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Riemann Liouville Fractional Integrals and Differential Formula Involving Multiindex Bessel Function

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Abstract: The paper is devoted to the study of some properties of the multiindex Bessel function $J_{(\beta_j)_m,k}^{(\alpha_j)_m,\gamma}(z)$ introduced by Choi et al. Certain relations that exist between multiindex Bessel function and the Riemann Liouville fractional integrals and derivatives are investigated. Fractional integration and differentiation operators transform are shown functions with power multipliers in to the same of the functions form.

Keywords: Fractional Calculus Operators, Multiindex Bessel function.

1 Introduction

The fractional integral is the most rapidly growing subject of mathematical analysis. The fractional integral operator involving various special functions has found significant importance and applications in various sub fields of applicable mathematical analysis. Since last three decades, a number of researchers like [1], [3], [10], [11], [12], [14], [21] and [24] and so forth have studied, in depth, the properties, applications, and different extensions of various operators of fractional calculus. A detailed account of fractional calculus operators along with their properties and applications can be found in the research monographs by [14], [16], and so forth.

Motivated by these avenues of applications, a remarkably large number of fractional integral formulas involving a variety of special functions have been developed by many authors (see, e.g., [19]-[20]). Fractional integration formulae for the Bessel function and generalized Bessel functions are given recently by [2], [12], [15] and [23]. A useful generalization of the Bessel function has been introduced and studied in [4],[5],[6] and [7].

Recently, the generalized multiindex Bessel function is defined by [8], as follow

$$J_{(\beta_j)_m,k}^{(\alpha_j)_m,\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{kn}}{\prod_{j=1}^{m}\Gamma\left(\alpha_j n+\beta_j+1\right)}\frac{(-z)^n}{n!},(m\in\mathbb{N}) \tag{1}$$

where $\alpha_j,\ \beta_j,\ \gamma\in\mathbb{C};\ \ (j=1,...,m),\ \ \Re(\gamma)>0,\ \Re(\beta)>-1,\ \ \sum_{j=1}^{m}\Re(\alpha)_j>\max\{0;\ \Re(k)-1\};\ \ k>0.$

If we put $k=0,\ m=1,\ \alpha_1=1,\ \beta_1=\upsilon$ and replace z by $z^2/4$ in (1), we obtain

$$J_{\upsilon,0}^{1,\gamma}\left[\frac{z^2}{4}\right]=\left(\frac{2}{z}\right)^{\upsilon}J_{\upsilon}\left[z\right], \tag{2}$$

where $J_{\upsilon}[z]$ is a well- known Bessel function of the first kind defined for complex $z\in\mathbb{C}$, $(z\neq 0)$ and $\upsilon\in\mathbb{C}$, $(\Re(\upsilon)>-1)$ by ([9], 7.2(2)) (see also [17])

$$J_{\upsilon}\left[z\right]=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\upsilon+k+1)}\frac{(z/2)^{\upsilon+2k}}{k!}. \tag{3}$$

A detail account of Bessel function, the reader may be referred to the earlier extensive works by [9] and [25].

The object of this paper is to derive the relations that exist between the generalized multiindex Bessel function defined by (1) and the left and right sided operators of

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Riemann-Liouville fractional calculus. The results derived in this paper are believed to be new.

The operators are defined by (see [22], Sect. 2) for $\alpha > 0$:

$$\left(I_{0+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt; \tag{4}$$

$$\left(I_{0-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{f(t)}{(t-x)^{1-\alpha}} dt; \tag{5}$$

$$\left(D_{0+}^{\alpha} f\right)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} \left(I_{0+}^{1-[\alpha]} f\right)(x);$$

$$= \frac{1}{\Gamma(1-\{\alpha\})} \left(\frac{d}{dx}\right)^{[\alpha]+1} \int_0^x \frac{f(t)}{(x-t)^{\{\alpha\}}} dt; \tag{6}$$

$$\left(D_{0-}^{\alpha} f\right)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} \left(I_{0-}^{1-[\alpha]} f\right)(x);$$

$$= \frac{1}{\Gamma(1-\{\alpha\})} \left(-\frac{d}{dx}\right)^{[\alpha]+1} \int_x^{\infty} \frac{f(t)}{(t-x)^{\{\alpha\}}} dt. \tag{7}$$

where $[\alpha]$ means the maximal integer not exceeding α and $\{\alpha\}$ is the fractional part of α.

2 Properties of Generalized Multiindex Bessel Function

In this section, we derive several interesting properties of the generalized multiindex Bessel function $J_{(\beta_j)_m,k}^{(\rho_j)_m,\gamma}(z)$ defined by (1) associated with Riemann-Liouville fractional integrals and derivatives.

Theorem 2.1. Let $\alpha > 0, \beta_j > 0, \rho_j > 0\,(j = 1,\dots,m)$ and $a \in \mathbb{R}$. Let I_{0+}^{α} be the left sided operator of Riemann-Liouville fractional integral (4). Then there holds the formula

$$\left(I_{0+}^{\alpha}\left[t^{\beta_1-1} J_{(\beta_j)_m,k}^{(\rho_j)_m,\gamma}(at^{\rho_1})\right]\right)(x) = x^{\alpha+\beta_1-1} J_{(\alpha+\beta_1),(\beta_j)_m,k}^{(\rho_j)_m,\gamma}(ax^{\rho_1}). \tag{8}$$

Proof. By virtue of (1) and (4), we have

$$L \equiv \left(I_{0+}^{\alpha}\left[t^{\beta_1-1} J_{(\beta_j)_m,k}^{(\rho_j)_m,\gamma}(at^{\rho_1})\right]\right)(x),$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}}{\prod_{j=1}^m \Gamma(\rho_j n+\beta_j+1)\, n!} \times (-a)^n t^{\rho_1 n+\beta_1-1} dt,$$

Now, interchanging the order of integration and summation is permissible under the conditions stated with the theorem dues to convergence of the integrals involved in the process and evaluating the inner integral by beta-function, it gives

$$L = x^{\alpha+\beta_1-1} \times \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}}{\prod_{j=2}^m \Gamma(\rho_j n+\beta_j+1)\Gamma(\alpha+\rho_1 n+\beta_1) n!} \left(-ax^{\rho_1}\right)^n,$$

$$= x^{\alpha+\beta_1-1} J_{(\alpha+\beta_1),(\beta_j)_m,k}^{(\rho_j)_m,\gamma}(ax^{\rho_1}).$$

This completes the proof of Theorem 2.1.

Lemma: For $a \in \mathbb{R}$ there holds the formula

$$ax^{\rho_1} J_{(\beta_j)_m,k}^{(\rho_j)_m,\gamma}(ax^{\rho_1}) = J_{(-\rho_1),(\beta_j)_m,k}^{(\rho_j)_m,\gamma}(ax^{\rho_1}) - J_{(-\rho_1),(\beta_j)_m,k}^{(\rho_j)_m,\gamma-1}(ax^{\rho_1}) \tag{9}$$

Proof. The formula (9) is easily verified by virtue of the relation

$$(nk+1)(\delta)_{nk} = (\delta)_{nk+1} - (\delta-1)_{nk+1}.$$

Theorem 2.1 and above Lemma imply.

Theorem 2.2. Let $\alpha > 0, \beta_j > 0, \rho_j > 0\ (j = 1,\dots,m)$ and $a \in \mathbb{R}$. Let I_{0+}^{α} be the left sided operator of Riemann-Liouville fractional integral (4). Then there holds the formula

$$\left(I_{0+}^{\alpha}\left[t^{\beta_1-1} J_{(\beta_j)_m,k}^{(\rho_j)_m,\gamma}(at^{\rho_1})\right]\right)(x) = \frac{1}{a} x^{\alpha+\beta_1-\rho_1-1} \times \left[J_{(\alpha+\beta_1-\rho_1),(\beta_j)_m,k}^{(\rho_j)_m,\gamma}(ax^{\rho_1}) - J_{(\alpha+\beta_1-\rho_1),(\beta_j)_m,k}^{(\rho_j)_m,\gamma-1}(ax^{\rho_1})\right]. \tag{10}$$

Theorem 2.3. Let $\alpha > 0, \beta_j > 0, \rho_j > 0\ (j = 1,\dots,m)$ and $a \in \mathbb{R}$. Let I_{0-}^{α} be the right sided operator of Riemann-Liouville fractional integral (5). Then there holds the formula

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\beta_1} J_{(\beta_j)_m,k}^{(\rho_j)_m,\gamma}(at^{\rho_1})\right]\right)(x) = x^{-\beta_1} J_{(\alpha+\beta_1),(\beta_j)_m,k}^{(\rho_j)_m,\gamma}(ax^{\rho_1}). \tag{11}$$

Proof. By virtue of (1) and (5) we have

$$L \equiv \left(I_{-}^{\alpha}\left[t^{-\alpha-\beta_1} J_{(\beta_j)_m,k}^{(\rho_j)_m,\gamma}(at^{\rho_1})\right]\right)(x)$$

$$= \frac{1}{\Gamma(\alpha)} \int_x^{\infty} t^{-\alpha-\beta_1}(t-x)^{\alpha-1} \times \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}}{\prod_{j=1}^m \Gamma(\rho_j n+\beta_j+1)\, n!} (-a)^n t^{\rho_1 n} dt,$$

Interchanging the order of integration and summation and evaluating the inner integral by beta-function, it gives

$$L = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}}{\prod_{j=2}^m \Gamma(\rho_j n+\beta_j+1)(\rho_1 n+\beta_1+\alpha)\, n!} \times (-a)^n x^{-\rho_1 n-\beta_1}$$

$$= x^{-\beta_1} J^{(\rho_j)_m,\gamma}_{(\alpha+\beta_1),(\beta_j)_m,k}(ax^{\rho_1}),$$

Theorem 2.4. Let $\alpha > 0, \beta_j > 0, \rho_j > 0$ $(j = 1,\ldots,m)$ and $a \in \mathbb{R}$. Let I^{α}_{0-} be the left sided operator of Riemann-Liouville fractional integral (5). Then there holds the formula

$$\left(I^{\alpha}_{-}\left[t^{-\alpha-\beta_1} J^{(\rho_j)_m,\gamma}_{(\beta_j)_m,k}(ax^{\rho_1})\right]\right)(x) = \frac{1}{a}x^{-\beta_1+\rho_1}$$

$$\times\left[J^{(\rho_j)_m,\gamma}_{(\alpha+\beta_1-\rho_1),(\beta_j)_m,k}(ax^{-\rho_1}) - J^{(\rho_j)_m,\gamma-1}_{(\alpha+\beta_1-\rho_1),(\beta_j)_m,k}(ax^{-\rho_1})\right]. \tag{12}$$

The proof can be developed on similar way to that of Theorem 2.3.

Now, we proceed to derive certain other properties of $J^{(\rho_j)_m,\gamma}_{(\beta_j)_m,k}(z)$ associated with the fractional derivative operators D^{α}_{0+} and D^{α}_{-} defined by (6) and (7) respectively.

Theorem 2.5. Let $\alpha > 0, \beta_j > 0, \rho_j > 0$ $(j = 1,\ldots,m)$ and $a \in \mathbb{R}$. Let D^{α}_{0+} be the left sided operator of Riemann-Liouville fractional integral (6). Then there holds the formula

$$\left(D^{\alpha}_{0+}\left[t^{\beta_1-1} J^{(\rho_j)_m,\gamma}_{(\beta_j)_m,k}(at^{\rho_1})\right]\right)(x)$$

$$= x^{\beta_1-\alpha-1} J^{(\rho_j)_m,\gamma}_{(\beta_1-\alpha),(\beta_j)_m,k}(ax^{\rho_1}). \tag{13}$$

Proof. By virtue of (1) and (6), we have

$$L \equiv \left(D^{\alpha}_{0+}\left[t^{\beta_1-1} J^{(\rho_j)_m,\gamma}_{(\beta_j)_m,k}(at^{\rho_1})\right]\right)(x)$$

$$= \left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I^{1-\{\alpha\}}_{0+}\left[t^{\beta_1-1} J^{(\rho_j)_m,\gamma}_{(\beta_j)_m,k}(at^{\rho_1})\right]\right)(x),$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-a)^n}{\prod_{j=1}^{m}\Gamma(\rho_j n+\beta_j+1)\Gamma(1-\{\alpha\})n!}$$

$$\times\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_0^x t^{\rho_1 n+\beta_1-1}(x-t)^{-\{\alpha\}}dt,$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-a)^n}{\prod_{j=2}^{m}\Gamma(\rho_j n+\beta_j+1)\Gamma(\rho_1 n+\beta_1+1-\{\alpha\})n!}$$

$$\times\left(\frac{d}{dx}\right)^{[\alpha]+1} x^{\rho_1 n+\beta_1-\{\alpha\}},$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-a)^n}{\prod_{j=2}^{m}\Gamma(\rho_j n+\beta_j+1)\Gamma(\rho_1 n+\beta_1-\alpha)n!}$$

$$\times x^{\rho_1 n+\beta_1-\alpha-1},$$

$$= x^{\beta_1-\alpha-1} J^{(\rho_j)_m,\gamma}_{(\beta_1-\alpha),(\beta_j)_m,k}(ax^{\rho_1}).$$

which proves the theorem.

Theorem 2.6. Let $\alpha > 0, \beta_j > 0, \rho_j > 0$ $(j = 1,\ldots,m)$ and $a \in \mathbb{R}$. Let D^{α}_{0+} be the left sided operator of Riemann-Liouville fractional integral (6). Then there holds the formula

$$\left(D^{\alpha}_{0+}\left[t^{\beta_j-1} J^{(\rho_j)_m,\gamma}_{(\beta_j)_m,k}(at^{\rho_j})\right]\right)(x) = \frac{1}{a}x^{\beta_1-\rho_1-\alpha-1}$$

$$\left[J^{(\rho_j)_m,\gamma}_{(\beta_1-\rho_1-\alpha),(\beta_j)_m,k}(ax^{\rho_1}) - J^{(\rho_j)_m,\gamma-1}_{(\beta_1-\rho_1-\alpha),(\beta_j)_m,k}(ax^{\rho_1})\right]. \tag{14}$$

Theorem 2.7. Let $\alpha > 0, \beta_j > 0,\ \beta_j - \alpha + \{\alpha\} > 1$ $(j = 1,\ldots,m)$ and $a \in \mathbb{R}$. Let D^{α}_{-} be the right sided operator of Riemann-Liouville fractional integral (7). Then there holds the formula

$$\left(D^{\alpha}_{-}\left[t^{\alpha-\beta_1} J^{(\rho_j)_m,\gamma}_{(\beta_j)_m,k}(at^{-\rho_1})\right]\right)(x)$$

$$= x^{-\beta_1} J^{(\rho_j)_m,\gamma}_{(\beta_1-\alpha),(\beta_j)_m,k}(ax^{-\rho_1}). \tag{15}$$

Proof. By virtue of (1) and (7) we have

$$L \equiv \left(D^{\alpha}_{-}\left[t^{\alpha-\beta_1} J^{(\rho_j)_m,\gamma}_{(\beta_j)_m,k}(at^{-\rho_1})\right]\right)(x),$$

$$= \left(-\frac{d}{dx}\right)^{[\alpha]+1}\left(I^{1-\{\alpha\}}_{-}\left[t^{\alpha-\beta_1} J^{(\rho_j)_m,\gamma}_{(\beta_j)_m,k}(at^{-\rho_1})\right]\right)(x),$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-a)^n}{\Gamma(\rho_j n+\beta_j+1)\Gamma(1-\{\alpha\})n!}$$

$$\times\left(-\frac{d}{dx}\right)^{[\alpha]+1}\int_x^{\infty} t^{-\rho_1 n-\beta_1+\alpha}(t-x)^{-\{\alpha\}}dt,$$

If we set $t = x/u$, then the above expression transforms in to the form

$$L = \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-a)^n}{\prod_{j=1}^{m}\Gamma(\rho_j n+\beta_j+1)\Gamma(1-\{\alpha\})n!}$$

$$\times\int_0^1 u^{\rho_1 n-\alpha+\beta_1+\{\alpha\}-2}(1-u)^{-\{\alpha\}}du$$

$$\times\left(\frac{d}{dx}\right)^{[\alpha]+1} x^{\alpha-\rho_1 n-\beta_1-\{\alpha\}+1},$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}(-a)^n}{\prod_{j=2}^{m}\Gamma(\rho_j n+\beta_j+1)\Gamma(\rho_1 n+\beta_1+\alpha)n!}x^{-\rho_1 n-\beta_1},$$

$$= x^{-\beta_1} J^{(\rho_j)_m,\gamma}_{(\beta_1-\alpha),(\beta_j)_m,k}(ax^{-\rho_1}).$$

which proves the theorem.

Theorem 2.8. Let $\alpha > 0,\ \rho_j > 0,\ \beta_j - [\alpha] > 1$ $(j = 1,\ldots,m)$ and $a \in \mathbb{R}$ $(a \neq 0)$. Let D^{α}_{-} be the right sided operator of Riemann-Liouville fractional integral (7). Then there holds the formula

$$\left(D^{\alpha}_{-}\left[t^{\alpha-\beta_1} J^{(\rho_j)_m,\gamma}_{(\beta_j)_m,k}(at^{-\rho_1})\right]\right)(x) = \frac{1}{a}x^{\rho_1-\beta_1}$$

$$\left[J^{(\rho_j)_m,\gamma}_{(\beta_1-\rho_1-\alpha),(\beta_j)_m,k}(ax^{-\rho_1}) - J^{(\rho_j)_m,\gamma-1}_{(\beta_1-\rho_1-\alpha),(\beta_j)_m,k}(ax^{-\rho_1})\right]. \tag{16}$$

3 Concluding remark

In this paper, we have obtained the certain relations that exist between multiindex Bessel function and the Riemann Liouville fractional integrals and derivatives are investigated. Fractional integration and differentiation operators transform are shown functions with power multipliers in to the same of the functions form. It is expected that the results are derived, may find applications in the solution of the fractional order integral and differential equations arising problems of physical sciences and engineering field.

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