{-(\frac{a}{x})^b}}\right)^{n-1} \times \alpha^{e^{-(\frac{a}{x})^b}+n-1}, \tag{22}$$

and

$$f_{X_{(n)}}(x) = \frac{n a^b b x^{-b-1}\log\alpha}{(\alpha-1)^n}\left(\alpha^{e^{-(\frac{a}{x})^b}}-1\right)^{n-1} \times \alpha^{e^{-(\frac{a}{x})^b}}. \tag{23}$$

5 Parameter Estimation

In this section, the parameters of the APTF distribution are estimated using the maximum-likelihood estimation method. Given a random sample $x_1, x_2, \ldots, x_n$ of size n from the APTF distribution with parameter vector $\xi = (a,b,\alpha)'$, then the log-likelihood function is given by:

$$\ell = n\log\left(\frac{a^b b\log\alpha}{\alpha-1}\right) - (b+1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\left(\frac{a}{x_i}\right)^b + \log(\alpha)\sum_{i=1}^{n} e^{-(\frac{a}{x_i})^b}. \tag{24}$$

Taking the partial derivatives of the log-likelihood function with respect to the parameters yields the following score functions:

$$\frac{\partial\ell}{\partial a} = \frac{nb}{a} - \sum_{i=1}^{n}\frac{b(\frac{a}{x_i})^{b-1}}{x_i} - \log(\alpha)\sum_{i=1}^{n}\frac{b(\frac{a}{x_i})^{b-1}e^{-(\frac{a}{x_i})^b}}{x_i}, \tag{25}$$

$$\frac{\partial\ell}{\partial b} = \frac{n a^{-b}(\alpha-1)\left(\frac{a^b\log(\alpha)}{\alpha-1} + \frac{a^b b\log(a)\log(\alpha)}{\alpha-1}\right)}{b\log(\alpha)} - \sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\left(\frac{a}{x_i}\right)^b\log\left(\frac{a}{x_i}\right) - \log(\alpha)\sum_{i=1}^{n}\left(\frac{a}{x_i}\right)^b e^{-(\frac{a}{x_i})^b}\log\left(\frac{a}{x_i}\right), \tag{26}$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n a^{-b}(\alpha-1)\left(\frac{a^b b}{\alpha(\alpha-1)} + \frac{a^b b\log(\alpha)}{(\alpha-1)^2}\right)}{b\log(\alpha)} + \frac{1}{\alpha}\sum_{i=1}^{n} e^{-(\frac{a}{x_i})^b}. \tag{27}$$

The estimates of the unknown parameters can be obtained by setting the score functions to zero and solving the system of nonlinear equations numerically by means of iterative techniques such as the Newton-Raphson algorithm. For the purpose of interval estimation of the model parameters, a 3×3 observed information matrix, $J(\xi) = \{J_{rs}\}$ (for $r,s = a,b,\alpha$) is required. Under the usual regularity condition, the multivariate normal distribution, $N_3(0, J(\hat{\xi})^{-1})$, can be employed to estimate approximate confidence intervals for the model parameters. Here, $J(\hat{\xi})$ is the total observed information matrix evaluated at $\hat{\xi}$. Using this multivariate normal approximation, the approximate $100(1-\rho)\%$ confidence intervals for the parameters can be determined.

6 Monte Carlo Simulation

In this section, Monte Carlo simulations are performed to examine the finite sample properties of the maximum likelihood estimators for the parameters of the APTF distribution. The results of the simulation are obtained from 2000 Monte Carlo replications. In each replication, a random sample of size $n = 25, 50, 75$ and 100 is generated from the APTF distribution using its quantile function. Table 3 presents the Average Estimate (AE), Average Bias (AB), Root Mean Square Error (RMSE) and Coverage Probability (CP) for the 95% confidence interval for the parameters of APTF distribution. From the results, it can be seen that the AE are close to the actual values while the AB and the RMSEs exhibit fluctuating pattern. That is for some parameters, AB and RMSE show upward and downward movements as the sample size increases. The CPs of the confidence intervals are quite close to the nominal 0.95 in most cases. Thus, the results indicate that the estimates for the parameters are stable and their asymptotic properties can be employed for constructing confidence intervals.

7 Applications

The applications of the APTF distribution are demonstrated in this section using real data sets. The performance of the APTF model is compared with that of its sub-models and the Transmuted Fréchet (TFr) distribution using the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and -2ℓ criterion. The smaller the values of the model selection criteria the better the distribution. The maximum-likelihood estimates for the parameters of the fitted models are obtained by maximizing the log-likelihood function. The PDF of the TFr distribution is given by:

$$f(x) = a^b b x^{-b-1} e^{-\left(\frac{a}{x}\right)^b}\left[1+\alpha-2\alpha e^{-\left(\frac{a}{x}\right)^b}\right], \quad a>0, b>0, |\alpha|\leq 1, x>0.$$

7.1 First Data Set

The data set is obtained from Smith and Naylor, and consists of the strength of 1.5cm glass fibers measured at the National Physical Laboratory, England [19]. The data are: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24. Table 4 displays the maximum-likelihood estimates for the parameters of the fitted distributions with their corresponding standard errors in parentheses and the model selection criteria. The parameters of the fitted models are significant at the 5% level of significance. Using the model selection criteria, the APTF distribution provides a more reasonable parametric fit to the data than its sub-models and the TFr model.
To further compare the APTF distribution with its sub-models, a Likelihood Ratio Test (LRT) is performed. The LRT results shown in Table 5 reveal that the APTF provides a more reasonable parameteric fit to the data than its sub-models.
The estimated variance-covariance matrix for the parameters of the APTF distribution for the data is:

$$J^{-1} = \begin{pmatrix} 1.7405\times 10^{-3} & 7.5932\times 10^{-3} & -3.3796\times 10^{-7} \\ 7.5932\times 10^{-3} & 9.6884\times 10^{-2} & -6.1003\times 10^{-6} \\ -3.3796\times 10^{-7} & -6.1003\times 10^{-6} & 4.0125\times 10^{-10} \end{pmatrix}.$$

Figure 4 shows the plot of the empirical and fitted densities for the data.

7.2 Second Data Set

The data set is made up of failure time in hours of kevlar 49/epoxy strands with pressure at 90% and and was already studied [20]. The data consists of 101 observations and the numbers are: 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89. The maximum-likelihood estimates for the parameters of the fitted models with their standard errors in parentheses and model selection criteria are given in Table 6. All the parameters of the fitted models are significant at the 5%

Table 3: Monte Carlo simulation results: AE, AB, RME and CP for APTF distribution

	Parameter			AE			AB			RMSE			CP		
n	a	b	α	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$
25	0.3	0.2	0.3	0.3495	0.2203	0.5045	0.0495	0.0203	0.2045	0.3152	0.0464	0.3562	0.8590	0.9975	0.9580
50	0.3	0.2	0.3	0.3525	0.2137	0.4823	0.0525	0.0137	0.1823	0.3121	0.0340	0.3400	0.8780	0.9965	0.9650
75	0.3	0.2	0.3	0.3702	0.2100	0.4632	0.0702	0.0100	0.1632	0.3174	0.0287	0.3305	0.8715	0.9975	0.9755
100	0.3	0.2	0.3	0.3745	0.2073	0.4549	0.0745	0.0073	0.1549	0.3162	0.0257	0.3193	0.8805	0.9995	0.9765
25	2.5	0.5	1.5	2.1170	0.5156	1.7043	-0.3830	0.0156	0.2043	1.2819	0.0943	1.1656	0.8640	0.9950	0.8945
50	2.5	0.5	1.5	2.1042	0.5015	1.6600	-0.3958	0.0015	0.1600	1.2602	0.0709	1.1507	0.8505	0.9870	0.8825
75	2.5	0.5	1.5	2.0344	0.4979	1.7051	-0.4656	-0.0021	0.2051	1.2482	0.0626	1.1455	0.8220	0.9885	0.8805
100	2.5	0.5	1.5	2.0662	0.4908	1.6438	-0.4338	-0.0092	0.1438	1.2298	0.0579	1.1115	0.8045	0.9925	0.8690
25	3.5	3.2	2.5	2.7064	3.1802	2.2378	-0.7936	-0.0198	-0.2622	0.9018	0.6033	1.6263	0.5805	0.9775	0.7875
50	3.5	3.2	2.5	2.6871	3.1179	2.2858	-0.8129	-0.0821	-0.2142	0.9050	0.5074	1.5934	0.4585	0.9640	0.7860
75	3.5	3.2	2.5	2.6742	3.0966	2.3591	-0.8258	-0.1034	-0.1409	0.9140	0.4650	1.5597	0.3685	0.9550	0.7910
100	3.5	3.2	2.5	2.6754	3.0613	2.3288	-0.08246	-0.1387	-0.1712	0.9131	0.4594	1.5349	0.3325	0.9405	0.7995
25	8.0	8.0	8.0	8.4371	7.7998	4.8474	0.4371	-0.2002	-3.1526	0.7395	1.4386	5.1178	0.9705	0.9745	0.6655
50	8.0	8.0	8.0	8.3613	7.7225	5.3863	0.3613	-0.2775	-2.6137	0.6480	1.2089	4.7861	0.9705	0.9620	0.7040
75	8.0	8.0	8.0	8.3117	7.7221	5.7483	0.3117	-0.2779	-2.2517	0.5944	1.0940	4.5138	0.9605	0.9545	0.7315
100	8.0	8.0	8.0	8.2848	7.6653	5.8425	0.2848	-0.3347	-2.1575	0.5641	1.0164	4.3940	0.9635	0.9465	0.7525

Table 4: Maximum-likelihood estimates and model selection criteria

	Estimates of parameters			Model selection criteria		
Model	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	-2ℓ	AIC	BIC
APTF	0.8734 (4.1720×10^{-2})	3.8900 (3.1126×10^{-1})	300.9964 (2.0031×10^{-5})	75.8685	81.8685	88.2979
Fréchet	1.2644 (0.0589)	2.8873 (0.2344)		93.7066	97.7066	101.9929
APTIE	0.2043 (2.7663×10^{-2})		1000.4897 (8.0745×10^{-7})	187.4785	191.4785	195.7648
APTIR	0.5286 (3.6926×10^{-2})		1000.4462 (1.4656×10^{-6})	113.3644	117.3644	121.6507
OAPTF		4.6591 (2.3289×10^{-1})	5000.2430 (6.0277×10^{-6})	116.3075	120.3075	124.5938
TFr	1.0937 (0.0561)	3.2217 (0.2564)	-0.7745 (0.1561)	86.3031	92.3031	98.7326

Table 5: LRT Statistics

Models	Hypotheses	LRT statistics	$p-$value
Fréchet	$H_0 : \alpha = 1$ vs $H_1 : H_0$ is false	17.8380	2.4050×10^{-5}
APTIE	$H_0 : b = 1$ vs $H_1 : H_0$ is false	111.6100	$< 2.2000\times10^{-16}$
APTIR	$H_0 : b = 2$ vs $H_1 : H_0$ is false	37.4960	9.1600×10^{-10}
OAPTF	$H_0 : a = 1$ vs $H_1 : H_0$ is false	40.4390	2.0280×10^{-10}

significance level. From the values of the model selection criteria, it is obvious that the APTF distribution provides a better fit than the other estimated models.

The LRT is performed to compare the performance of the APTF model with its sub-models. From Table 7, the APTF model provides a better fit to the data than its sub-models.

The computed variance-covariance matrix for the

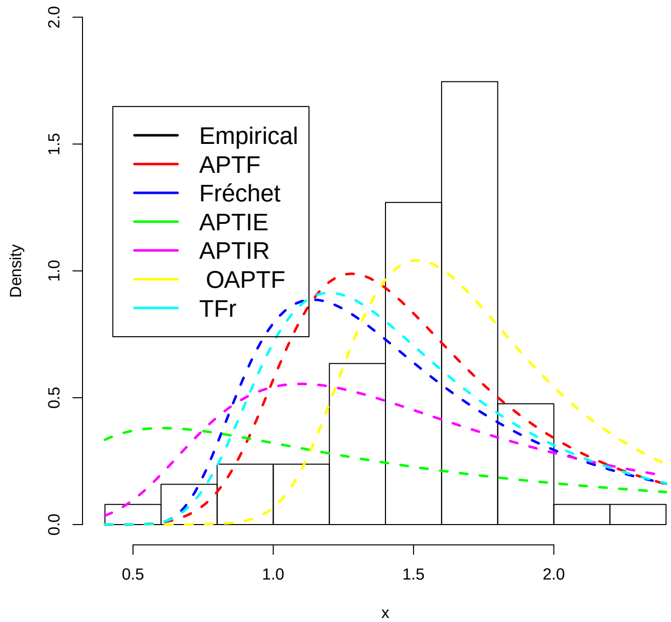

Fig. 4: Plot of the empirical and fitted densities

Table 6: Maximum-likelihood estimates and model selection criteria

	Estimates of parameters			Model selection criteria		
Model	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	-2ℓ	AIC	BIC
APTF	0.0491	0.7797	100.5303	249.3418	255.3418	263.1871
	(9.7346×10^{-3})	(5.6106×10^{-2})	(4.1294×10^{-6})			
Fréchet	0.2425	0.6132		264.8788	268.8788	274.1091
	(0.0419)	(0.0424)				
APTIE	0.0654		100.5336	264.4213	268.4213	273.6515
	(7.9847×10^{-3})		(9.2054×10^{-7})			
APTIR	0.0465		1437.101	615.0299	619.0299	624.2602
	(2.6164×10^{-3})		(2.2585×10^{-9})			
OAPTF		0.4467	0.0638	262.5252	266.5252	271.7555
		(0.0246)	(0.0261)			
TFr	0.1341	0.6752	-0.6664	257.7661	263.7661	271.6114
	(0.0293)	(0.0461)	(0.1647)			

parameters of the APTF model for the data set is:

$$J^{-1} = \begin{pmatrix} 9.4763\times10^{-5} & 3.2333\times10^{-4} & 3.1802\times10^{-8} \\ 3.2333\times10^{-4} & 3.1479\times10^{-3} & 2.2272\times10^{-7} \\ 3.1802\times10^{-8} & 2.2272\times10^{-7} & 1.7052\times10^{-11} \end{pmatrix}.$$

The empirical and fitted densities plot for the data are shown in Figure 5.

Table 7: LRT Statistics

Models	Hypotheses	LRT statistics	$p-$value
Fréchet	$H_0 : \alpha = 1$ vs $H_1 : H_0$ is false	15.5370	8.0900×10^{-5}
APTIE	$H_0 : b = 1$ vs $H_1 : H_0$ is false	15.0790	0.0001
APTIR	$H_0 : b = 2$ vs $H_1 : H_0$ is false	365.6900	$< 2.2000 \times 10^{-16}$
OAPTF	$H_0 : a = 1$ vs $H_1 : H_0$ is false	13.1840	0.0003

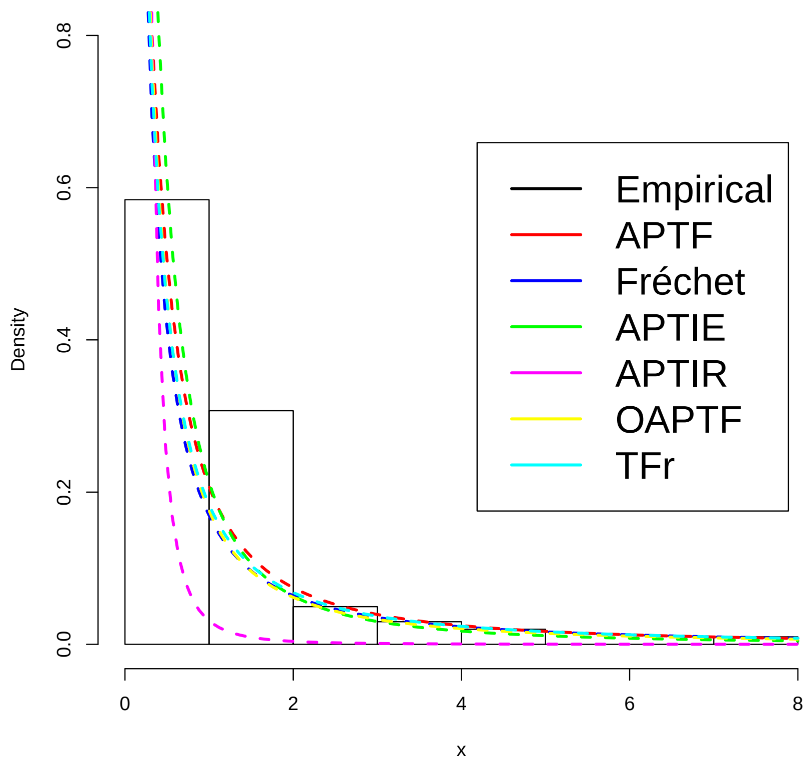

Fig. 5: Plot of the empirical and fitted densities

8 Bivariate and Multivariate Extensions

This section presents some bivariate and multivariate extensions of the APTF distribution using copulas. Let (X_1, X_2) be a random pair, then a copula C associated with the pair is simply a joint distribution of the random vector $(F_{X_1}(x_1), F_{X_2}(x_2))$. Given that $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$ are marginal CDFs of the random variables X_1 and X_2 respectively. According Sklar [21], if C is the copula associated with (X_1, X_2), then the joint CDF of the pair (X_1, X_2) is given by:

$$F_{X_1X_2}(x_1, x_2) = C(F_{X_1}(x_1), F_{X_2}(x_2)).$$

Using this concept, the bivariate and multivariate extensions of the APTF distribution are proposed.

8.1 Ali-Mikhail-Haq Bivariate and Multivariate APTF

Suppose $X_1 \sim APTF(a_1, b_1, \alpha_1)$, $X_2 \sim APTF(a_2, b_2, \alpha_2)$ and the copula associated with the random pair (X_1, X_2) belong to the Ali-Mikhail-Haq (AMH) family. Then,

$$C(u,v) = \frac{uv}{1-\phi(1-u)(1-v)}, \ |\phi| < 1.$$

The joint CDF of the AMH Bivariate APTF (AMHBAPTF) distribution is given by:

$$F_{X_1X_2}(x_1,x_2) = \frac{F_{X_1}(x_1)F_{X_2}(x2)}{1-\phi(1-F_{X_1}(x_1))(1-F_{X_2}(x_2))} = \begin{cases} \dfrac{\left(\alpha_1^{e^{-(\frac{a_1}{x_1})^{b_1}}}-1\right)\left(\alpha_2^{e^{-(\frac{a_2}{x_2})^{b_2}}}-1\right)}{(\alpha_1-1)(\alpha_2-1)-\phi\left(\alpha_1-e^{-(\frac{a_1}{x_1})^{b_1}}\right)\left(\alpha_2-e^{-(\frac{a_2}{x_2})^{b_2}}\right)}, \alpha_i \neq 1 \\ \dfrac{e^{-\left[(\frac{a_1}{x_1})^{b_1}+(\frac{a_2}{x_2})^{b_2}\right]}}{1-\phi\left(1-e^{-(\frac{a_1}{x_1})^{b_1}}\right)\left(1-e^{-(\frac{a_2}{x_2})^{b_2}}\right)}, \alpha_i = 1 \end{cases}, \quad (28)$$

where, $a_i > 0, i = 1,2, b_1 > 0$ and $b_2 > 0$ are marginal parameters. The parameter ϕ is the dependence parameter. The $p-$ variate extension is:

$$F_{X_1X_2\ldots X_p}(x_1,x_2,\ldots x_p) = \begin{cases} (1-\phi)\left[\prod_{i=1}^{p}\left(\dfrac{(\alpha_i-1)(1-\phi)}{\alpha_i^{e^{-(\frac{a_i}{x_i})^{b_i}}}}+\phi\right)-\phi\right]^{-1}, \alpha_i \neq 1 \\ (1-\phi)\left[\prod_{i=1}^{p}\left(\dfrac{1-\phi}{e^{-(\frac{a_i}{x_i})^{b_i}}}+\phi\right)-\phi\right]^{-1}, \alpha_i = 1 \end{cases},$$

where, $a_i > 0$ and $b_i > 0$ are marginal parameters.

8.2 Clayton Bivariate and Multivariate APTF

Suppose $X_1 \sim APTF(a_1, b_1, \alpha_1)$, $X_2 \sim APTF(a_2, b_2, \alpha_2)$ and the copula associated with the random pair (X_1, X_2) belong to the Clayton family. Then,

$$C(u,v) = \left[u^{-\phi}+v^{-\phi}-1\right]^{\frac{-1}{\phi}}, \phi \geq 0.$$

The joint CDF of the Clayton Bivariate APTF (CBAPTF) distribution is given by:

$$F_{X_1X_2}(x_1,x_2) = \left[(F_{X_1}(x_1))^{-\phi}+(F_{X_2}(x_2))^{-\phi}-1\right]^{\frac{-1}{\phi}} = \begin{cases} \left[\sum_{i=1}^{2}\left(\dfrac{\alpha_i^{e^{-(\frac{a_i}{x_i})^{b_i}}}-1}{\alpha_i-1}\right)^{-\phi}-1\right]^{\frac{-1}{\phi}}, \alpha_1 \neq 1, \alpha_2 \neq 1 \\ \left[\sum_{i=1}^{2}\left(e^{-(\frac{a_i}{x_i})^{b_i}}\right)^{-\phi}-1\right]^{\frac{-1}{\phi}}, \alpha_1 = 1, \alpha_2 = 1 \end{cases},$$

where, $a_1 > 0, a_2 > 0, b_1 > 0$, $b_2 > 0$ are marginal parameters and ϕ is the dependence parameter. The $p-$variate extension is given by:

$$F_{X_1X_2\ldots X_p}(x_1,x_2,\ldots x_p) = \begin{cases} \left[\sum_{i=1}^{p}\left(\dfrac{\alpha_i^{e^{-(\frac{a_i}{x_i})^{b_i}}}-1}{\alpha_i-1}\right)^{-\phi}-1\right]^{\frac{-1}{\phi}}, \alpha_i \neq 1, \\ \left[\sum_{i=1}^{p}\left(e^{-(\frac{a_i}{x_i})^{b_i}}\right)^{-\phi}-1\right]^{\frac{-1}{\phi}}, \alpha_i = 1 \end{cases}, \quad (29)$$

where, $a_i > 0$ and $b_i > 0$ are marginal parameters.

9 Conclusion

In this study, a three-parameter model called APTF distribution is proposed and its statistical properties are derived. The estimators for the parameters of the distribution are developed using maximum-likelihood estimation and Monte Carlo simulation are performed to assess the properties of the estimators. The applications of the APTF distribution are demonstrated by using real-data sets. The performance of the APFT distribution with regards to providing good fit to the data sets is assessed by comparing it with other models. The results reveal that the APTF model provides a more reasonable parametric fit to the data sets. Finally, bivariate and multivariate extensions of the model are proposed using copulas. Future extensions of this work requires using the APTF distribution to model-censored data and also developed regression model using the APTF distribution.

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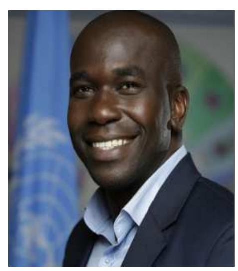

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