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Activated Approximation by Fractional Smooth Singular Operators

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Abstract: In this article we study the fractional smooth activated singular integral operators on the real line, regarding their convergence to the unit operator with fractional rates in the uniform norm. The related established inequalities involve the higher order moduli of smoothness of the associated right and left Caputo fractional derivatives of the engaged function. Furthermore we produce fractional Voronocskaya type results giving the fractional asymptotic expansion of the basic error of our approximations. Our operators are not in general positive. We are mainly motivated and based on [6], Chapter 17.

Keywords: activated fractional singular integrals, modulus of smoothness, Caputo fractional derivative

1 Fractional Background

We mention

Definition 1. *Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number, $\lfloor \cdot \rfloor$ is the integral part), $f \in C^n(\mathbb{R})$. We call left Caputo fractional derivative the function*

$$D_{*x_0}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_{x_0}^{x} (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{1}$$

$\forall\, x \geq x_0 \in \mathbb{R}$ fixed, where Γ is the gamma function $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$, $\nu > 0$.
We set $D_{*x_0}^{0} f(x) = f(x)$, $\forall\, x \geq x_0$.
We assume $D_{*x_0}^{\nu} f(x) = 0$, for $x < x_0$.

We need

Lemma 1. *([2]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^n(\mathbb{R})$, $\left\| f^{(n)} \right\|_\infty < \infty$, $x_0 \in \mathbb{R}$ fixed. Then $D_{*x_0}^{\nu} f(x_0) = 0$.*

We need the following left Caputo fractional Taylor formula

Theorem 1. *([1], [7]) Let $f \in C^m(\mathbb{R})$, $m = \lceil \gamma \rceil$, $\gamma > 0$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \frac{1}{\Gamma(\gamma)} \int_{x_0}^{x} (x-\zeta)^{\gamma-1} D_{*x_0}^{\gamma} f(\zeta)\, d\zeta, \tag{2}$$

$\forall\, x \in \mathbb{R} : x \geq x_0$.

We also mention

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Definition 2.([9], [10]) *Let $f \in C^m(\mathbb{R})$, $\gamma > 0$, $m = \lceil \gamma \rceil$. The right Caputo fractional derivative of order $\gamma > 0$ is given by*

$$D_{x_0-}^{\gamma} f(x) = \frac{(-1)^m}{\Gamma(m-\gamma)} \int_x^{x_0} (\zeta - x)^{m-\gamma-1} f^{(m)}(\zeta)\, d\zeta, \tag{3}$$

$\forall\, x \leq x_0 \in \mathbb{R}$ fixed.

We assume $D_{x_0-}^{\gamma} f(x) = 0$, $\forall\, x > x_0$.

We need

Lemma 2.([2]) *Let $\gamma > 0$, $\gamma \notin \mathbb{N}$, $m = \lceil \gamma \rceil$, $f \in C^m(\mathbb{R})$, $\left\| f^{(m)} \right\|_{\infty} < \infty$, $x_0 \in \mathbb{R}$ fixed. Then $D_{x_0-}^{\gamma} f(x_0) = 0$.*

We need the following right Caputo fractional Taylor formula

Theorem 2.([1]) *Let $f \in C^m(\mathbb{R})$, $m = \lceil \gamma \rceil$, $\gamma > 0$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{\Gamma(\gamma)} \int_x^{x_0} (\zeta - x)^{\gamma-1} D_{x_0-}^{\gamma} f(\zeta)\, d\zeta, \tag{4}$$

$\forall\, x \leq x_0$.

We further need

Theorem 3.([2]) *Let $g \in C_b(\mathbb{R})$ (continuous and bounded), $0 < c < 1$, $x, x_0 \in \mathbb{R}$. Define*

$$L(x, x_0) = \int_{x_0}^{x} (x - t)^{c-1} g(t)\, dt, \ \ \text{for } x \geq x_0,$$

and $L(x, x_0) = 0$, for $x < x_0$.

Then L is jointly continuous in $(x, x_0) \in \mathbb{R}^2$.

Theorem 4.([2]) *Let $g \in C_b(\mathbb{R})$, $0 < c < 1$, $x, x_0 \in \mathbb{R}$. Define*

$$K(x, x_0) = \int_x^{x_0} (\zeta - x)^{c-1} g(\zeta)\, d\zeta, \ \ \text{for } x \leq x_0,$$

and $K(x, x_0) = 0$, for $x > x_0$.

Then $K(x, x_0)$ is jointly continuous from $\mathbb{R}^2$ into $\mathbb{R}$.

Based on Theorems 3, 4 we get

Proposition 1.([2]) *Let $f \in C^m(\mathbb{R})$, with $\left\| f^{(m)} \right\|_{\infty} < \infty$, $m = \lceil \gamma \rceil$, $\gamma \notin \mathbb{N}$, $\gamma > 0$, $x, x_0 \in \mathbb{R}$. Then $D_{*x_0}^{\gamma} f(x)$, $D_{x_0-}^{\gamma} f(x)$ are jointly continuous functions in (x, x_0) from $\mathbb{R}^2$ into $\mathbb{R}$.*

We need

Definition 3. *Let $f \in C^m(\mathbb{R})$, $\left\| f^{(m)} \right\|_{\infty} < \infty$, $m = \lceil \gamma \rceil$, $\gamma \notin \mathbb{N}$, $\gamma > 0$, $r \in \mathbb{N}$, $x, x_0 \in \mathbb{R}$. We define the difference*

$$\left(\Delta_w^r \left(\Delta_{*x_0}^{\gamma} f\right)\right)(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \left(D_{*x_0}^{\gamma} f\right)(x + jw), \tag{5}$$

$\forall\, w \in \mathbb{R}$,

and the rth modulus of smoothness,

$$\omega_r\left(D_{*x_0}^{\gamma} f, h\right) := \sup_{|t| \leq h} \left\| \left(\Delta_t^r \left(\Delta_{*x_0}^{\gamma} f\right)\right)(x) \right\|_{\infty, x, \mathbb{R}}. \tag{6}$$

Notice that

$$\left| \left(\Delta_w^r \left(\Delta_{*x_0}^{\gamma} f\right)\right)(x_0) \right| \leq \left\| \left(\Delta_w^r \left(\Delta_{*x_0}^{\gamma} f\right)\right)(x) \right\|_{\infty, x, \mathbb{R}} \leq \omega_r\left(D_{*x_0}^{\gamma} f, |w|\right). \tag{7}$$

Similarly, we define the difference

$$\left(\Delta_w^r\left(\Delta_{x_0-}^{\gamma}f\right)\right)(x):=\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\left(D_{x_0-}^{\gamma}f\right)(x+jw), \tag{8}$$

$\forall\, w\in\mathbb{R}$, *and the rth modulus of smoothness,*

$$\omega_r\left(D_{x_0-}^{\gamma}f,h\right):=\sup_{|t|\leq h}\left\|\left(\Delta_t^r\left(\Delta_{x_0-}^{\gamma}f\right)\right)(x)\right\|_{\infty,x,\mathbb{R}}. \tag{9}$$

Notice again that

$$\left|\left(\Delta_w^r\left(\Delta_{x_0-}^{\gamma}f\right)\right)(x_0)\right|\leq\left\|\left(\Delta_w^r\left(\Delta_{x_0-}^{\gamma}f\right)\right)(x)\right\|_{\infty,x,\mathbb{R}}\leq\omega_r\left(D_{x_0-}^{\gamma}f,|w|\right). \tag{10}$$

As a related result we mention

Proposition 2.*([2]) Let $f:\mathbb{R}^2\to\mathbb{R}$ be jointly continuous. Consider*

$$G(x)=\omega_r\left(f(\cdot,x),\delta\right)_{[x,+\infty)},\quad \delta>0,\ x\in\mathbb{R}.$$

(Here ω_r is defined over $[x,+\infty)$ instead of $\mathbb{R}$.)
Then G is continuous on $\mathbb{R}$.

Proposition 3.*([2])Let $f:\mathbb{R}^2\to\mathbb{R}$ be jointly continuous. Consider*

$$H(x)=\omega_r\left(f(\cdot,x),\delta\right)_{(-\infty,x]},\quad \delta>0,\ x\in\mathbb{R}.$$

(Here ω_r is defined over $(-\infty,x]$ instead of $\mathbb{R}$.)
Then H is continuous on $\mathbb{R}$.

From Propositions 1, 2, 3 we derive

Proposition 4.*([2]) Let $f\in C^m(\mathbb{R})$, $\left\|f^{(m)}\right\|_{\infty}<\infty$, $m=\lceil\gamma\rceil$, $\gamma\notin\mathbb{N}$, $\gamma>0$, $r\in\mathbb{N}$, $x\in\mathbb{R}$. Then $\omega_r\left(D_{*x}^{\gamma},h\right)_{[x,+\infty)}$, $\omega_r\left(D_{x-}^{\gamma},h\right)_{(-\infty,x]}$ are continuous functions of $x\in\mathbb{R}$, $h>0$ fixed.*

We make

Remark.(2) Let g continuous and bounded from $\mathbb{R}$ to $\mathbb{R}$. Then we know that

$$\omega_r(g,t)\leq 2^r\|g\|_{\infty}<\infty.$$

Assuming that $\left(D_{*x}^{\gamma}f\right)(t)$, $\left(D_{x-}^{\gamma}f\right)(t)$, are both continuous and bounded in $(x,t)\in\mathbb{R}^2$, i.e.

$\left\|D_{*x}^{\gamma}f\right\|_{\infty}\leq K_1,\ \ \forall\, x\in\mathbb{R};$

$\left\|D_{x-}^{\gamma}f\right\|_{\infty}\leq K_2,\ \ \forall\, x\in\mathbb{R},$

where $K_1,K_2>0$, we get

$\omega_r\left(D_{*x}^{\gamma}f,\xi\right)\leq 2^rK_1;$

$\omega_r\left(D_{x-}^{\gamma}f,\xi\right)\leq 2^rK_2,\ \ \forall\,\xi\geq 0,$

for each $x\in\mathbb{R}$.

Therefore, for any $\xi\geq 0$,

$$\sup_{x\in\mathbb{R}}\left[\max\left(\omega_r\left(D_{*x}^{\gamma}f,\xi\right),\omega_r\left(D_{x-}^{\gamma}f,\xi\right)\right)\right]\leq 2^r\max(K_1,K_2)<\infty. \tag{11}$$

So in our setting for $f\in C^m(\mathbb{R})$, $\left\|f^{(m)}\right\|_{\infty}<\infty$, $m=\lceil\gamma\rceil$, $\gamma\notin\mathbb{N}$, $\gamma>0$, by Proposition 1, both $\left(D_{*x}^{\gamma}f\right)(t)$, $\left(D_{x-}^{\gamma}f\right)(t)$ are jointly continuous in (t,x) on $\mathbb{R}^2$. Assuming further that they are both bounded on $\mathbb{R}^2$ we get (11) valid. In particular, each of $\omega_r\left(D_{*x}^{\gamma}f,\xi\right),\omega_r\left(D_{x-}^{\gamma}f,\xi\right)$ is finite for any $\xi\geq 0$.

We need

Remark.([2]) Again let $f \in C^m(\mathbb{R})$, $m = \lceil \gamma \rceil$, $\gamma \notin \mathbb{N}$, $\gamma > 0, f^{(m)}(x) = 1, \forall x \in \mathbb{R}$; $x_0 \in \mathbb{R}$. Notice $0 < m - \gamma < 1$. Then

$$D_{*x_0}^{\gamma} f(x) = \frac{(x - x_0)^{m-\gamma}}{\Gamma(m - \gamma + 1)}, \ \forall x \geq x_0.$$

Let us consider $x, y \geq x_0$, then

$$\left| D_{*x_0}^{\gamma} f(x) - D_{*x_0}^{\gamma} f(y) \right| \leq \frac{|x - y|^{m-\gamma}}{\Gamma(m - \gamma + 1)}.$$

So it is not strange to assume that

$$\left| D_{*x_0}^{\gamma} f(x_1) - D_{*x_0}^{\gamma} f(x_2) \right| \leq K |x_1 - x_2|^{\mu}, \tag{12}$$

$K > 0$, $0 < \mu \leq 1$, $\forall x_1, x_2 \in \mathbb{R}$, any $x_0 \in \mathbb{R}$, here more generally $\left\| f^{(m)} \right\|_{\infty} < \infty$.

In general, one may assume

$$\begin{gathered} \omega_r \left(D_{x-}^{\gamma} f, \xi \right) \leq M_1 \xi^{r-1+\mu_1}, \\ \text{and} \\ \omega_r \left(D_{*x}^{\gamma} f, \xi \right) \leq M_2 \xi^{r-1+\mu_2}, \end{gathered} \tag{13}$$

where $0 < \mu_1, \mu_2 \leq 1$, $\forall \xi > 0$, $r \in \mathbb{N}$; $M_1, M_2 > 0$; any $x \in \mathbb{R}$.

Setting $\mu = \min(\mu_1, \mu_2)$ and $M = \max(M_1, M_2)$, in that case we obtain

$$\sup_{x \in \mathbb{R}} \left\{ \max \left(\omega_r \left(D_{x-}^{\gamma} f, \xi \right), \omega_r \left(D_{*x}^{\gamma} f, \xi \right) \right) \right\} \leq M \xi^{r-1+\mu} \to 0, \text{ as } \xi \to 0+. \tag{14}$$

2 General Fractional Singular Integrals Background

We need

Definition 4.([2]) *Let $r \in \mathbb{N}$, $\gamma > 0$. We mention the numbers*

$$\alpha_j := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-\gamma}, & j = 1, \dots, r. \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-\gamma}, & j = 0, \end{cases} \tag{15}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$.

Also denote

$$\delta_k = \sum_{j=1}^{r} \alpha_j j^k, \ k = 1, \dots, m-1, \tag{16}$$

where $m = \lceil \gamma \rceil$.

We mention

Theorem 5.([2]) *Let $f \in C^m(\mathbb{R})$, $m = \lceil \gamma \rceil$, $\gamma > 0$, $\left\| f^{(m)} \right\|_{\infty} < \infty$, $x_0 \in \mathbb{R}$ fixed, $\xi > 0$. Then*

i) if $t \geq 0$ we get

$$A := A(t, x_0) := \sum_{j=0}^{r} \alpha_j [f(x_0 + jt) - f(x_0)] - \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k =$$

$$\frac{1}{\Gamma(\gamma)} \int_0^t (t - w)^{\gamma-1} \left(\Delta_w^r \left(D_{*x_0}^{\gamma} f \right) \right)(x_0) \, dw, \tag{17}$$

and

$$|A| \leq \omega_r \left(D_{*x_0}^{\gamma} f, \xi \right) \left(\sum_{k=0}^{r} \frac{r!}{(r-k)!} \frac{t^{\gamma+k}}{\xi^k \Gamma(\gamma + k + 1)} \right), \tag{18}$$

ii) if $t<0$ we obtain

$$B:=B(t,x_0):=\sum_{j=0}^{r}\alpha_j\left[f(x_0+jt)-f(x_0)\right]-\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)}{k!}\delta_k t^k= \frac{1}{\Gamma(\gamma)}\int_t^0 (w-t)^{\gamma-1}\left(\Delta_w^r\left(D_{x_0-}^{\gamma}f\right)\right)(x_0)\,dw, \tag{19}$$

and

$$|B|\le\omega_r\left(D_{x_0-}^{\gamma}f,\xi\right)\left(\sum_{k=0}^{r}\frac{r!}{(r-k)!}\frac{|t|^{\gamma+k}}{\xi^k\Gamma(\gamma+k+1)}\right). \tag{20}$$

In the next, let $\xi>0$, $x,x_0\in\mathbb{R}$, $f\in C^m(\mathbb{R})$, $m=\lceil\gamma\rceil$, $\gamma>0$, with $\left\|f^{(m)}\right\|_\infty<\infty$. Here μ_ξ is a probability Borel measure on $\mathbb{R}$, $\forall\,\xi>0$.

Consider the integral (see also [6])

$$\Theta_{r,\xi}(f,x):=\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+jt)\right)d\mu_\xi(t). \tag{21}$$

where $\Theta_{r,\xi}(f,x)\in\mathbb{R}$, $\forall\,x\in\mathbb{R}$.

See that

$$\Theta_{r,\xi}(c,x)=c,\ c\text{ constant}, \tag{22}$$

and

$$\Theta_{r,\xi}(f,x_0)-f(x_0)=\int_{-\infty}^{\infty}\sum_{j=0}^{r}\alpha_j\left(f(x_0+jt)-f(x_0)\right)d\mu_\xi(t)$$
$$=\int_{-\infty}^{0}\sum_{j=1}^{r}\alpha_j\left(f(x_0+jt)-f(x_0)\right)d\mu_\xi(t)+ \int_{0}^{\infty}\sum_{j=1}^{r}\alpha_j\left(f(x_0+jt)-f(x_0)\right)d\mu_\xi(t)=:\Lambda. \tag{23}$$

We need

Theorem 6.([6], Ch. 17) *Let $f\in C^m(\mathbb{R})$, $m=\lceil\gamma\rceil$, $\gamma>0$, with $\left\|f^{(m)}\right\|_\infty<\infty$, $\xi>0$, $x_0\in\mathbb{R}$. Assume existence of*

$$c_{k,\xi}:=\int_{-\infty}^{\infty}t^k d\mu_\xi(t),\quad k=1,\dots,m-1. \tag{24}$$

Suppose also existence of $\int_{-\infty}^{\infty}|t|^{\gamma+k}d\mu_\xi(t)$, $k=0,1,\dots,r$. Then

1)

$$\left|\Theta_{r,\xi}(f,x_0)-f(x_0)-\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)\,\delta_k c_{k,\xi}}{k!}\right|\le \left[\sum_{k=0}^{r}\frac{r!}{(r-k)!\Gamma(\gamma+k+1)\xi^k}\int_{-\infty}^{\infty}|t|^{\gamma+k}d\mu_\xi(t)\right] \max\left\{\omega_r\left(D_{x_0-}^{\gamma}f,\xi\right),\omega_r\left(D_{*x_0}^{\gamma}f,\xi\right)\right\}. \tag{25}$$

2)

$$\left\|\Theta_{r,\xi}(f,\cdot)-f(\cdot)-\sum_{k=1}^{m-1}\frac{f^{(k)}(\cdot)\,\delta_k c_{k,\xi}}{k!}\right\|_\infty\le \left[\sum_{k=0}^{r}\frac{r!}{(r-k)!\Gamma(\gamma+k+1)\xi^k}\int_{-\infty}^{\infty}|t|^{\gamma+k}d\mu_\xi(t)\right] \sup_{x\in\mathbb{R}}\left\{\max\left[\omega_r\left(D_{x_0-}^{\gamma}f,\xi\right),\omega_r\left(D_{*x_0}^{\gamma}f,\xi\right)\right]\right\}. \tag{26}$$

(Above if $m=1$ the sum disappears).

Next we mention also a Voronovskaya type result regarding fractional general singular integral operators.

Theorem 7.
([6], Ch. 17) Here $f \in C^m(\mathbb{R})$, $m \in \mathbb{N}$, $m = \lceil \gamma \rceil$, $\gamma > 0$, $\left\| f^{(m)} \right\|_\infty < \infty$, and $\left\| D^\gamma_{x-} f(y) \right\|_\infty \le M_1$, $\left\| D^\gamma_{*x} f(y) \right\|_\infty \le M_2$, where $M_1, M_2 > 0$, for any $x, y \in \mathbb{R}$. Suppose $\xi^{-\gamma} \int_{-\infty}^{\infty} |t|^\gamma d\mu_\xi(t) \le \rho$, $\rho > 0$, $\forall\ \xi > 0$. Assume the existence of $c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t)$, $k = 1, \ldots, m-1$. Then

$$\Theta_{r,\xi}(f,x) - f(x) - \sum_{k=1}^{m-1} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi} = o\left(\xi^{\gamma-\eta}\right), \tag{27}$$

$0 < \eta < \gamma$, as $\xi \to 0+$.

I.e.

$$\Theta_{r,\xi}(f,x) - f(x) = \sum_{k=1}^{m-1} \frac{f^{(k)}(x)}{k!} \left(\sum_{j=1}^{r} \alpha_j j^k \right) c_{k,\xi} + o\left(\xi^{\gamma-\eta}\right), \tag{28}$$

where $0 < \eta < \gamma$.

(Above if $m = 1$ the sum disappears).

3 Background on Activation functions

Here all come from [5].

3.1 About Richards's curve

Here we follow [4], Chapter 1.

A Richards's curve is

$$\varphi(x) = \frac{1}{1 + e^{-\mu x}}; \quad x \in \mathbb{R},\ \mu > 0, \tag{29}$$

which is strictly increasing on $\mathbb{R}$, and it is a sigmoid function, in particular this is a generalized logistic function. And it is an activation function in neural networks, see [4], chapter 1.

It is

$$\lim_{x \to +\infty} \varphi(x) = 1 \quad \text{and} \quad \lim_{x \to -\infty} \varphi(x) = 0. \tag{30}$$

We consider the function

$$G(x) = \frac{1}{2}\left(\varphi(x+1) - \varphi(x-1)\right),\ x \in \mathbb{R}, \tag{31}$$

which is $G(x) > 0$, all $x \in \mathbb{R}$.

It is

$$\varphi(0) = \frac{1}{2}, \quad \varphi(x) = 1 - \varphi(-x), \tag{32}$$

and

$$G(x) = G(-x), \forall\ x \in \mathbb{R}. \tag{33}$$

We also have

$$G(0) = \frac{e^\mu - 1}{2(e^\mu + 1)}. \tag{34}$$

We also get

$$\lim_{x \to +\infty} G(x) = \lim_{x \to -\infty} G(x) = 0, \tag{35}$$

and G is a bell symmetric function with maximum

$$G(0) = \frac{e^\mu - 1}{2(e^\mu + 1)}. \tag{36}$$

Theorem 8. *It holds*

$$\sum_{i=-\infty}^{\infty} G(x-i) = 1, \ \forall x \in \mathbb{R}. \tag{37}$$

Theorem 9. *It holds*

$$\int_{-\infty}^{\infty} G(x)\,dx = 1. \tag{38}$$

So G is a density function.

We make

Remark. So we have

$$G(x) = \frac{1}{2}(\varphi(x+1) - \varphi(x-1)), \ \forall x \in \mathbb{R}. \tag{39}$$

i) Let $x \geq 1$. That is $0 \leq x-1 < x+1$. Applying the mean value theorem we get:

$$G(x) = \frac{1}{2}2\varphi'(\eta) = \varphi'(\eta) = \frac{\mu e^{-\mu\eta}}{(1+e^{-\mu\eta})^2}, \ \mu > 0, \tag{40}$$

where $0 \leq x-1 < \eta < x+1$.

Notice that

$$G(x) < \mu e^{-\mu\eta} < \mu e^{-\mu(x-1)}, \ \forall x \geq 1. \tag{41}$$

ii) Let now $x \leq -1$. That is $x-1 < x+1 \leq 0$. Applying again the mean value theorem we get:

$$G(x) = \frac{1}{2}2\varphi'(\eta) = \varphi'(\eta) = \frac{\mu e^{-\mu\eta}}{(1+e^{-\mu\eta})^2}, \tag{42}$$

where $x-1 < \eta < x+1 \leq 0$.

Hence, we derive that

$$G(x) < \mu e^{-\mu\eta} < \mu e^{-\mu(x-1)}, \ \forall x \leq -1. \tag{43}$$

Consequently, we proved that

$$G(x) < \mu e^{-\mu(x-1)}, \ \forall x \in (-\infty, -1] \cup [1, +\infty) = \mathbb{R} - (-1, 1). \tag{44}$$

Let $0 < \xi \leq 1$, it holds

$$G\left(\frac{x}{\xi}\right) < \mu e^{-\mu\left(\frac{x}{\xi}-1\right)}, \ \forall x \geq \xi, \text{or} \ \forall x \leq -\xi. \tag{45}$$

Clearly, by Theorem 9 we have that

$$\frac{1}{\xi}\int_{-\infty}^{\infty} G\left(\frac{x}{\xi}\right) dx = 1, \ \xi > 0. \tag{46}$$

So that $\frac{1}{\xi}G\left(\frac{x}{\xi}\right)$ is a density function, and let $d\mu_\xi(x) := \frac{1}{\xi}G\left(\frac{x}{\xi}\right) dx$, $\xi > 0$, that is μ_ξ is a Borel probability measure.

We give the following important result.

Theorem 10. *Let $\xi > 0$, and*

$$c^*_{k,\xi} := \frac{1}{\xi}\int_{-\infty}^{\infty} x^k G\left(\frac{x}{\xi}\right) dx, \ k = 1, \ldots, n \in \mathbb{N}. \tag{47}$$

Then $c^*_{k,\xi}$ are finite and $c^*_{k,\xi} \to 0$, as $\xi \to 0$.

Infact it holds

$$\frac{1}{\xi}\int_{-\infty}^{\infty} |x|^k G\left(\frac{x}{\xi}\right) dx \leq \left[1 + 2\mu^{-k}e^{\mu}k!\right]\xi^k < \infty, \tag{48}$$

for $k = 1, \ldots, n$.

3.2 *About the q-Deformed and λ-Parametrized Hyperbolic tangent function* $g_{q,\lambda}$

We consider the activation function $g_{q,\lambda}$ and study its related properties, all the basics come from [4], ch. 17.

Let the activation function

$$g_{q,\lambda}(x) = \frac{e^{\lambda x} - qe^{-\lambda x}}{e^{\lambda x} + qe^{-\lambda x}}, \quad \lambda, q > 0, x \in \mathbb{R}. \tag{49}$$

It is

$$g_{q,\lambda}(0) = \frac{1-q}{1+q},$$

and

$$g_{q,\lambda}(-x) = -g_{\frac{1}{q},\lambda}(x), \quad \forall\, x \in \mathbb{R}, \tag{50}$$

with

$$g_{q,\lambda}(+\infty) = 1, \quad g_{q,\lambda}(-\infty) = -1.$$

We consider the function

$$M_{q,\lambda}(x) := \frac{1}{4}\left(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right) > 0, \tag{51}$$

$\forall\, x \in \mathbb{R}$, $q, \lambda > 0$. We have $M_{q,\lambda}(\pm\infty) = 0$, so that the x-axis is a horizontal asymptote.

It holds

$$M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \quad \forall\, x \in \mathbb{R}, q, \lambda > 0, \tag{52}$$

and

$$M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x), \quad \forall\, x \in \mathbb{R}.$$

The $M_{q,\lambda}$ maximum is

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = \frac{\tanh(\lambda)}{2}, \quad \lambda > 0. \tag{53}$$

Theorem 11. *We have that*

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i) = 1, \quad \forall\, x \in \mathbb{R}, \forall\, \lambda, q > 0. \tag{54}$$

Theorem 12. *It holds*

$$\int_{-\infty}^{\infty} M_{q,\lambda}(x)\, dx = 1, \quad \lambda, q > 0. \tag{55}$$

So that $M_{q,\lambda}$ is a density function on $\mathbb{R}$; $\lambda, q > 0$.

Remark. i) Let $x \geq 1$. That is $0 \leq x-1 < x+1$. By mean value theorem we obtain

$$M_{q,\lambda}(x) = \frac{1}{4}\left[g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right] = \frac{1}{4} \cdot 2 \cdot \frac{4q\lambda e^{2\lambda\xi}}{\left(e^{2\lambda\xi} + q\right)^2} = \frac{2q\lambda e^{2\lambda\xi}}{\left(e^{2\lambda\xi} + q\right)^2}, \tag{56}$$

for some $0 \leq x-1 < \xi < x+1$; $\lambda, q > 0$.

But $e^{2\lambda\xi} < e^{2\lambda\xi} + q$, and

$$M_{q,\lambda}(x) < \frac{2q\lambda\left(e^{2\lambda\xi} + q\right)}{\left(e^{2\lambda\xi} + q\right)^2} = \frac{2q\lambda}{\left(e^{2\lambda\xi} + q\right)} < \frac{2q\lambda}{\left(e^{2\lambda(x-1)} + q\right)} < \frac{2q\lambda}{e^{2\lambda(x-1)}}, \tag{57}$$

$x \geq 1$.

That is

$$M_{q,\lambda}(x) < 2q\lambda e^{-2\lambda(x-1)}, \quad \forall\, x \geq 1. \tag{58}$$

Set $\mu := 2\lambda$, then

$$M_{q,\lambda}(x) < q\mu e^{-\mu(x-1)}, \quad \forall\, x \geq 1. \tag{59}$$

ii) Let now $x \leq -1$. That is $x-1 < x+1 \leq 0$. Again we have

$$M_{q,\lambda}(x) < \frac{2q\lambda}{(e^{2\lambda\xi}+q)}, \tag{60}$$

$x-1 < \xi < x+1 \leq 0$; $\lambda, q > 0$.

We have

$$e^{2\lambda(x-1)} < e^{2\lambda\xi} < e^{2\lambda(x+1)},$$

and

$$e^{2\lambda(x-1)} + q < e^{2\lambda\xi} + q < e^{2\lambda(x+1)} + q. \tag{61}$$

Hence

$$\frac{1}{e^{2\lambda\xi}+q} < \frac{1}{e^{2\lambda(x-1)}+q}. \tag{62}$$

Therefore it holds

$$M_{q,\lambda}(x) < \frac{2q\lambda}{e^{2\lambda(x-1)}+q} < \frac{2q\lambda}{e^{2\lambda(x-1)}}, \quad x \leq -1. \tag{63}$$

That is

$$M_{q,\lambda}(x) < 2q\lambda e^{-2\lambda(x-1)}, \quad \forall\, x \leq -1. \tag{64}$$

Set $\mu := 2\lambda$, then

$$M_{q,\lambda}(x) < q\mu e^{-\mu(x-1)}, \quad \forall\, x \leq -1. \tag{65}$$

We have proved that

$$M_{q,\lambda}(x) < q\mu e^{-\mu(x-1)}, \tag{66}$$

$\forall\, x \in (-\infty, -1] \cup [1, +\infty) = \mathbb{R} - (-1, 1)$.

Let $0 < \xi \leq 1$, it holds

$$M_{q,\lambda}\left(\frac{x}{\xi}\right) < q\mu e^{-\mu\left(\frac{x}{\xi}-1\right)}, \ \forall\, x \geq \xi, \text{ or } \forall\, x \leq -\xi. \tag{67}$$

By Theorem 12 we have

$$\frac{1}{\xi}\int_{-\infty}^{\infty} M_{q,\lambda}\left(\frac{x}{\xi}\right) dx = 1, \ \ \xi > 0. \tag{68}$$

So that $\frac{1}{\xi} M_{q,\lambda}\left(\frac{x}{\xi}\right)$ is a density function and let

$$d\mu_{\xi}(x) := \frac{1}{\xi} M_{q,\lambda}\left(\frac{x}{\xi}\right) dx, \ \ \xi > 0, \tag{69}$$

that is μ_{ξ} is a Borel probability measure.

We give

Theorem 13. *Let $\xi > 0$, and*

$$\overline{c}_{k,\xi} := \frac{1}{\xi}\int_{-\infty}^{\infty} x^k M_{q,\lambda}\left(\frac{x}{\xi}\right) dx, \ \ k = 1, \ldots, n \in \mathbb{N}. \tag{70}$$

Then $\overline{c}_{k,\xi}$ are finite and $\overline{c}_{k,\xi} \to 0$, as $\xi \to 0$.

In fact it holds

$$\frac{1}{\xi}\int_{-\infty}^{\infty} |x|^k M_{q,\lambda}\left(\frac{x}{\xi}\right) dx \leq \left[1 + \left(q + \frac{1}{q}\right)\mu^{-k} e^{\mu} k!\right] \xi^k < \infty, \tag{71}$$

$k = 1, \ldots, n$.

3.3 About the Gudermannian generated activation function

Here we follow [3], Ch. 2.

Let the related normalized generator sigmoid function:

$$f(x) := \frac{8}{\pi}\int_0^x \frac{1}{e^t + e^{-t}}dt, \quad x \in \mathbb{R}, \tag{72}$$

and the neural network activation function:

$$\psi(x) := \frac{1}{4}(f(x+1) - f(x-1)) > 0, \quad x \in \mathbb{R}. \tag{73}$$

We mention

Theorem 14. *It holds*

$$\int_{-\infty}^{\infty} \psi(x)\,dx = 1. \tag{74}$$

So that $\psi(x)$ is a density function.

By [3], p. 49, we found that

$$\psi(x) < \frac{2}{\pi \cosh(x-1)}, \quad \forall\, x \geq 1. \tag{75}$$

But

$$\frac{1}{\cosh(x-1)} = \frac{2}{e^{x-1} + e^{-(x-1)}} < \frac{2}{e^{x-1}} = 2e^{-(x-1)}, \tag{76}$$

$\forall\, x \in \mathbb{R}$.

Therefore it is

$$\psi(x) < \frac{4}{\pi}e^{-(x-1)} = \frac{4}{\pi}ee^{-x}, \quad \forall\, x \geq 1. \tag{77}$$

So here it is

$$d\mu_\xi(x) = \frac{1}{\xi}\psi\left(\frac{x}{\xi}\right)dx, \quad \xi > 0,$$

the related Borel probability measure.

We give the following result, its proof as similar to Theorem 10 is omitted.

Theorem 15. *Let $\xi > 0$, and*

$$\gamma_{k,\xi} := \frac{1}{\xi}\int_{-\infty}^{\infty} x^k \psi\left(\frac{x}{\xi}\right)dx, \quad k = 1, ..., n \in \mathbb{N}. \tag{78}$$

Then $\gamma_{k,\xi}$ and $\frac{1}{\xi}\int_{-\infty}^{\infty}|x|^k \psi\left(\frac{x}{\xi}\right)dx$, are finite and $\gamma_{k,\xi} \to 0$, as $\xi \to 0$.

3.4 About the q-deformed and λ-parametrized logistic type activation function

Here all come from [4], Ch. 15.

The activation function now is

$$\varphi_{q,\lambda}(x) := \frac{1}{1 + qe^{-\lambda x}}, \quad x \in \mathbb{R}, \tag{79}$$

where $q, \lambda > 0$.

The density function here will be

$$G_{q,\lambda}(x) := \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right) > 0, \quad x \in \mathbb{R}. \tag{80}$$

We mention

Theorem 16. *It holds*

$$\int_{-\infty}^{\infty} G_{q,\lambda}(x)\,dx = 1. \tag{81}$$

By [4], p. 373, we have

$$G_{q,\lambda}(x) < q\lambda e^{-\lambda(x-1)}, \quad \forall\, x \geq 1.$$

So here it is

$$d\mu_{\xi}(x) = \frac{1}{\xi} G_{q,\lambda}\left(\frac{x}{\xi}\right) dx, \quad \xi > 0,$$

the related Borel probability measure.

We give the following result, its proof as similar to Theorem 13 is omitted.

Theorem 17. *Let* $\xi > 0$ *and*

$$\overline{\delta}_{k,\xi} := \frac{1}{\xi}\int_{-\infty}^{\infty} x^k G_{q,\lambda}\left(\frac{x}{\xi}\right) dx, \quad k = 1, \ldots, n \in \mathbb{N}. \tag{82}$$

Then $\overline{\delta}_{k,\xi}$ *and* $\frac{1}{\xi}\int_{-\infty}^{\infty} |x|^k G_{q,\lambda}\left(\frac{x}{\xi}\right) dx$*, are finite and* $\overline{\delta}_{k,\xi} \to 0$*, as* $\xi \to 0$.

3.5 About the q-Deformed and β-Parametrized Half Hyperbolic Tangent function $\varphi_{q,\beta}$

Here all come from [4], Ch. 19.

The activation function now is

$$\varphi_{q,\beta}(x) := \frac{1 - qe^{-\beta t}}{1 + qe^{-\beta t}}, \quad \forall\, t \in \mathbb{R}, \tag{83}$$

where $q, \beta > 0$.

The corresponding density function will be

$$\Phi_{q,\beta}(x) := \frac{1}{4}\left(\varphi_{q,\beta}(x+1) - \varphi_{q,\beta}(x-1)\right) > 0, \quad \forall\, x \in \mathbb{R}. \tag{84}$$

It holds

Theorem 18.

$$\int_{-\infty}^{\infty} \Phi_{q,\beta}(x)\,dx = 1. \tag{85}$$

By [4], p. 481, we have that

$$\Phi_{q,\beta}(x) < \beta q e^{-\beta(x-1)}, \quad \forall\, x \geq 1. \tag{86}$$

Thus here it is

$$d\mu_{\xi}(x) = \frac{1}{\xi}\Phi_{q,\beta}\left(\frac{x}{\xi}\right) dx, \quad \xi > 0, \tag{87}$$

the related Borel probability measure.

We state the following result, its proof as similar to Theorem 13 is omitted.

Theorem 19. *Let* $\xi > 0$ *and*

$$\varepsilon_{k,\xi} := \frac{1}{\xi}\int_{-\infty}^{\infty} x^k \Phi_{q,\beta}\left(\frac{x}{\xi}\right) dx, \quad k = 1, \ldots, n \in \mathbb{N}. \tag{88}$$

Then $\varepsilon_{k,\xi}$ *and* $\frac{1}{\xi}\int_{-\infty}^{\infty} |x|^k \Phi_{q,\beta}\left(\frac{x}{\xi}\right) dx$*, are finite and* $\varepsilon_{k,\xi} \to 0$*, as* $\xi \to 0$.

In the next we define the following five activated singular integral operators.

These are applications to general singular operator $\Theta_{r,\xi}$.

Definition 5. *Let $f:\mathbb{R}\to\mathbb{R}$ be a Borel measurable function and α_j as in (15), $x\in\mathbb{R}$, $0<\xi\leq 1$.*

We call

1)

$$\Theta_{1,r,\xi}(f,x)=\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j f(x+jt)\right)G\left(\frac{t}{\xi}\right)dt, \tag{89}$$

2)

$$\Theta_{2,r,\xi}(f,x)=\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j f(x+jt)\right)M_{q,\lambda}\left(\frac{t}{\xi}\right)dt,\quad q,\lambda>0, \tag{90}$$

3)

$$\Theta_{3,r,\xi}(f,x)=\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j f(x+jt)\right)\psi\left(\frac{t}{\xi}\right)dt, \tag{91}$$

4)

$$\Theta_{4,r,\xi}(f,x)=\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j f(x+jt)\right)G_{q,\lambda}\left(\frac{t}{\xi}\right)dt,\quad q,\lambda>0, \tag{92}$$

and

5)

$$\Theta_{5,r,\xi}(f,x)=\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j f(x+jt)\right)\Phi_{q,\beta}\left(\frac{t}{\xi}\right)dt,\quad q,\beta>0. \tag{93}$$

4 Main Results

We need the following supporting results.

Theorem 20. *Here ($\xi>0$), $\gamma>0$ and $k\in\mathbb{N}$. Then*

$$\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma+k}G\left(\frac{t}{\xi}\right)dt\leq \tag{94}$$

$$2^{\lceil\gamma+k\rceil-1}\left[1+\left[1+2\mu^{-\lceil\gamma+k\rceil}e^{\mu}\lceil\gamma+k\rceil!\right]\xi^{\lceil\gamma+k\rceil}\right]<\infty.$$

Proof. We have that

$$\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma+k}G\left(\frac{t}{\xi}\right)dt\leq\frac{1}{\xi}\int_{-\infty}^{\infty}(1+|t|)^{\gamma+k}G\left(\frac{t}{\xi}\right)dt\leq$$

$$\frac{1}{\xi}\int_{-\infty}^{\infty}(1+|t|)^{\lceil\gamma+k\rceil}G\left(\frac{t}{\xi}\right)dt\leq$$

$$\frac{1}{\xi}2^{\lceil\gamma+k\rceil-1}\int_{-\infty}^{\infty}\left(1+|t|^{\lceil\gamma+k\rceil}\right)G\left(\frac{t}{\xi}\right)dt\overset{(46)}{=}$$

$$2^{\lceil\gamma+k\rceil-1}\left[1+\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\lceil\gamma+k\rceil}G\left(\frac{t}{\xi}\right)dt\right]\overset{(48)}{\leq} \tag{95}$$

$$2^{\lceil\gamma+k\rceil-1}\left[1+\left[1+2\mu^{-\lceil\gamma+k\rceil}e^{\mu}\lceil\gamma+k\rceil!\right]\xi^{\lceil\gamma+k\rceil}\right]<\infty.$$

Theorem 21. *Let $\xi>0$ and $\gamma>0$. Then*

i)

$$\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma}G\left(\frac{t}{\xi}\right)dt\leq\left[1+2\mu^{-\gamma}e^{\mu}\Gamma(\gamma+1)\right]\xi^{\gamma}<\infty, \tag{96}$$

and

ii)

$$\xi^{-\gamma}\left(\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma}G\left(\frac{t}{\xi}\right)dt\right)\leq\left(1+2\mu^{-\gamma}e^{\mu}\Gamma(\gamma+1)\right)<\infty. \tag{97}$$

Proof. We have that

$$\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma}G\left(\frac{t}{\xi}\right)dt =$$

$$\frac{1}{\xi}\left[\int_{-\infty}^{-\xi}|t|^{\gamma}G\left(\frac{t}{\xi}\right)dt+\int_{-\xi}^{\xi}|t|^{\gamma}G\left(\frac{t}{\xi}\right)dt+\int_{\xi}^{\infty}|t|^{\gamma}G\left(\frac{t}{\xi}\right)dt\right]. \tag{98}$$

We notice that

$$\int_{-\xi}^{\xi}|t|^{\gamma}G\left(\frac{t}{\xi}\right)dt \le \xi^{\gamma}\int_{-\xi}^{\xi}G\left(\frac{t}{\xi}\right)dt =$$

$$\xi^{\gamma+1}\left(\frac{1}{\xi}\int_{-\xi}^{\xi}G\left(\frac{t}{\xi}\right)dt\right) \le \xi^{\gamma+1}\left(\frac{1}{\xi}\int_{-\infty}^{\infty}G\left(\frac{t}{\xi}\right)dt\right) = \xi^{\gamma+1} < \infty. \tag{99}$$

Next, we have that

$$\int_{\xi}^{\infty}|x|^{\gamma}G\left(\frac{x}{\xi}\right)dx = \int_{\xi}^{\infty}x^{\gamma}G\left(\frac{x}{\xi}\right)dx =$$

$$\xi^{\gamma+1}\int_{\xi}^{\infty}\left(\frac{x}{\xi}\right)^{\gamma}G\left(\frac{x}{\xi}\right)d\frac{x}{\xi}$$

$$\left(\xi \le x < \infty \Leftrightarrow 1 \le \frac{x}{\xi} =: y < \infty\right) \tag{100}$$

$$= \xi^{\gamma+1}\int_{1}^{\infty}y^{\gamma}G(y)\,dy \overset{(41)}{\le} \mu\xi^{\gamma+1}\int_{1}^{\infty}y^{\gamma}e^{-\mu(y-1)}dy$$

(we have the gamma function $\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt,\ z > 0$)

$$\le \mu\xi^{\gamma+1}e^{\mu}\int_{0}^{\infty}y^{\gamma}e^{-\mu y}dy = \frac{\mu\xi^{\gamma+1}e^{\mu}}{\mu^{\gamma+1}}\int_{0}^{\infty}(\mu y)^{\gamma}e^{-\mu y}d(\mu y) =$$

$$\mu^{-\gamma}e^{\mu}\xi^{\gamma+1}\int_{0}^{\infty}z^{\gamma}e^{-z}dz = \mu^{-\gamma}e^{\mu}\Gamma(\gamma+1)\,\xi^{\gamma+1}. \tag{101}$$

So, we have established that

$$\int_{\xi}^{\infty}|x|^{\gamma}G\left(\frac{x}{\xi}\right)dx \le \mu^{-\gamma}e^{\mu}\Gamma(\gamma+1)\,\xi^{\gamma+1}. \tag{102}$$

Finally, we observe that

$$\int_{-\infty}^{-\xi}|x|^{\gamma}G\left(\frac{x}{\xi}\right)dx = \xi^{\gamma+1}\int_{-\infty}^{-\xi}\left|\frac{x}{\xi}\right|^{\gamma}G\left(\frac{x}{\xi}\right)d\frac{x}{\xi} \tag{103}$$

$$\left(-\infty < x \le -\xi \Leftrightarrow -\infty < \frac{x}{\xi} \le -1\right)$$

$$= \xi^{\gamma+1}\int_{-\infty}^{-1}|z|^{\gamma}G(z)\,dz = \xi^{\gamma+1}\left[-\int_{-\infty}^{-1}|-z|^{\gamma}G(-z)\,d(-z)\right]$$

$$(-\infty < z \le -1 \Leftrightarrow \infty > -z =: y \ge 1)$$

$$= \xi^{\gamma+1}\left[-\int_{\infty}^{1}|y|^{\gamma}G(y)\,dy\right] = \xi^{\gamma+1}\left[\int_{1}^{\infty}|y|^{\gamma}G(y)\,dy\right] = \tag{104}$$

$$\xi^{\gamma+1}\left[\int_{1}^{\infty}y^{\gamma}G(y)\,dy\right] \overset{(41)}{\le} \xi^{\gamma+1}\mu\left(\int_{1}^{\infty}y^{\gamma}e^{-\mu(y-1)}dy\right) \le \tag{105}$$

$$\xi^{\gamma+1}\mu e^{\mu}\int_{0}^{\infty}y^{\gamma}e^{-\mu y}dy \le \mu^{-\gamma}e^{\mu}\Gamma(\gamma+1)\,\xi^{\gamma+1}.$$

That is

$$\int_{-\infty}^{-\xi}|x|^{\gamma}G\left(\frac{x}{\xi}\right)dx \le \mu^{-\gamma}e^{\mu}\Gamma(\gamma+1)\,\xi^{\gamma+1}. \tag{106}$$

Therefore, by (98) we get that

$$\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma}G\left(\frac{t}{\xi}\right)dt\leq$$

$$\frac{1}{\xi}\left[\xi^{\gamma+1}+2\mu^{-\gamma}e^{\mu}\Gamma(\gamma+1)\xi^{\gamma+1}\right]=\left[1+2\mu^{-\gamma}e^{\mu}\Gamma(\gamma+1)\right]\xi^{\gamma}<\infty. \tag{107}$$

The claim is proved.

We continue with the following.

Theorem 22. *Let $\xi>0$, $\gamma>0$ and $k\in\mathbb{N}$; $\lambda,q>0$. Then*

$$\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma+k}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt\leq$$

$$2^{\lceil\gamma+k\rceil-1}\left[1+\left[1+\left(q+\frac{1}{q}\right)\mu^{-\lceil\gamma+k\rceil}e^{\mu}\lceil\gamma+k\rceil!\right]\xi^{\lceil\gamma+k\rceil}\right]<\infty. \tag{108}$$

Proof. We have that

$$\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma+k}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt\leq\frac{1}{\xi}\int_{-\infty}^{\infty}(1+|t|)^{\gamma+k}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt\leq$$

$$\frac{1}{\xi}\int_{-\infty}^{\infty}(1+|t|)^{\lceil\gamma+k\rceil}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt\leq \tag{109}$$

$$\frac{1}{\xi}2^{\lceil\gamma+k\rceil-1}\int_{-\infty}^{\infty}\left(1+|t|^{\lceil\gamma+k\rceil}\right)M_{q,\lambda}\left(\frac{t}{\xi}\right)dt\overset{(68)}{=}$$

$$2^{\lceil\gamma+k\rceil-1}\left[1+\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\lceil\gamma+k\rceil}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt\right]\overset{(71)}{\leq}$$

$$2^{\lceil\gamma+k\rceil-1}\left[1+\left[1+\left(q+\frac{1}{q}\right)\mu^{-\lceil\gamma+k\rceil}e^{\mu}\lceil\gamma+k\rceil!\right]\xi^{\lceil\gamma+k\rceil}\right]<\infty.$$

Theorem 23. *Let $\xi>0$, $\gamma>0$, $q>0$, $\lambda>0$. Then*

i)

$$\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt\leq\left[1+\left(q+\frac{1}{q}\right)\mu^{-\gamma}e^{\mu}\Gamma(\gamma+1)\right]\xi^{\gamma}<\infty, \tag{110}$$

and

ii)

$$\xi^{-\gamma}\left(\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt\right)\leq\left(1+\left(q+\frac{1}{q}\right)\mu^{-\gamma}e^{\mu}\Gamma(\gamma+1)\right)<\infty. \tag{111}$$

Proof. We have that

$$\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt=$$

$$\frac{1}{\xi}\left[\int_{-\infty}^{-\xi}|t|^{\gamma}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt+\int_{-\xi}^{\xi}|t|^{\gamma}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt+\int_{\xi}^{\infty}|t|^{\gamma}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt\right]. \tag{112}$$

Clearly, we have that

$$\int_{-\xi}^{\xi}|t|^{\gamma}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt\leq\xi^{\gamma+1}<\infty. \tag{113}$$

Next we see

$$\int_{\xi}^{\infty}|t|^{\gamma}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt=\int_{\xi}^{\infty}t^{\gamma}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt=$$

$$\xi^{\gamma+1}\int_{\xi}^{\infty}\left(\frac{t}{\xi}\right)^{\gamma}M_{q,\lambda}\left(\frac{t}{\xi}\right)d\frac{t}{\xi} \tag{114}$$

$$\left(\xi \le t < \infty \Leftrightarrow 1 \le \frac{t}{\xi} =: y < \infty\right)$$

$$= \xi^{\gamma+1} \int_1^\infty y^\gamma M_{q,\lambda}(y)\,dy \overset{(59)}{\le} q\mu\xi^{\gamma+1} \int_1^\infty y^\gamma e^{-\mu(y-1)} dy$$

$$= q\mu e^\mu \xi^{\gamma+1} \int_1^\infty y^\gamma e^{-\mu y} dy \le \frac{q\mu e^\mu}{\mu^{\gamma+1}} \xi^{\gamma+1} \int_0^\infty (\mu y)^\gamma e^{-\mu y} d(\mu y) = \tag{115}$$

$$\mu^{-\gamma} q e^\mu \xi^{\gamma+1} \int_0^\infty z^\gamma e^{-z} dz = \mu^{-\gamma} q e^\mu \xi^{\gamma+1} \Gamma(\gamma+1).$$

So, we have established that

$$\int_\xi^\infty |t|^\gamma M_{q,\lambda}\left(\frac{t}{\xi}\right) dt \le q\mu^{-\gamma} e^\mu \Gamma(\gamma+1)\,\xi^{\gamma+1} < \infty. \tag{116}$$

Finally, we observe that

$$\int_{-\infty}^{-\xi} |t|^\gamma M_{q,\lambda}\left(\frac{t}{\xi}\right) dt = \xi^{\gamma+1} \int_{-\infty}^{-\xi} \left|\frac{t}{\xi}\right|^\gamma M_{\frac{1}{q},\lambda}\left(\frac{t}{\xi}\right) d\frac{t}{\xi}$$

$$\left(-\infty < t \le -\xi \Leftrightarrow -\infty < \frac{t}{\xi} \le -1\right)$$

$$= \xi^{\gamma+1} \int_{-\infty}^{-1} |z|^\gamma M_{q,\lambda}(z)\,dz = \xi^{\gamma+1} \left[-\int_{-\infty}^{-1} |-z|^\gamma M_{\frac{1}{q},\lambda}(-z)\,d(-z)\right]$$

$$(-\infty < z \le -1 \Leftrightarrow \infty > -z =: y \ge 1)$$

$$= \xi^{\gamma+1} \left[-\int_\infty^1 |y|^\gamma M_{\frac{1}{q},\lambda}(y)\,dy\right] = \xi^{\gamma+1} \left(\int_1^\infty |y|^\gamma M_{\frac{1}{q},\lambda}(y)\,dy\right) = \tag{117}$$

$$\xi^{\gamma+1} \left(\int_1^\infty y^\gamma M_{\frac{1}{q},\lambda}(y)\,dy\right) \overset{(59)}{\le} \frac{1}{q}\mu\xi^{\gamma+1} \left(\int_1^\infty y^\gamma e^{-\mu(y-1)} dy\right) =$$

$$\frac{1}{q}\mu\xi^{\gamma+1} e^\mu \int_1^\infty y^\gamma e^{-\mu y} dy \le \frac{1}{q}\mu\xi^{\gamma+1} e^\mu \int_0^\infty y^\gamma e^{-\mu y} dy = \tag{118}$$

$$\frac{1}{q}\frac{\mu}{\mu^{\gamma+1}} \xi^{\gamma+1} e^\mu \int_0^\infty (\mu y)^\gamma e^{-\mu y} d(\mu y) =$$

$$\frac{1}{q}\mu^{-\gamma}\xi^{\gamma+1} e^\mu \int_0^\infty z^\gamma e^{-z} dz = \frac{1}{q}\mu^{-\gamma}\xi^{\gamma+1} e^\mu \Gamma(\gamma+1).$$

Therefore, we found that

$$\int_{-\infty}^{-\xi} |t|^\gamma M_{q,\lambda}\left(\frac{t}{\xi}\right) dt \le \frac{1}{q}\mu^{-\gamma} e^\mu \Gamma(\gamma+1)\,\xi^{\gamma+1} < \infty. \tag{119}$$

Consequently now, by (112) we obtain

$$\frac{1}{\xi}\int_{-\infty}^\infty |t|^\gamma M_{q,\lambda}\left(\frac{t}{\xi}\right) dt \le \tag{120}$$

$$\frac{1}{\xi}\left[\xi^{\gamma+1} + \left(q + \frac{1}{q}\right)\mu^{-\gamma} e^\mu \Gamma(\gamma+1)\,\xi^{\gamma+1}\right] =$$

$$\left[1 + \left(q + \frac{1}{q}\right)\mu^{-\gamma} e^\mu \Gamma(\gamma+1)\right]\xi^\gamma < \infty.$$

The claim is proved.

More similar results follow, as their proofs are similar are omitted.

Theorem 24. *Here $\xi > 0$, $\gamma > 0$, $k \in \mathbb{N}$. Then*

i)

$$\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma+k}\psi\left(\frac{t}{\xi}\right)dt < \infty, \tag{121}$$

ii)

$$\xi^{-\gamma}\left(\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma}\psi\left(\frac{t}{\xi}\right)dt\right) \leq \rho_1, \tag{122}$$

where $\rho_1 > 0$.

Theorem 25. *Here $\xi > 0$, $\gamma > 0$, $k \in \mathbb{N}$, $q, \lambda > 0$. Then*

i)

$$\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma+k}G_{q,\lambda}\left(\frac{t}{\xi}\right)dt < \infty, \tag{123}$$

ii)

$$\xi^{-\gamma}\left(\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma}G_{q,\lambda}\left(\frac{t}{\xi}\right)dt\right) \leq \rho_2, \tag{124}$$

where $\rho_2 > 0$.

Theorem 26. *Here $\xi > 0$, $\gamma > 0$, $k \in \mathbb{N}$, $q, \beta > 0$. Then*

i)

$$\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma+k}\Phi_{q,\beta}\left(\frac{t}{\xi}\right)dt < \infty, \tag{125}$$

ii)

$$\xi^{-\gamma}\left(\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{\gamma}\Phi_{q,\beta}\left(\frac{t}{\xi}\right)dt\right) \leq \rho_3, \tag{126}$$

where $\rho_3 > 0$.

Next follow our main results based on the above preparations.

Theorem 27. *Let $f \in C^m(\mathbb{R})$, $m = \lceil\gamma\rceil$, $\gamma > 0$, with $\left\|f^{(m)}\right\|_{\infty} < \infty$, $\xi > 0$, $x_0 \in \mathbb{R}$. Then*

1)

$$\left|\Theta_{1,r,\xi}(f,x_0) - f(x_0) - \sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)\,\delta_k c^*_{k,\xi}}{k!}\right| \leq$$

$$\left[\sum_{k=0}^{r}\frac{r!}{(r-k)!\,\Gamma(\gamma+k+1)\,\xi^{k+1}}\int_{-\infty}^{\infty}|t|^{\gamma+k}G\left(\frac{t}{\xi}\right)dt\right]$$

$$\max\left\{\omega_r\left(D^{\gamma}_{x_0-}f,\xi\right),\omega_r\left(D^{\gamma}_{*x_0}f,\xi\right)\right\}. \tag{127}$$

2)

$$\left\|\Theta_{1,r,\xi}(f,\cdot) - f(\cdot) - \sum_{k=1}^{m-1}\frac{f^{(k)}(\cdot)\,\delta_k c^*_{k,\xi}}{k!}\right\|_{\infty} \leq$$

$$\left[\sum_{k=0}^{r}\frac{r!}{(r-k)!\,\Gamma(\gamma+k+1)\,\xi^{k+1}}\int_{-\infty}^{\infty}|t|^{\gamma+k}G\left(\frac{t}{\xi}\right)dt\right]$$

$$\sup_{x\in\mathbb{R}}\left\{\max\left[\omega_r\left(D^{\gamma}_{x-}f,\xi\right),\omega_r\left(D^{\gamma}_{*x}f,\xi\right)\right]\right\}. \tag{128}$$

(Above if $m = 1$ the sum disappears).

Proof. By Theorems 6, 10, 20.

Theorem 28. *Here* $f \in C^m(\mathbb{R})$, $m \in \mathbb{N}$, $m = \lceil \gamma \rceil$, $\gamma > 0$, $\left\| f^{(m)} \right\|_{\infty} < \infty$, *and* $\left\| D^{\gamma}_{x-} f(y) \right\|_{\infty} \leq M_1$, $\left\| D^{\gamma}_{*x} f(y) \right\|_{\infty} \leq M_2$, *where* $M_1, M_2 > 0$, *for any* $x, y \in \mathbb{R}$. *Then*

$$\Theta_{1,r,\xi}(f,x) - f(x) - \sum_{k=1}^{m-1} \frac{f^{(k)}(x)}{k!} \delta_k c^*_{k,\xi} = o\left(\xi^{\gamma-\eta}\right), \tag{129}$$

$0 < \eta < \gamma$, *as* $\xi \to 0+$.

I.e.

$$\Theta_{1,r,\xi}(f,x) - f(x) = \sum_{k=1}^{m-1} \frac{f^{(k)}(x)}{k!} \left(\sum_{j=1}^{r} \alpha_j j^k \right) c^*_{k,\xi} + o\left(\xi^{\gamma-\eta}\right), \tag{130}$$

where $0 < \eta < \gamma$.

(Above if $m = 1$ *the sum disappears).*

Proof. By Theorems 7, 10, 21.

We continue with

Theorem 29. *All as in Theorem 27. Then*

1)

$$\left| \Theta_{2,r,\xi}(f,x_0) - f(x_0) - \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)\, \delta_k \overline{c}_{k,\xi}}{k!} \right| \leq$$

$$\left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\,\Gamma(\gamma+k+1)\,\xi^{k+1}} \int_{-\infty}^{\infty} |t|^{\gamma+k} M_{q,\lambda}\left(\frac{t}{\xi}\right) dt \right]$$

$$\max\left\{ \omega_r\left(D^{\gamma}_{x_0-} f, \xi\right), \omega_r\left(D^{\gamma}_{*x_0} f, \xi\right) \right\}. \tag{131}$$

2)

$$\left\| \Theta_{2,r,\xi}(f,\cdot) - f(\cdot) - \sum_{k=1}^{m-1} \frac{f^{(k)}(\cdot)\, \delta_k \overline{c}_{k,\xi}}{k!} \right\|_{\infty} \leq$$

$$\left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\,\Gamma(\gamma+k+1)\,\xi^{k+1}} \int_{-\infty}^{\infty} |t|^{\gamma+k} M_{q,\lambda}\left(\frac{t}{\xi}\right) dt \right]$$

$$\sup_{x\in\mathbb{R}} \left\{ \max\left[\omega_r\left(D^{\gamma}_{x-} f, \xi\right), \omega_r\left(D^{\gamma}_{*x} f, \xi\right) \right] \right\}. \tag{132}$$

(Above if $m = 1$ *the sum disappears).*

Proof. By Theorems 6, 13, 22.

Theorem 30. *All as in Theorem 28. Then*

$$\Theta_{2,r,\xi}(f,x) - f(x) - \sum_{k=1}^{m-1} \frac{f^{(k)}(x)}{k!} \delta_k \overline{c}_{k,\xi} = o\left(\xi^{\gamma-\eta}\right), \tag{133}$$

$0 < \eta < \gamma$, *as* $\xi \to 0+$.

I.e.

$$\Theta_{2,r,\xi}(f,x) - f(x) = \sum_{k=1}^{m-1} \frac{f^{(k)}(x)}{k!} \left(\sum_{j=1}^{r} \alpha_j j^k \right) \overline{c}_{k,\xi} + o\left(\xi^{\gamma-\eta}\right), \tag{134}$$

where $0 < \eta < \gamma$.

(Above if $m = 1$ *the sum disappears).*

Proof. By Theorems 7, 13, 23.

Furthermore we have

Theorem 31. *All as in Theorem 27. Then*

1)

$$\left|\Theta_{3,r,\xi}(f,x_0)-f(x_0)-\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)\,\delta_k\gamma_{k,\xi}}{k!}\right|\leq$$
$$\left[\sum_{k=0}^{r}\frac{r!}{(r-k)!\Gamma(\gamma+k+1)\,\xi^{k+1}}\int_{-\infty}^{\infty}|t|^{\gamma+k}\,\psi\left(\frac{t}{\xi}\right)dt\right]$$
$$\max\left\{\omega_r\left(D_{x_0-}^{\gamma}f,\xi\right),\omega_r\left(D_{*x_0}^{\gamma}f,\xi\right)\right\}. \tag{135}$$

2)

$$\left\|\Theta_{3,r,\xi}(f,\cdot)-f(\cdot)-\sum_{k=1}^{m-1}\frac{f^{(k)}(\cdot)\,\delta_k\gamma_{k,\xi}}{k!}\right\|_{\infty}\leq$$
$$\left[\sum_{k=0}^{r}\frac{r!}{(r-k)!\Gamma(\gamma+k+1)\,\xi^{k+1}}\int_{-\infty}^{\infty}|t|^{\gamma+k}\,\psi\left(\frac{t}{\xi}\right)dt\right]$$
$$\sup_{x\in\mathbb{R}}\left\{\max\left[\omega_r\left(D_{x-}^{\gamma}f,\xi\right),\omega_r\left(D_{*x}^{\gamma}f,\xi\right)\right]\right\}. \tag{136}$$

(Above if $m=1$ the sum disappears).

Proof. By Theorems 6, 15, 24.

Theorem 32. *All as in Theorem 28. Then*

$$\Theta_{3,r,\xi}(f,x)-f(x)-\sum_{k=1}^{m-1}\frac{f^{(k)}(x)}{k!}\delta_k\gamma_{k,\xi}=o\left(\xi^{\gamma-\eta}\right), \tag{137}$$

$0<\eta<\gamma$, *as* $\xi\to 0+$.

I.e.

$$\Theta_{3,r,\xi}(f,x)-f(x)=\sum_{k=1}^{m-1}\frac{f^{(k)}(x)}{k!}\left(\sum_{j=1}^{r}\alpha_j j^k\right)\gamma_{k,\xi}+o\left(\xi^{\gamma-\eta}\right), \tag{138}$$

where $0<\eta<\gamma$.

(Above if $m=1$ the sum disappears).

Proof. By Theorems 7, 15, 24.

Next come the following results.

Theorem 33. *All as in Theorem 27. Then*

1)

$$\left|\Theta_{4,r,\xi}(f,x_0)-f(x_0)-\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)\,\delta_k\overline{\delta}_{k,\xi}}{k!}\right|\leq$$
$$\left[\sum_{k=0}^{r}\frac{r!}{(r-k)!\Gamma(\gamma+k+1)\,\xi^{k+1}}\int_{-\infty}^{\infty}|t|^{\gamma+k}\,G_{q,\lambda}\left(\frac{t}{\xi}\right)dt\right]$$
$$\max\left\{\omega_r\left(D_{x_0-}^{\gamma}f,\xi\right),\omega_r\left(D_{*x_0}^{\gamma}f,\xi\right)\right\}. \tag{139}$$

2)

$$\left\|\Theta_{4,r,\xi}(f,\cdot)-f(\cdot)-\sum_{k=1}^{m-1}\frac{f^{(k)}(\cdot)\,\delta_k\overline{\delta}_{k,\xi}}{k!}\right\|_{\infty}\leq$$
$$\left[\sum_{k=0}^{r}\frac{r!}{(r-k)!\Gamma(\gamma+k+1)\,\xi^{k+1}}\int_{-\infty}^{\infty}|t|^{\gamma+k}\,G_{q,\lambda}\left(\frac{t}{\xi}\right)dt\right]$$
$$\sup_{x\in\mathbb{R}}\left\{\max\left[\omega_r\left(D_{x-}^{\gamma}f,\xi\right),\omega_r\left(D_{*x}^{\gamma}f,\xi\right)\right]\right\}. \tag{140}$$

(Above if $m=1$ the sum disappears).

Proof. By Theorems 6, 17, 25.

Theorem 34. *All as in Theorem 28. Then*

$$\Theta_{4,r,\xi}(f,x) - f(x) - \sum_{k=1}^{m-1} \frac{f^{(k)}(x)}{k!} \delta_k \overline{\delta}_{k,\xi} = o\left(\xi^{\gamma-\eta}\right), \tag{141}$$

$0 < \eta < \gamma$, *as* $\xi \to 0+$.

I.e.

$$\Theta_{4,r,\xi}(f,x) - f(x) = \sum_{k=1}^{m-1} \frac{f^{(k)}(x)}{k!} \left(\sum_{j=1}^{r} \alpha_j j^k \right) \overline{\delta}_{k,\xi} + o\left(\xi^{\gamma-\eta}\right), \tag{142}$$

where $0 < \eta < \gamma$.

(Above if $m = 1$ *the sum disappears).*

Proof. By Theorems 7, 17, 25.

We finish with the following results.

Theorem 35. *All as in Theorem 27. Then*

1)

$$\left| \Theta_{5,r,\xi}(f,x_0) - f(x_0) - \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)\, \delta_k \varepsilon_{k,\xi}}{k!} \right| \leq$$
$$\left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\,\Gamma(\gamma+k+1)\,\xi^{k+1}} \int_{-\infty}^{\infty} |t|^{\gamma+k}\, \Phi_{q,\beta}\left(\frac{t}{\xi}\right) dt \right]$$
$$\max\left\{ \omega_r\left(D_{x_0-}^{\gamma} f, \xi\right), \omega_r\left(D_{*x_0}^{\gamma} f, \xi\right) \right\}. \tag{143}$$

2)

$$\left\| \Theta_{5,r,\xi}(f,\cdot) - f(\cdot) - \sum_{k=1}^{m-1} \frac{f^{(k)}(\cdot)\, \delta_k \varepsilon_{k,\xi}}{k!} \right\|_{\infty} \leq$$
$$\left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\,\Gamma(\gamma+k+1)\,\xi^{k+1}} \int_{-\infty}^{\infty} |t|^{\gamma+k}\, \Phi_{q,\beta}\left(\frac{t}{\xi}\right) dt \right]$$
$$\sup_{x\in\mathbb{R}} \left\{ \max\left[\omega_r\left(D_{x-}^{\gamma} f, \xi\right), \omega_r\left(D_{*x}^{\gamma} f, \xi\right) \right] \right\}. \tag{144}$$

(Above if $m = 1$ *the sum drops).*

Proof. By Theorems 6, 19, 26.

Theorem 36. *All as in Theorem 28. Then*

$$\Theta_{5,r,\xi}(f,x) - f(x) - \sum_{k=1}^{m-1} \frac{f^{(k)}(x)}{k!} \delta_k \varepsilon_{k,\xi} = o\left(\xi^{\gamma-\eta}\right), \tag{145}$$

$0 < \eta < \gamma$, *as* $\xi \to 0+$.

I.e.

$$\Theta_{5,r,\xi}(f,x) - f(x) = \sum_{k=1}^{m-1} \frac{f^{(k)}(x)}{k!} \left(\sum_{j=1}^{r} \alpha_j j^k \right) \varepsilon_{k,\xi} + o\left(\xi^{\gamma-\eta}\right), \tag{146}$$

where $0 < \eta < \gamma$.

(Above if $m = 1$ *the sum drops).*

Proof. By Theorems 7, 19, 26.

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