A General Dynamic Inequality of Opial Type

Ravi Agarwal¹, Martin Bohner², Donal O’Regan³, Mahmoud Osman⁴ and Samir Saker⁴,*

¹ Texas A&M University–Kingsville, Department of Mathematics, Kingsville, TX 78363, USA
² Missouri S&T, Department of Mathematics and Statistics, Rolla, MO 65409-0020, USA
³ National University of Ireland, School of Mathematics, Statistics, and Applied Mathematics, Galway, Ireland
⁴ Mansoura University, Department of Mathematics, Faculty of Science, Mansoura, Egypt

Abstract: We present a new general dynamic inequality of Opial type. This inequality is new even in both the continuous and discrete cases. The inequality is proved by making use of a recently introduced new technique for Opial dynamic inequalities, the time scales integration by parts formula, the time scales chain rule, and classical as well as time scales versions of Hölder’s inequality.

Keywords: Opial’s inequality, Hölder’s inequality, time scales.

1 Introduction

In 1960, Olech [8] extended an inequality of Opial [9] and proved that if \( f \in C^1([0,h],\mathbb{R}) \) with \( h > 0 \) satisfies \( f(0) = 0 \), then

\[
\int_0^h |f(t)f'(t)| \, dt \leq \frac{h}{2} \int_0^h |f'(t)|^2 \, dt. \tag{1}
\]

This inequality created a lot of research activity, which was summarized in the monograph [2], both for the continuous and the discrete cases. In [3] (see also [6, Theorem 6.23]), the authors extended (1) to an arbitrary time scale \( \mathbb{T} \) and proved that if \( f \in C^1_{\alpha}(\{0,h\};\mathbb{R}) \) with \( h > 0 \) satisfies \( f(0) = 0 \), then

\[
\int_0^h \left| (f^2)\Delta(t) \right| \Delta t \leq h \int_0^h \left( f^2(t) \right)^2 \Delta t. \tag{2}
\]

For extensions and generalizations of (2), we refer the reader to the monograph [1]. Over the last sixty years, the study of Opial inequalities (continuous and discrete) or related Hardy operators focused on the investigations of new inequalities or operators with weighted functions. These inequalities have natural applications in applied mathematics, especially in the theory of differential equations in elasticity (ordinary or partial) and led to many interesting questions and connections between different areas of mathematical analysis. For example, Hardy operators are closely related to quasiadditivity properties of capacities and were recently used with Opial-type inequalities to find the gaps between zeros of differential equations that appear in the binding of beams [10].

Here we will not give an introduction to time scales calculus but instead refer the reader to [6, 7]. We only remark that the delta derivative is the usual derivative if \( \mathbb{T} = \mathbb{R} \) and the forward difference if \( \mathbb{T} = \mathbb{Z} \), and the delta integral is the usual integral if \( \mathbb{T} = \mathbb{R} \) and a sum if \( \mathbb{T} = \mathbb{Z} \), and that the theory can be applied to any nonempty closed set \( \mathbb{T} \subset \mathbb{R} \), the so-called underlying time scale. We note that plugging \( \mathbb{T} = \mathbb{R} \) in (2) results in (1).

Using a novel technique in [4], the following generalization of (2) was established, involving two different weight functions \( s \) and \( r \), see [4, Theorem 5.2].

**Theorem 1.** Assume that \( a \in \mathbb{T}, b \in (a,\infty)_\mathbb{T} \),

\[
r_s \in C_{\alpha}([a,b];\mathbb{T}), (0,\infty), \quad \text{and} \quad f \in C^1_{\alpha}([a,b];\mathbb{R}).
\]

If \( f(a) = 0 \), then

\[
\int_a^b s(t) \left| (f^2)\Delta(t) \right| \Delta t \leq K \int_a^b r(t) \left( f^2(t) \right)^2 \Delta t,
\]

where

\[
K = \sqrt{\int_a^b s(t)(R^2)\Delta(t) \Delta t} \quad \text{with} \quad R(t) = \int_a^t \frac{\Delta \tau}{r(\tau)}.
\]

* Corresponding author e-mail: shsaker@mans.edu.eg

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We note that plugging $a = 0$, $b = h$, and $r = s = 1$ in Theorem 1 results in (2).

Refining the technique from [4], the same authors proved in [5] the following generalization of Theorem 1.

**Theorem 2.** Assume that $a \in \mathbb{T}$, $b \in (a, \infty)_T$,

$$r, s \in C_{\alpha}((a, b)_T, (0, \infty)), \quad \text{and} \quad f \in C_{\alpha}^1((a, b)_T, \mathbb{R}).$$

Let $\alpha > 1$ and $\beta > 0$. If $f(a) = 0$, then

$$\int_a^b s(t) \left| \left( f^\alpha \right)^{\Delta} (t) (f^\Delta (t))^\beta \right| \Delta t \leq K \int_a^b \left| f^\Delta (t) \right|^{\alpha + \beta} \Delta t,$$

where

$$K = \frac{\alpha(\beta + 1)}{\alpha + \beta} \left( \frac{b}{s(r(t))} \right)^{\alpha + \beta} \left( \frac{\beta}{s(r(t))} \right)^{\alpha + \beta} \Delta t$$

with

$$R(t) = \int_a^t \frac{\Delta \tau}{(r(\tau))^{\alpha + \beta}}.$$

We note that plugging $a = 0$, $b = h$, and $r = s = 1$ in Theorem 3 results in Theorem 2.

The purpose of this paper is to apply the new technique developed in [4, 5] to prove the following generalization of Theorem 2.

**Theorem 3.** Assume that $a \in \mathbb{T}$, $b \in (a, \infty)_T$,

$$r, s \in C_{\alpha}((a, b)_T, (0, \infty)), \quad \text{and} \quad f \in C_{\alpha}^1((a, b)_T, \mathbb{R}).$$

Let $\alpha \geq 1$, $\beta \geq 0$, and $k > \beta + 1$. If $f(a) = 0$, then

$$\int_a^b s(t) \left| \left( f^\alpha \right)^{\Delta} (t) (f^\Delta (t))^\beta \right| \Delta t \leq K \left( \int_a^b \left| f^\Delta (t) \right|^k \Delta t \right)^{\alpha + \beta},$$

where

$$K = \frac{\alpha(\beta + 1)}{\alpha + \beta} \left( \frac{b}{s(r(t))} \right)^{\alpha + \beta} \left( \frac{\beta}{s(r(t))} \right)^{\alpha + \beta} \Delta t$$

with

$$R(t) = \int_a^t \frac{\Delta \tau}{(r(\tau))^{\alpha + \beta}}.$$

We note that plugging $k = \alpha + \beta$ in Theorem 3 results in Theorem 2.

The paper is organized as follows: In Section 2, we present the basic definitions of time scales calculus that will be used in the sequel. In Section 3, we prove Theorem 3 and give some remarks. We prove our main result by using the time scales chain rule, the time scales integration by parts formula, and classical continuous and discrete as well as time scales versions of Hölder's inequality.

## 2 Time Scales Preliminaries

In this section, we briefly present some basic definitions and results concerning the delta calculus on time scales that will be used in this article. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf \{ s \in \mathbb{T} : s > t \}$ for $t \in \mathbb{T}$. For any function $f : \mathbb{T} \to \mathbb{R}$, we put $f^\sigma = f \circ \sigma$. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, denoted by $f \in C_{\sigma}$, if it is continuous at each right-dense point (i.e., $\sigma(t) = t$) and there exists a finite left-sided limit at all left-dense points (i.e., $\rho(t) = t$, where the backward jump $\rho$ is defined in a similar way as the forward jump $\sigma$). For the definition of the delta derivative and the delta integral, we refer to [6, 7]. If $f \in C_{\alpha}^1(\mathbb{R}, \mathbb{R})$ and $g : \mathbb{T} \to \mathbb{R}$ is rd-differentiable, then the time scales chain rule, see [6, Theorem 1.90], states that

$$(f \circ g)^\Delta = g^\Delta \int_0^1 f'(t)h^\sigma + (1 - h)g^\Delta \, dh,$$

and a special case, which we will use in this paper, is given by

$$(f \gamma)^\Delta = \gamma f^\Delta \int_0^1 (h \gamma f^\sigma + (1 - h)\gamma)^{\gamma - 1} \, dh \quad \text{for} \quad \gamma \in \mathbb{R}.$$ 

The time scales Hölder inequality, see [6, Theorem 6.13], says

$$\int_a^b |f(t)g(t)|\Delta t \leq \left( \int_a^b |f(t)|^\gamma \Delta t \right)^{\frac{1}{\gamma}} \left( \int_a^b |g(t)|^\nu \Delta t \right)^{\frac{1}{\nu}},$$

where $a, b \in \mathbb{T}$, $f, g \in C_{\alpha}((a, b)_T, \mathbb{R})$, $\gamma > 1$, and $\nu = \gamma/(\gamma - 1)$.

## 3 Proof of the Opial Inequality

In this section, we present the proof of our main result, Theorem 3, and give some corollaries and concluding remarks.

**Proof.** Define

$$g(t) := \int_a^t f(r(\tau)) \left| f^\Delta (\tau) \right|^k \Delta \tau.$$
Then $g(a) = 0$,

$$g^\Delta = r f^\Delta \bigg|_a^k$$

so that $|f^\Delta| = \left( \frac{g^\Delta}{r} \right)^{1/2}$, and

$$|f(t)| = \left| \int_a^t \frac{1}{(r(\tau))^k} |f^\Delta(\tau)| \Delta \tau \right|$$

$$\leq \int_a^t \frac{1}{(r(\tau))^k} |f^\Delta(\tau)| \Delta \tau$$

$$\leq \left\{ \int_a^t \frac{\Delta t}{(r(\tau))^k} \right\}^{1/2} \left\{ \int_a^t r(\tau)|f^\Delta(\tau)| \Delta t \right\}^{1/2}$$

$$= (R(t))^{1/2} (g(t))^{1/2},$$

where we have used the time scales Hölder inequality with conjugate exponents $\frac{k}{k-1}$ and $k > 1$. Thus, for $h \in [0, 1]$, we obtain

$$|h f^\sigma + (1-h)f| \leq h |f^\sigma| + (1-h)|f|$$

$$\leq h (R^\sigma)^{1/2} (g^\sigma)^{1/2} + (1-h)R^{1/2} g^{1/2}$$

$$= (R^\sigma)^{1/2} (h g^\sigma)^{1/2} + (1-h)(R^\sigma + (1-h)R)^{1/2}$$

$$\leq (R^\sigma + (1-h)R)^{1/2} (hg^\sigma + (1-h)g)^{1/2},$$

where we have used the classical Hölder inequality for sums with conjugate exponents $\frac{k}{k-1}$ and $k > 1$. Hence

$$\left| \int_0^1 (h f^\sigma + (1-h)f)^{\alpha - 1} dh \right|$$

$$\leq \int_0^1 |h f^\sigma + (1-h)f|^{\alpha - 1} dh$$

$$\leq \int_0^1 (h R^\sigma + (1-h)R)^{\frac{k-1}{k} (\alpha - 1)} (hg^\sigma + (1-h)g)^{\frac{\alpha}{k+1}} dh$$

$$\leq \left\{ \int_0^1 (h R^\sigma + (1-h)R)^{\frac{k-1}{k} (\alpha - 1)} dh \right\}^{\frac{k-1}{k}}$$

$$\times \left\{ \int_0^1 (hg^\sigma + (1-h)g)^{\frac{\alpha}{k+1}} dh \right\}^{\frac{k+1}{k}}.$$
with
\[ R(t) = \int_a^t \frac{\Delta \tau}{(r(\tau))^{\alpha}}. \]

The next result follows from Corollary 1 by choosing \( k = \alpha \) (see also [5, Corollary 3.2]).

**Corollary 2.** Assume that \( a \in \mathbb{T}, b \in (a, \infty)_\mathbb{T}, \)
\( r, s \in C^1_{\mathbb{T}^+}([a, b] \mathbb{T}, (0, \infty)), \) and \( f \in C^1_{\mathbb{T}^+}([a, b] \mathbb{T}, \mathbb{R}). \)

Let \( \alpha > 1. \) If \( f(a) = 0, \) then
\[
\int_a^b s(t) \left| (f^\alpha)'(t) \right| \Delta t \leq K \int_a^b r(t) \left| f^\alpha(t) \right| \Delta t,
\]
where
\[ K = \left\{ \int_a^b (s(t))^{\alpha-1} (R(t))^{\frac{\alpha-1}{\alpha}} \right\} \frac{\alpha-1}{\alpha}, \]
with
\[ R(t) = \int_a^t \frac{\Delta \tau}{(r(\tau))^{\alpha-1}}. \]

**References**


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**Ravi Agarwal** is the head of the Department of Mathematics at Texas A&M University–Kingsville in USA. He is the author of many books and papers on fixed point theory, operator, integral, differential, partial and difference equations, oscillation theory, inequalities, and critical point methods. He serves as the managing editor and is on the editorial boards of numerous mathematical journals.

**Martin Bohner** is the Curators’ Professor of Mathematics and Statistics at Missouri University of Science and Technology in Rolla, Missouri, USA. He received the BS (1989) and MS (1993) in Econo-mathematics and PhD (1995) from Universitität Ulm, Germany, and MS (1992) in Applied Mathematics from San Diego State University. His research interests center around differential, difference, and dynamic equations as well as their applications to economics, finance, biology, physics, and engineering. He is the author of five textbooks and more than 200 publications, Editor-in-Chief of two international journals, Associate Editor for more than 50 international journals, and President of ISDE, the International Society of Difference Equations. Professor Bohner’s honors at Missouri S&T include five Faculty Excellence Awards, one Faculty Research Award, and eight Teaching Awards.

**Donal O’Regan** is a Professor of Mathematics at the National University of Ireland in Galway. He is the author of many books and papers on fixed point theory, operator, integral, differential, partial and difference equations, oscillation theory, inequalities and critical point methods. He serves on the editorial board of numerous mathematical journals.
Mahmoud Osman is a Lecturer of Mathematics at Mansoura University of Egypt in Mansoura. His interests are qualitative analysis of dynamic equations and inequalities on time scales. Up to now, he has co-authored one paper on Opial inequalities in the journal *Mathematical Inequalities and Applications*.

Samir Saker is a Professor of Mathematics at Mansoura University of Egypt in Mansoura. His interests are qualitative analysis of dynamic equations, inequalities on time scales, and qualitative behavior of delay models. He is an author of four books and more than 200 papers. He serves on the editorial board of many mathematical journals.