

# Chaotic Jerk Circuit: Existence and Stability of Solutions for a Fractional Model

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Received: 12 Oct. 2021, Revised: 12 Dec. 2021, Accepted: 17 Jan. 2022

Published online: 1 Jul. 2023

**Abstract:** We propose a new model of Jerk equation involving fractional derivatives. We begin by presenting an existence and uniqueness result for the problem. Then, by means of Schaefer fixed point theorem, another main result, that deals with the existence of at least one solution, is established. Beside this, Ulam type stabilities are discussed. At the end, two illustrative examples are also discussed.

**Keywords:** Jerk fractional model, integral boundary conditions, uniqueness, fixed point, Ulam stability.

## 1 Introduction

Fractional differential equations have been a powerful tool for modeling several phenomena and have emerged in many disciplines, such as probability and statistics, visco-elasticity, chemistry, fluid flow, electrical networks, optics, control theory of dynamical systems, electrical circuits and, so on. For more details and some concrete applications of this theory, we refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

Let us now, recall some papers that have motivated the present work. We begin by the references [16], where J.C. Sportt identified further chaotic systems by extensive computational research on this phenomenon, with either five terms and two quadratic non-linearities or six terms and a simple quadratic non-linearity.

Then, H.P.W. Gottlieb [17] discovered and studied the following "chaotic" problem

$x''' = -x's + x''(x + x'')/x'$ , which he called a Jerk function.

Then, J.C. Sportt [18] discovered a particular case of a simplest Jerk function that gives chaos. It is the following:

$x''' + ax'' \pm x'^2 + x = 0$ , which has three terms in its Jerk description or five terms in its dynamical system version with a unique quadratic nonlinearity and a only one parameter.  $a$ .

Subsequently, Z. Fu and J. Heidel [19] proved that there can be no simpler system with a quadratic nonlinearity.

Then, B. Munmuangsaen et al. [20] studied several simple chaotic systems of the form:

$x''' + x'' + x = f(x')$ .

Other papers dealing with Jerk equations for chaotic behaviours can be found in [21, 22, 23, 24, 25].

In this work we try to find a suitable fractional presentation for a simple Jerk circuit that allows studying chaotic dynamics which was modeled by the above problem of [20].

So, let us consider the following fractional differential problem:

$$D^\alpha(D^2 + \lambda^2 D^\alpha)y(t) = f(t, y(t), D^\alpha y(t)), \quad t \in [0, T], \quad T > 0, \quad (1)$$

subject to a nonlocal and integral boundary condition and initial conditions as follows:

$$y(0) = 0, \quad D^{1-\alpha} D^\alpha y(0) = 0, \quad y(T) = \beta J^\gamma y(\eta), \quad 0 < \eta \leq T, \quad (2)$$

where  $D^\alpha$  Caputo fractional derivatives order  $\alpha \in [0, 1]$ ,  $J^\gamma$  is a Riemann-Liouville fractional integral order  $\gamma \in [0, \infty[$ ,  $\lambda \in \mathbb{R}_+$  and  $\beta \in \mathbb{R}$ .

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The embedded integral boundary condition of the above problem is a right way to understand the vascular hemodynamics, so we conduct to the same process on the charge in electric circuits which is not usually defensible see [26], while the non local condition is suitable physical measurement to describe processes that have different positions [1].

This paper is organized as follows: In the second section, we recall some useful definitions, notations, and some auxiliary results about fractional derivatives. In the third section, we establish the existence and uniqueness as well as the existence of at least one solution for the problem (1)-(2). Moreover, we discuss Ulam type stabilities for problem(1)-(2). In the last section, we provide two illustrative examples.

## 2 Preliminaries and relevant lemmas

We introduce in this section the essential definitions of the fractional calculus. For more details, one can consult [16].

**Definition 1.** The Riemann-Liouville fractional integral of order  $\alpha \geq 0$  for a continuous function  $f : [a, b] \rightarrow \mathbb{R}$  is defined as

$$J_a^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, a < t \leq b, \\ f(t), & \alpha = 0, a < t \leq b, \end{cases}$$

where,  $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$ .

For  $\alpha > 0$ ,  $\beta > 0$ , we note that:

$$J_a^\alpha J_a^\beta f(t) = J_a^{\alpha+\beta} f(t).$$

**Definition 2.** The Caputo fractional derivative of order  $\alpha \geq 0$  for  $f \in C^n([a, b], \mathbb{R})$ ,  $n = [\alpha] + 1$ , is defined by:

$$D^\alpha f(t) = J^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds.$$

Note also that for  $\alpha \leq \beta$ , we have :

$$D_a^\alpha J_a^\beta f(t) = J_a^{\beta-\alpha} f(t).$$

**Lemma 1.** For all  $t \in [a, b]$  and  $\alpha > 0$ ,  $n = [\alpha] + 1$ , the general solution of  $D^\alpha y(t) = 0$ , is

$$y(t) = \sum_{i=0}^{n-1} c_i (t-a)^i,$$

where  $c_i \in \mathbb{R}$ ,  $i = \overline{0, n-1}$ .

**Lemma 2.** Let  $\alpha > 0$ ,  $n = [\alpha] + 1$ . Then

$$J_a^\alpha D^\alpha y(t) = y(t) + \sum_{i=0}^{n-1} c_i (t-a)^i, \quad t \in [a, b],$$

for  $c_i \in \mathbb{R}$ ,  $i = \overline{0, n-1}$ .

**Lemma 3.** Then the unique solution of the problem:

$$\begin{cases} D^\alpha (D^2 + \lambda^2 D^\alpha) y(t) = f(t), & t \in [0, T], \quad 0 < \alpha \leq 1, \quad T > 0 \\ y(0) = 0, \quad D^{1-\alpha} D^\alpha y(0) = 0, \quad y(T) = \beta J^\gamma y(\eta), \quad \gamma \geq 0, \end{cases}$$

is represented by the following formula:

$$\begin{aligned}
 y(t) = & \int_0^t \frac{(t-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} f(\tau) d\tau - \lambda^2 \int_0^t \frac{(t-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} y(\tau) d\tau \\
 & - \mathcal{C}_2 \left( t^2 - \mathcal{C}_1 t^{2-\alpha} \right) \left( \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} f(\tau) d\tau - \lambda^2 \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} y(\tau) d\tau \right) \\
 & + \mathcal{C}_2 \left( t^2 - \mathcal{C}_1 t^{2-\alpha} \right) \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} y(\tau) d\tau,
 \end{aligned} \quad (3)$$

where  $\mathcal{C}_1 = \frac{2\lambda^2}{\Gamma(3-\alpha)}$ ,  $\mathcal{C}_2 = \frac{1}{T^2 - \mathcal{C}_1 T^{2-\alpha}}$  and  $\lambda^2 \neq \frac{\Gamma(3-\alpha)T^{-\alpha}}{2}$ .

*Proof.* By lemmas 3 and 4, the general solution can be written as:

$$y(t) = \int_0^t \frac{(t-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} f(\tau) d\tau - \lambda^2 \int_0^t \frac{(t-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} y(\tau) d\tau - \frac{c_0}{2} \left( t^2 - \left( \frac{2\lambda^2}{\Gamma(3-\alpha)} \right) t^{2-\alpha} \right) - c_1 - c_2 t, \quad (4)$$

where  $c_i \in \mathbb{R}$ ,  $i = 0 \dots 2$  are arbitrary constants to be determined.

Thanks to the imposed conditions, we get :  $c_1 = c_2 = 0$ , and

$$\frac{c_0}{2\mathcal{C}_2} = \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} f(\tau) d\tau - \lambda^2 \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} y(\tau) d\tau - \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} y(\tau) d\tau.$$

Substituting the values in (4), we get (3).

### 3 Main results

Within this section, we discuss two results. First outcome deals with the existence of a unique solution for the considered problem. In the second one, we will be concerned to study the of existence of at least one solution for (1)-(2). To do that, we need to use the fixed point theory.

So, let us consider the Banach space:

$$\mathbb{H} := \{ y \in \mathcal{C}([0, T], \mathbb{R}), D^\alpha x \in \mathcal{C}([0, T], \mathbb{R}) \}.$$

We equipped it with the norm

$$\|y\|_{\mathbb{H}} = \|y\|_{\infty} + \|D^\alpha y\|_{\infty},$$

where

$$\|y\|_{\infty} = \sup_{t \in [0, T]} |y(t)|, \quad \|D^\alpha y\|_{\infty} = \sup_{t \in [0, T]} |D^\alpha y(t)|.$$

In view of Lemma 5, we change problem (1), as  $y(t) = \Psi(y)(t)$ , where the operator  $\Psi$  defined by:

$$\begin{aligned}
 \Psi y(t) = & \int_0^t \frac{(t-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} f(t, y(t), D^\alpha y(t)) d\tau - \lambda^2 \int_0^t \frac{(t-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} y(\tau) d\tau \\
 & - \left( \mathcal{C}_2 \left( t^2 - \mathcal{C}_1 t^{2-\alpha} \right) \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} f(t, y(t), D^\alpha y(t)) d\tau + \lambda^2 \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} y(\tau) d\tau \right) \\
 & + \mathcal{C}_2 \left( t^2 - \mathcal{C}_1 t^{2-\alpha} \right) \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} y(\tau) d\tau.
 \end{aligned}$$

In the following, we need the assumptions:

(H1): There exist two positive constants  $L_1, L_2 > 0$ . For each  $t \in [0, T]$  and  $u_i, v_i \in \mathbb{R}, i = 1, 2$ , satisfy

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L_1 |u_1 - v_1| + L_2 |u_2 - v_2|, \quad L := \max(L_1, L_2)$$

(H2): Let  $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a jointly continuous function.

(H3): There exist positive real numbers  $\mathcal{B}_f, \mathcal{B}_y$ , for all  $t \in [0, T], u_i \in \mathbb{R}, i = 1, 2$

$$|f(t, u_1, u_2)| \leq \mathcal{B}_f, \quad |y(t)| \leq \mathcal{B}_y.$$

And for calculation convenience, we bring the notes:

$$S_1 = \frac{T^{\alpha+2}}{\Gamma(3+\alpha)} \left( 1 + |\mathcal{C}_2| T^2 + |\mathcal{C}_2 \mathcal{C}_1| T^{2-\alpha} \right),$$

$$S_2 = \frac{\lambda^2 T^{2-\alpha}}{\Gamma(3-\alpha)} \left( 1 + |\mathcal{C}_2| T^2 + |\mathcal{C}_2 \mathcal{C}_1| T^{2-\alpha} \right),$$

$$S_3 = \beta \frac{\eta^\gamma}{\Gamma(\gamma+1)} \left( |\mathcal{C}_2| T^2 + |\mathcal{C}_2 \mathcal{C}_1| T^{2-\alpha} \right),$$

and

$$R_1 = \frac{T^2}{\Gamma(3)} + \frac{2 |\mathcal{C}_2| T^4}{\Gamma(3-\alpha)\Gamma(3+\alpha)} + \frac{\Gamma(3-\alpha) |\mathcal{C}_2 \mathcal{C}_1| T^{4-\alpha}}{\Gamma(3-2\alpha)\Gamma(3+\alpha)},$$

$$R_2 = \frac{\lambda^2 T^{2-2\alpha}}{\Gamma(3-2\alpha)} + \frac{2 |\mathcal{C}_2| \lambda^2 T^{4-2\alpha}}{\Gamma(3-\alpha)^2} + \frac{\lambda^2 |\mathcal{C}_2 \mathcal{C}_1| T^{4-3\alpha}}{\Gamma(3-2\alpha)},$$

$$R_3 = \beta \frac{\eta^\gamma}{\Gamma(\gamma+1)} \left( \frac{2 |\mathcal{C}_2| T^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|\mathcal{C}_2 \mathcal{C}_1| \Gamma(3-\alpha) T^{2-2\alpha}}{\Gamma(3-2\alpha)} \right).$$

$$\Delta := L (S_1 + R_1) + S_2 + R_2 + S_3 + R_3.$$

$$F_y(t) := f(t, y(t), D^\alpha y(t)).$$

### 3.1 A unique solution

**Theorem 1.** Assume that (H1) holds. Then problem(1)-(2) has a unique solution provided that  $\Delta < 1$ .

*Proof.* The procedure is performed in two steps:

**First step:** For  $y_1, y_2 \in \mathbb{H}$ , and  $t \in [0, T]$ , we have

$$\begin{aligned} |\psi y_1(t) - \psi y_2(t)| \leq & \int_0^t \frac{(t-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} |F_{y_1}(\tau) - F_{y_2}(\tau)| d\tau + \lambda^2 \int_0^t \frac{(t-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} |y_1(\tau) - y_2(\tau)| d\tau \\ & + \left| \mathcal{C}_2 \mathcal{C}_1 t^{2-\alpha} \right| \left( \lambda^2 \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} |y_1(\tau) - y_2(\tau)| d\tau + \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} |y_1(\tau) - y_2(\tau)| d\tau \right) \\ & + \left| \mathcal{C}_2 t^2 \right| \left( \lambda^2 \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} |y_1(\tau) - y_2(\tau)| d\tau + \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} |y_1(\tau) - y_2(\tau)| d\tau \right) \\ & + \left| \mathcal{C}_2 \mathcal{C}_1 t^{2-\alpha} \right| \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} |F_{y_1}(\tau) - F_{y_2}(\tau)| d\tau \\ & + \left| \mathcal{C}_2 t^2 \right| \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} |F_{y_1}(\tau) - F_{y_2}(\tau)| d\tau. \end{aligned}$$

Under the above assumptions, we can write:

$$\begin{aligned}
 |\psi_{y_1}(t) - \psi_{y_2}(t)| \leq & \frac{LT^{\alpha+2}}{\Gamma(3+\alpha)} (\|y_1 - y_2\|_\infty + \|D^\alpha y_1 - D^\alpha y_2\|_\infty) + \frac{\lambda^2 T^{2-\alpha}}{\Gamma(3-\alpha)} \|y_1 - y_2\|_\infty \\
 & + |\mathcal{C}_2| \left( \frac{LT^{\alpha+2}}{\Gamma(3+\alpha)} (\|y_1 - y_2\|_\infty + \|D^\alpha y_1 - D^\alpha y_2\|_\infty) + \frac{\lambda^2 T^{2-\alpha}}{\Gamma(3-\alpha)} \|y_1 - y_2\|_\infty \right) T^2 \\
 & + |\mathcal{C}_2 \mathcal{C}_1| \left( \frac{LT^{\alpha+2}}{\Gamma(3+\alpha)} (\|y_1 - y_2\|_\infty + \|D^\alpha y_1 - D^\alpha y_2\|_\infty) + \frac{\lambda^2 T^{2-\alpha}}{\Gamma(3-\alpha)} \|y_1 - y_2\|_\infty \right) T^{2-\alpha} \\
 & + |\mathcal{C}_2| \left( \beta \frac{\eta^\gamma}{\Gamma(\gamma+1)} T^2 + \beta \frac{\eta^\gamma}{\Gamma(\gamma+1)} \mathcal{C}_1 T^{2-\alpha} \right) \|y_1 - y_2\|_\infty.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 |\psi_{y_1}(t) - \psi_{y_2}(t)| \leq & L \frac{T^{\alpha+2}}{\Gamma(3+\alpha)} \left( 1 + |\mathcal{C}_2| T^2 + |\mathcal{C}_2 \mathcal{C}_1| T^{2-\alpha} \right) \|y_1 - y_2\|_\mathbb{H} \\
 & + \frac{\lambda^2 T^{2-\alpha}}{\Gamma(3-\alpha)} \left( 1 + |\mathcal{C}_2| T^2 + |\mathcal{C}_2 \mathcal{C}_1| T^{2-\alpha} \right) \|y_1 - y_2\|_\infty \\
 & + \beta \frac{\eta^\gamma}{\Gamma(\gamma+1)} \left( |\mathcal{C}_2| T^2 + |\mathcal{C}_2 \mathcal{C}_1| T^{2-\alpha} \right) \|y_1 - y_2\|_\infty.
 \end{aligned}$$

In consequence, we get

$$\|\psi_{y_1} - \psi_{y_2}\|_\infty \leq (L S_1 + S_2 + S_3) \|y_1 - y_2\|_\mathbb{H}. \quad (5)$$

**Second step:** To facilitate the proof, we calculate  $D^\alpha \psi$ , we obtain

$$\begin{aligned}
 |D^\alpha \psi_y(t)| = & \int_0^t \frac{(t-\tau)}{\Gamma(2)} F_y(\tau) d\tau + \lambda^2 \int_0^t \frac{(t-\tau)^{2-2\alpha-1}}{\Gamma(2-2\alpha)} y(\tau) d\tau \\
 & + \left| \frac{\mathcal{C}_2 \mathcal{C}_1 \Gamma(3-\alpha) t^{2-2\alpha}}{\Gamma(3-2\alpha)} \right| \left( \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} F_y(\tau) d\tau + \lambda^2 \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} y(\tau) d\tau \right) \\
 & + \left| \frac{2 \mathcal{C}_2 t^{2-\alpha}}{\Gamma(3-\alpha)} \right| \left( \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} F_y(\tau) d\tau + \lambda^2 \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} y(\tau) d\tau \right) \\
 & + \left| \frac{\mathcal{C}_2 \mathcal{C}_1 \Gamma(3-\alpha) t^{2-2\alpha}}{\Gamma(3-2\alpha)} \right| \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} y(\tau) d\tau \\
 & + \left| \frac{2 \mathcal{C}_2 t^{2-\alpha}}{\Gamma(3-\alpha)} \right| \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} y(\tau) d\tau.
 \end{aligned}$$

In the same manner,

$$\begin{aligned}
 |D^\alpha \psi_{y_1}(t) - D^\alpha \psi_{y_2}(t)| \leq & L \left( \frac{T^2}{2} + |\mathcal{C}_2| \frac{2T^4}{\Gamma(3-\alpha)\Gamma(3+\alpha)} + |\mathcal{C}_2 \mathcal{C}_1| \frac{\Gamma(3-\alpha) T^{4-\alpha}}{\Gamma(3-2\alpha)\Gamma(3+\alpha)} \right) \|y_1 - y_2\|_\mathbb{H} \\
 & + \lambda^2 \left( \frac{T^{2-2\alpha}}{\Gamma(3-2\alpha)} + |\mathcal{C}_2| \frac{2T^{4-2\alpha}}{\Gamma(3-\alpha)^2} + |\mathcal{C}_2 \mathcal{C}_1| \frac{T^{4-3\alpha}}{\Gamma(3-2\alpha)} \right) \|y_1 - y_2\|_\infty \\
 & + \beta \frac{\eta^\gamma}{\Gamma(\gamma+1)} \left( |\mathcal{C}_2| \frac{2T^{2-\alpha}}{\Gamma(3-\alpha)} + |\mathcal{C}_2 \mathcal{C}_1| \frac{\Gamma(3-\alpha) T^{2-2\alpha}}{\Gamma(3-2\alpha)} \right) \|y_1 - y_2\|_\infty.
 \end{aligned}$$

And consequently,

$$\|D^\alpha \psi_{y_1} - D^\alpha \psi_{y_2}\|_\infty \leq (LR_1 + R_2 + R_3) \|y_1 - y_2\|_\mathbb{H}. \quad (6)$$

Therefore, by (5)-(6)

$$\|\psi y_1 - \psi y_2\|_{\mathbb{H}} \leq \Delta \|y_1 - y_2\|_{\mathbb{H}}.$$

Thanks to the condition on  $L$ , the operator  $\psi$  is a contractive. Therefore, problem (1)-(2) has a unique solution.

### 3.2 At least one solution

**Theorem 2.** Suppose that (H2) and (H3) are valid. Then the problem (1)-(2) has at least one solution on  $[0, T]$ .

*Proof.* We elaborate the demonstration in four steps:

(A:) Firstly, we shall show that the operator  $\psi$  is completely continuous.

Via the continuity of  $f$ , (see (H2)) it follows that the operator  $\psi$  is continuous. This proof is thus omitted.

(B:) Lets take for  $r > 0$ ,  $\Theta_r := \{y \in \mathbb{H} : \|y\|_{\mathbb{H}} \leq r\}$ . Then, we prove that the operator  $\psi$  maps bounded sets into a bounded sets in  $\mathbb{H}$ . By (H3), for any  $y \in \Theta_r$ , we have:

$$\begin{aligned} |\psi y(t)| &= \int_0^t \frac{(t-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} |F_y(\tau)| d\tau + \lambda^2 \int_0^t \frac{(t-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} |y(\tau)| d\tau \\ &+ (|\mathcal{C}_2 t^2| + |\mathcal{C}_2 \mathcal{C}_1 t^{2-\alpha}|) \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} |F_y(\tau)| d\tau \\ &+ (|\mathcal{C}_2 t^2| + |\mathcal{C}_2 \mathcal{C}_1 t^{2-\alpha}|) \lambda^2 \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} |y(\tau)| d\tau \\ &+ (|\mathcal{C}_2 t^2| + |\mathcal{C}_2 \mathcal{C}_1 t^{2-\alpha}|) \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} |y(\tau)| d\tau. \end{aligned}$$

Therefore,

$$\|\psi y\|_{\infty} \leq \mathcal{B}_f S_1 + (S_2 + S_3) \mathcal{B}_y. \quad (7)$$

With the same arguments as before, we can write:

$$\|D^\alpha \psi y\|_{\infty} \leq \mathcal{B}_f R_1 + (R_2 + R_3) \mathcal{B}_y. \quad (8)$$

Hence, by (7), (8), we obtain

$$\|\psi y\|_{\mathbb{H}} \leq \mathcal{B}_f (S_1 + R_1) + \mathcal{B}_y (S_2 + R_2 + S_3 + R_3).$$

(C:) Let  $t_1, t_2 \in [0, T]; t_1 < t_2$ . Then, we get:

$$\begin{aligned} |\psi y(t_1) - \psi y(t_2)| &\leq \frac{\lambda^2}{\Gamma(2-\alpha)} \left( \int_0^{t_1} |(s-t_1)^{1-\alpha} - (s-t_2)^{1-\alpha}| |y(s)| ds + \int_{t_1}^{t_2} |(s-t_2)^{1-\alpha}| |y(s)| ds \right) \\ &+ |\mathcal{C}_2 \mathcal{C}_1| \left( \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} |F_y(\tau)| d\tau + \lambda^2 \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} |y(\tau)| d\tau \right) |t_1^{2-\alpha} - t_2^{2-\alpha}| \\ &+ |\mathcal{C}_2| \left( \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} |F_y(\tau)| d\tau + \lambda^2 \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} |y(\tau)| d\tau \right) |t_1^2 - t_2^2| \\ &+ |\mathcal{C}_2 \beta| (|t_1^2 - t_2^2| + |\mathcal{C}_1| |t_1^{2-\alpha} - t_2^{2-\alpha}|) \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} |y(\tau)| d\tau \\ &+ \frac{1}{\Gamma(2+\alpha)} \int_0^{t_1} |(s-t_1)^{1+\alpha} - (s-t_2)^{1+\alpha}| |F_y(\tau)| ds \\ &+ \frac{1}{\Gamma(2+\alpha)} \int_{t_1}^{t_2} |(s-t_2)^{1+\alpha}| |F_y(\tau)| ds. \end{aligned} \quad (9)$$

Also, we have:

$$\begin{aligned}
 |D^\alpha \psi y(t_1) - D^\alpha \psi y(t_2)| \leq & \frac{2|\mathcal{C}_2| |t_1^{2-\alpha} - t_2^{2-\alpha}|}{\Gamma(3-\alpha)} \left( \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} |F_y(\tau)| d\tau \right) \\
 & + \frac{2|\mathcal{C}_2| |t_1^{2-\alpha} - t_2^{2-\alpha}|}{\Gamma(3-\alpha)} \left( \lambda^2 \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} y(\tau) d\tau \right) \\
 & + \frac{|\mathcal{C}_2 \mathcal{C}_1| \Gamma(3-\alpha) |t_1^{2-2\alpha} - t_2^{2-2\alpha}|}{\Gamma(3-2\alpha)} \left( \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} |F_y(\tau)| d\tau \right) \\
 & + \frac{|\mathcal{C}_2 \mathcal{C}_1| \Gamma(3-\alpha) |t_1^{2-2\alpha} - t_2^{2-2\alpha}|}{\Gamma(3-2\alpha)} \left( \lambda^2 \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} y(\tau) d\tau \right) \\
 & + \frac{|\mathcal{C}_1 \mathcal{C}_2 \beta| \Gamma(3-\alpha) |t_1^{2-2\alpha} - t_2^{2-2\alpha}|}{\Gamma(3-2\alpha)} \left( \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} y(\tau) d\tau \right) \\
 & + \frac{2|\mathcal{C}_2 \beta| |t_1^{2-\alpha} - t_2^{2-\alpha}|}{\Gamma(3-\alpha)} \left( \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} y(\tau) d\tau \right) \\
 & + \int_0^{t_1} |(s-t_1) - (s-t_2)| |F_y(\tau)| ds + \int_{t_1}^{t_2} |(s-t_2)| |F_y(\tau)| ds.
 \end{aligned} \tag{10}$$

The right hand sides of (9) and (10) tend to zero as  $t_1 \rightarrow t_2$ .

This implies that  $\psi$  is equicontinuous.

As a consequence of above three steps (A),(B),and (C), and thanks to Arzela-Ascoli theorem, we conclude that  $\psi$  is completely continuous.

(D:) Finally, setting  $\Omega(\psi) := \{y \in \mathbb{H}; y = \theta \psi y, \quad 0 < \theta < 1\}$ .

Let  $y \in \mathbb{H}$ , we will show that  $\Omega$  is a bounded set.

For each  $t \in [0, T]$ , we have:

$$|y(t)| = |\theta \psi y(t)| \leq |\psi y(t)| \leq \mathcal{B}_f S_1 + (S_2 + S_3) \mathcal{B}_y,$$

and

$$|D^\alpha y(t)| = |\theta D^\alpha \psi y(t)| \leq |D^\alpha \psi y(t)| \leq \mathcal{B}_f R_1 + (R_2 + R_3) \mathcal{B}_y.$$

Consequently, we obtain

$$\|y\|_{\mathbb{H}} \leq \mathcal{B}_f (S_1 + R_1) + \mathcal{B}_y (S_2 + R_2 + S_3 + R_3).$$

Therefore,  $\Omega$  is bounded.

Thanks to Schaefer fixed point theorem and taking into account the above four steps, the problem (1) has at least one which is a solution on  $[0, T]$ .

## 4 Ulam stability analysis

Now, we present definitions criteria of Ulam stabilities for the problem (1)-(2).

Let  $\varepsilon > 0$ . We consider the following inequalities:

$$|D^\alpha (D^2 + \lambda^2 D^\alpha) y(t) - f(t, y(t), D^\alpha y(t))| \leq \varepsilon, \quad t \in [0, T], \tag{11}$$

$$|D^\alpha (D^2 + \lambda^2 D^\alpha) y(t) - f(t, y(t), D^\alpha y(t))| \leq \varepsilon \chi(t), \quad t \in [0, T], \tag{12}$$

$$|D^\alpha (D^2 + \lambda^2 D^\alpha) y(t) - f(t, y(t), D^\alpha y(t))| \leq \chi(t), \quad t \in [0, T]. \tag{13}$$

**Definition 3.** The problem (1) is Ulam-Hyers stable if there exists a real number  $M_1 > 0$ , such that for each  $\varepsilon > 0$  and for all solution  $z \in \mathbb{H}$  to the inequality (11), there exists a solution  $y \in \mathbb{H}$  of the problem (1), with:

$$\|z - y\|_{\mathbb{H}} \leq M_1 \varepsilon.$$

**Definition 4.** The problem (1) is generalized Ulam-Hyers stable if there exists a function  $\chi \in \mathbb{H}$ ,  $\chi(0) = 0$ , such that for each solution  $z \in \mathbb{H}$  to the inequality (11), there exists a solution  $y \in \mathbb{H}$  of the problem (1), with:

$$\|z - y\|_{\mathbb{H}} \leq \chi(\varepsilon).$$

**Definition 5.** The problem (1) is Ulam-Hyers-Rassias stable with respect to function  $\chi \in \mathbb{H}$  if there exists a real number  $M_2 > 0$  such that for each  $\varepsilon > 0$  and for each solution  $z \in \mathbb{H}$  to the inequality (12), there exists a solution  $y \in \mathbb{H}$  of the problem (1), with:

$$\|z - y\|_{\mathbb{H}} \leq M_2 \varepsilon \chi(t).$$

**Definition 6.** The problem (1) is generalized Ulam-Hyers-Rassias stable with respect to the function  $\chi \in \mathbb{H}$  if there exists a real number  $M_2 > 0$  such that for each  $\varepsilon > 0$  and for each solution  $z \in \mathbb{H}$  to the inequality (13), there exists a solution  $y \in \mathbb{H}$  of the problem (1), with:

$$\|z - y\|_{\mathbb{H}} \leq M_2 \chi(t).$$

Now, we introduce a further hypothesis:

(H4): There exists an increasing functions  $\chi \in \mathbb{H}$  and there exists  $\lambda_{\chi_1}, \lambda_{\chi_2} > 0$ .  
For any  $t \in [0, T]$

$$J^{\alpha+2} |\chi(t)| \leq \lambda_{\chi_1} \chi(t), \quad J^2 |\chi(t)| \leq \lambda_{\chi_2} \chi(t),$$

We are ready to prove the following theorem.

**Theorem 3.** Assume that The hypotheses of Theorem 1 and (H4) holds. Then problem (1) is generalised Ulam-Hyers-Rassias stable.

*Proof.* We need to proceed the proof of this theorem in three steps.

**Step 1:** Let  $z$  be solution of (13), we suppose that

$$\begin{aligned} z(t) = & \int_0^t \frac{(t-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} F_z(\tau) d\tau - \lambda^2 \int_0^t \frac{(t-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} z(\tau) d\tau \\ & + \mathcal{C}_2 \left( t^2 - \mathcal{C}_1 t^{2-\alpha} \right) \left( \lambda^2 \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} z(\tau) d\tau + \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} z(\tau) d\tau \right) \\ & - \mathcal{C}_2 \left( t^2 - \mathcal{C}_1 t^{2-\alpha} \right) \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} F_z(\tau) d\tau + \int_0^t \frac{(t-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} \chi(\tau) d\tau. \end{aligned}$$

Thanks to (H4), we have

$$\begin{aligned} & \left| z(t) - \int_0^t \frac{(t-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} F_z(\tau) d\tau + \mathcal{C}_2 \left( t^2 - \mathcal{C}_1 t^{2-\alpha} \right) \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} F_z(\tau) d\tau \right. \\ & \left. + \mathcal{C}_2 \left( t^2 - \mathcal{C}_1 t^{2-\alpha} \right) \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} z(\tau) d\tau - \mathcal{C}_2 \left( t^2 - \mathcal{C}_1 t^{2-\alpha} \right) \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} z(\tau) d\tau \right. \\ & \left. + \lambda^2 \int_0^t \frac{(t-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} z(\tau) d\tau + \int_0^t \frac{(t-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} \chi(\tau) d\tau \right| \leq \lambda_\chi \chi(t) \end{aligned} \quad (14)$$

By theorem 1, there  $y$  is unique solution of the problem (1).

It follows then that

$$|z(t) - y(t)| \leq \left| \begin{aligned} & z(t) - \int_0^t \frac{(t-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} F_y(\tau) d\tau - \lambda^2 \int_0^t \frac{(t-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} y(\tau) d\tau \\ & + \mathcal{C}_2 \left( t^2 - \mathcal{C}_1 t^{2-\alpha} \right) \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} y(\tau) d\tau + \mathcal{C}_2 \left( t^2 - \mathcal{C}_1 t^{2-\alpha} \right) \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} y(\tau) d\tau \\ & - \mathcal{C}_2 \left( t^2 - \mathcal{C}_1 t^{2-\alpha} \right) \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} F_y(\tau) d\tau \end{aligned} \right|.$$

Using equation (14), we obtain:

$$\begin{aligned} |z(t) - y(t)| &\leq (|\mathcal{C}_2 t^2| + |\mathcal{C}_2 \mathcal{C}_1 t^{2-\alpha}|) \left( \int_0^T \frac{(T-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} |F_z(\tau) - F_y(\tau)| d\tau \right) \\ &+ (|\mathcal{C}_2 t^2| + |\mathcal{C}_2 \mathcal{C}_1 t^{2-\alpha}|) \left( \int_0^T \frac{(T-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} |z(\tau) - y(\tau)| d\tau \right) \\ &+ (|\mathcal{C}_2 t^2| + |\mathcal{C}_2 \mathcal{C}_1 t^{2-\alpha}|) \left( \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} |z(\tau) - y(\tau)| d\tau \right) \\ &+ \int_0^t \frac{(t-\tau)^{2+\alpha-1}}{\Gamma(2+\alpha)} |F_z(\tau) - F_y(\tau)| d\tau \\ &+ \lambda^2 \int_0^t \frac{(t-\tau)^{2-\alpha-1}}{\Gamma(2-\alpha)} |z(\tau) - y(\tau)| d\tau + \lambda \chi(t). \end{aligned}$$

In consequence, we get

$$\|z - y\|_\infty \leq (L S_1 + S_2 + S_3) \|z - y\|_\mathbb{H} + \lambda_{\chi_1} \chi(t).$$

Similarly, we get

$$\|D^\alpha z - D^\alpha y\|_\infty \leq (L R_1 + R_2 + R_3) \|z - y\|_\mathbb{H} + \lambda_{\chi_2} \chi(t).$$

Therefore,

$$\|z - y\|_\mathbb{H} \leq M_2 \chi(t),$$

where,

$$M_2 := \frac{\lambda_{\chi_1} + \lambda_{\chi_2}}{1 - \Delta}.$$

Thus, problem (1) is generalized Ulam-Hyers-Rassias stable.

*Remark..*

- For  $\chi(t) = 1$ , Ulam-Hyers-Rassias stable  $\Rightarrow$  Ulam-Hyers stable.
- Ulam-Hyers stable  $\Rightarrow$  Generalized Ulam-Hyers stable.
- Ulam-Hyers-Rassias stable  $\Rightarrow$  Generalized Ulam-Hyers-Rassias stable.
- Under the assumption of Theorem 1, we consider the inequalitys (11), (12). We can repeat the same process to prove that problem (1) is Ulam-Hyers, Ulam-Hyers-Rassias stable (respectively).

## 5 Illustrative examples

### 5.1 Example

We consider the following problem:

$$\begin{cases} D^\alpha \left( D^2 + \lambda^2 D^\alpha \right) y(t) = -(D^\alpha y(t))^2 + 0.5 (D^\alpha y(t))^3 - 0.5 y(t), & t \in [0, T], \quad 0 < \alpha \leq 1, \quad T > 0 \\ y(0) = 0, \quad D^{1-\alpha} D^\alpha y(0) = 0, \quad y(T) = \beta J^\gamma y(\eta), \quad \gamma \geq 0, \end{cases} \quad (15)$$

Here,

$$f(t, y(t), D^\alpha y(t)) = -(D^\alpha y(t))^2 + 0.5 (D^\alpha y(t))^3 - 0.5 y(t),$$

$$T = 1, \alpha = 0.99, \gamma = 0.75, \beta = 10^{-2}, \lambda = 0.2, \eta = \frac{T}{2}.$$

Clearly, for all  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ , and  $t \in [0, 1]$ , by Taylor's formula, we have:

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq \left| -(x_2 - y_2) \left( (x_2 + y_2) - 0.5(x_2 + y_2)^2 \right) \right| + 0.5|x_1 - y_1|,$$

$$\leq 0.5 \left( |x_1 - y_1| + x_2 - y_2 \right).$$

consequently,

$$S_1 = 0.3668, S_2 = 0.0866, S_3 = 0.0076, R_1 = 0.8800,$$

$$R_2 = 0.1307, R_3 = 0.0146, \mathcal{C}_1 = 0.0797, \mathcal{C}_2 = 1.0866,$$

For  $L < \Delta = 0.8629$ , it follows by Theorem 1 that problem (15) has a unique solution. on  $[0, 1]$ .

## 5.2 Example

We consider the following problem:

$$\begin{cases} D^\alpha \left( D^2 + \lambda^2 D^\alpha \right) y(t) = \pm 0.2 \tan(\mp D^\alpha y(t)) - 0.3 y(t), & t \in [0, T], 0 < \alpha \leq 1, T > 0 \\ y(0) = 0, \quad D^{1-\alpha} D^\alpha y(0) = 0, \quad y(T) = \beta J^\gamma y(\eta), \quad \gamma \geq 0, \end{cases} \quad (16)$$

Here,

$$f(t, y(t), D^\alpha y(t)) = \pm 0.2 \tan(\mp D^\alpha y(t)) - 0.3 y(t),$$

$$T = 1, \alpha = 0.85, \gamma = 0.9, \beta = 10^{-3}, \lambda = 0.2, \eta = \frac{T}{4}.$$

Clearly, for all  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ , and  $t \in [0, 1]$ , by Taylor's formula, we have:

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq 0.3 \left( |x_1 - y_1| + x_2 - y_2 \right).$$

consequently,

$$S_1 = 0.4820, S_2 = 0.2016, S_3 = 4.1896e-4, S_1 = 0.9975,$$

$$R_2 = 0.3500, R_3 = 7.4070e-04, \mathcal{C}_1 = 0.1678, \mathcal{C}_2 = 1.2016.$$

For  $L < \Delta = 0.9966$ , it follows by Theorem 1 that problem (16) has a unique solution. on  $[0, 1]$ .

## Acknowledgement

Praveen Agarwal was very thankful to the NBHM (project 02011/12/ 2020NBHM(R.P)/R&D II/7867) for their necessary support and for providing the necessary facility.

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