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Hadamard-Simpson Type Integral Inequalities In Post Quantum Calculus For tgs-Convex Functions

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Abstract: In this paper, we derive some new Hadamard-Simpson integral inequalities for (p,q)-differentiable tgs-convex functions by using Hadamard-Simpson type (p,q)-integral equality . For special values of λ and μ, we obtain some new q-Hadamard and q-Simpson type integral inequalities in the case of $p=1$. We recapture some new results when $q\to 1$.

Keywords: Integral inequalities, (p,q)-Integral, Hadamard integral inequality, Simpson integral inequality, tgs-convex functions

1 Introduction

Let $J\subset\mathbb{R}$ is a nonempty interval. The function $f:J\to\mathbb{R}$ is said to be convex on J, if inequality

$$f(tu+(1-t)v)\leq tf(u)+(1-t)f(v) \tag{1}$$

holds for all $u,v\in J$ and $t\in[0,1]$. The function $f:J\to\mathbb{R}$ is said to be tgs-convex on J, if inequality

$$f(tu+(1-t)v)\leq t(1-t)[f(u)+f(v)] \tag{2}$$

holds for all $u,v\in J$ and $t\in(0,1)$.

One of the most extensively used inequality in the literature is Hermite-Hadamard type inequality. Let $f:J\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function and $u,v\in J$ with $u<v$. The following double inequality holds:

$$f\left(\frac{u+v}{2}\right)\leq\frac{1}{v-u}\int_u^v f(x)\,dx\leq\frac{f(u)+f(v)}{2}. \tag{3}$$

In [1], Dragomir and Agarval obtained inequalities for differantiable convex mappings which are engaged with Hermite-Hadamard type inequality. The following lemma was used to proved their theorems.

Lemma 1.*Let $f:J^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differantiable mapping on J°, $u,v\in J$ with $u<v$. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(u)+f(v)}{2}-\frac{1}{v-u}\int_u^v f(x)\,dx \tag{4}$$
$$=\frac{v-u}{2}\int_0^1(1-2t)f'(ta+(1-t)b)\,dt.$$

Some inequalities for differantiable convex and quasi-convex mappings connected with midpoint type inequlity are obtained in [2] and [3] respectively. They used the following lemma to prove their theorems.

Lemma 2.*Let $f:J^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differantiable mapping on J°, $u,v\in J$ with $u<v$. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{1}{v-u}\int_u^v f(x)\,dx-f\left(\frac{u+v}{2}\right) \tag{5}$$
$$=(v-u)\left[\int_0^{\frac{1}{2}}tf'(ta+(1-t)b)\,dt+\int_{\frac{1}{2}}^1(t-1)f'(ta+(1-t)b)\,dt\right].$$

The inequality given below is well known in the literature as Simpson's inequality:

$$\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]$$
$$\leq\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4$$

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where the mapping $f:[a,b]\to\mathbb{R}$ is assumed to be four times continuously differentiable on the interval and $f^{(4)}$ to be bounded on (a,b), that is,

$$\left\|f^{(4)}\right\|_\infty=\sup_{t\in(a,b)}\left|f^{(4)}\right|<\infty. \tag{6}$$

In [4], Alomari et al. obtained Simpson type integral inequalities for $s-$convex functions derived from the following equality.

Lemma 3. *Let $f:J\subseteq\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on J° where $a,b\in J$ with $a<b$. Then the following equality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(a)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx \tag{7}$$
$$=(b-a)\int_0^1 p(t)\,f'(tb+(1-t)a)\,dt$$

where

$$p(t)=\begin{cases} t-\frac{1}{6}\ ,\ t\in\left[0,\frac{1}{2}\right) \\ t-\frac{5}{6}\ ,\ t\in\left[\frac{1}{2},1\right)\end{cases}. \tag{8}$$

In recent years quantum calculus has been actively studied. There are numerous applications in many mathematical areas like special functions, integral transforms, quantum mechanics, information technology and mathematical inequalities. At present q analogues of many inequalities have been established. In the view of these developments q-convexity and convexity of q analogues of the inequalities has also been considered, see [5,6,7,8,9,10,11]. In [12], Tunç et al. obtained q-Simpson type integral inequalities for convex functions by using the following equality.

Lemma 4. *Let $f:J\to\mathbb{R}$ be a continuous function and $0<q<1$. If ${}_aD_qf$ is an integrable function on J° (the interior of J), then the following inequality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx \tag{9}$$
$$=(b-a)\int_0^1 p(t)\,{}_aD_qf((1-t)a+tb)\,{}_0d_qt$$

where

$$p(t)=\begin{cases} qt-\frac{1}{6}\ ,\ t\in\left[0,\frac{1}{2}\right) \\ qt-\frac{5}{6}\ ,\ t\in\left[\frac{1}{2},1\right)\end{cases}. \tag{10}$$

Recently, Tunç and Göv studied on (p,q)-derivative, (p,q)-integral and (p,q)-analogues of some of the well known integral inequalities in [13,14,15]. Göv and Taşbozan obtained (p,q)-Ostrowski type integral inequalities for convex and tgs-convex functions in [16]. Some (p,q)-Hermite-Hadamard inequalities and generalized (p,q)-Hermite-Hadamard inequality are proved in [17].

The aim of this paper is to establish (p,q)-analogues of Hadamard-Simpson type inequalities based on tgs-convexity. The consequences of Hadamard-Simpson type inequalities for tgs-convex functions are given as special cases when $p=1$ and $q\to 1^-$.

2 Materials and Methods

In this section, we recall some previously known concepts and basic results.

Let $J=[a,b]\subset\mathbb{R}$ be an interval and $0<q<p\le 1$ be a constant. We define (p,q)-derivative of a function $f:J\to\mathbb{R}$ at a point $x\in J$ as follows.

Definition 1. *Assume $f:J\to\mathbb{R}$ is a continuous function and let $x\in J$. Then the expression, for $x\neq a$*

$$ {}_aD_{p,q}f(x) = \frac{f(px+(1-p)a)-f(qx+(1-q)a)}{(p-q)(x-a)} \tag{11}$$
$$ {}_aD_{p,q}f(a) = \lim_{x\to a}{}_aD_{p,q}f(x)\,, \tag{12}$$

is called the (p,q)-derivative on J of function f at x. Also if $a=0$ in (11), then ${}_0D_{p,q}f(a)=D_{p,q}f$, where $D_{p,q}$ is the (p,q)-derivative of the function $f(x)$ defined by

$$D_{p,q}f(x)=\frac{f(px)-f(qx)}{(p-q)x}. \tag{13}$$

For more details, see [13].

Definition 2. *Let $f:J\subset\mathbb{R}\to\mathbb{R}$ be a continuous function. Then (p,q)-integral on J is defined as*

$$\int_a^x f(t)\,{}_ad_{p,q}t$$
$$=(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)$$

for $x\in J$. If $a=0$ in (14), then we have the classical (p,q)–integral [13].

Moreover, if $v\in(a,x)$ then the definite (p,q)–integral on J is defined by

$$\int_v^x f(t)\,{}_ad_{p,q}t$$
$$=\int_a^x f(t)\,{}_ad_{p,q}t-\int_a^v f(t)\,{}_ad_{p,q}t$$
$$=(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)$$
$$-(p-q)(v-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}v+\left(1-\frac{q^n}{p^{n+1}}\right)a\right).$$

The values of such defined (p,q)-integrals of the polynomials have very similar form to those in the standard integral calculus. So, for example, we have

$$\int_a^b x^n d_{p,q}x=\frac{b^{n+1}-a^{n+1}}{[n+1]_{p,q}}, \tag{14}$$

where

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} \tag{15}$$

for $n \in \mathbb{N}$.

Theorem 1. *[13] Let f and g be two functions defined on J, $0 < q < p \leq 1$ and $s_1, s_2 > 1$ with $\frac{1}{s_1} + \frac{1}{s_2} = 1$. Then*

$$\int_a^b |f(t)\, g(t)|\ {}_a d_{p,q}t \tag{16}$$

$$\leq \left(\int_a^b |f(t)|^{s_1}\ {}_a d_{p,q}t\right)^{\frac{1}{s_1}} \left(\int_a^b |g(t)|^{s_2}\ {}_a d_{p,q}t\right)^{\frac{1}{s_2}}. \tag{17}$$

In recent years, many authors have studied integral inequalities extensively either in classical analysis or quantum one; see [18,19,20,21,22,23,24] and references cited therein.

In this paper we will establish some new (p,q)-analogue of the Hadamard-Simpson type integral inequalities for functions whose first (p,q)-derivatives are of convex and quasi-convex.

Göv and Taşbozan proved $(p,q)-$Hadamard-Simpson equality and derive some consequences via Hermite-Hadamard and Simpson integral inequalities in [25] as follows:

Lemma 5. *[25] Let $f : J \to \mathbb{R}$ be a (p,q)-differantiable function on J° (the interior of J) and ${}_aD_{p,q}f$ be continuous and integrable function on J for $0 < q < 1$ and $0 < t < p \leq 1$. For $0 \leq \lambda$, $\mu < 1$, the following inequality holds:*

$$\frac{p}{b-a}\int_a^b f(x)\ {}_a d_{p,q}x - p\left[(\lambda-\mu) f\left(\frac{b+a(2p-1)}{2p}\right) + (1-\lambda) f(a) + \mu f\left(\frac{b+(p-1)a}{p}\right)\right] \tag{18}$$

$$= (b-a)\int_0^{\frac{1}{2}} (1-\lambda-qt) \times {}_aD_{p,q} f\left(\frac{tb+(p-t)a}{p}\right)\ {}_0 d_{p,q}t + (b-a)\int_{\frac{1}{2}}^1 (1-\mu-qt) \times {}_bD_{p,q} f\left(\frac{tb+(p-t)a}{p}\right)\ {}_0 d_{p,q}t\,.$$

Proof. Utilizing Definition 1 and Definition 2, we have

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} (1-\lambda-qt) \times {}_aD_{p,q} f\left(\frac{tb}{p} + \left(1-\frac{t}{p}\right)a\right)\ {}_0 d_{p,q}t \\
&= \frac{p(1-\lambda)}{(b-a)(p-q)} \times \int_0^{\frac{1}{2}} \frac{f((1-t)a+tb) - f\left(\left(1-\frac{qt}{p}\right)a + \frac{qtb}{p}\right)}{t}\ {}_0 d_{p,q}t \\
&\quad - \frac{qp}{(b-a)(p-q)} \times \int_0^1 \left(f((1-t)a+tb) - f\left(\left(1-\frac{qt}{p}\right)a + \frac{qtb}{p}\right)\right)\ {}_0 d_{p,q}t \\
&= \frac{p(1-\lambda)}{b-a}\left[\sum_{n=0}^{\infty} f\left(\left(1-\frac{q^n}{p^{n+1}}\right)a + \frac{q^n}{p^{n+1}}b\right) - \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\left(1-\frac{q^{n+1}}{p^{n+2}}\right)a + \frac{q^{n+1}}{p^{n+2}}b\right)\right] \\
&\quad + \frac{qp}{(b-a)}\sum_{n=0}^{\infty}\frac{q^n}{2p^{n+1}} \times \left[f\left(\left(1-\frac{q^n}{2p^{n+1}}\right)a + \frac{q^n}{2p^{n+1}}b\right) - f\left(\left(1-\frac{q^{n+1}}{2p^{n+2}}\right)a + \frac{q^{n+1}}{2p^{n+2}}b\right)\right] \\
&= \frac{p(1-\lambda)}{b-a}\left[f\left(\left(1-\frac{1}{2p}\right)a + \frac{b}{2p}\right) - f(a)\right] \\
&\quad - \frac{qp}{b-a}\left[\left(1-\frac{p}{q}\right)\sum_{n=0}^{\infty}\frac{q^n}{2p^{n+1}} \times f\left(\left(1-\frac{q^n}{2p^{n+1}}\right)a + \frac{q^n b}{2p^{n+1}}\right) + \frac{1}{2q} f\left(\left(1-\frac{1}{2p}\right)a + \frac{b}{2p}\right)\right] \\
&= \frac{p(1-\lambda) - p/2}{b-a} f\left(\left(1-\frac{1}{2p}\right)a + \frac{b}{2p}\right) - \frac{p(1-\lambda)}{b-a} f(a) \\
&\quad + \frac{p(p-q)}{b-a}\sum_{n=0}^{\infty}\frac{q^n}{2p^{n+1}} \times f\left(\left(1-\frac{q^n}{2p^{n+1}}\right)a + \frac{q^n b}{2p^{n+1}}\right).
\end{aligned} \tag{19}$$

Similarly, we get

$$\begin{aligned}
I_2 &= \int_{\frac{1}{2}}^{1}(1-\mu-qt) \\
&\quad \times {}_bD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)b\right)\ {}_0d_{p,q}t \\
&= \int_0^1(1-\mu-qt) \\
&\quad \times {}_bD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)b\right)\ {}_0d_{p,q}t \\
&\quad -\int_0^{\frac{1}{2}}(1-\mu-qt) \\
&\quad \times {}_bD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)b\right)\ {}_0d_{p,q}t \\
&= \frac{p}{(b-a)^2}\int_a^b f(x)\ {}_ad_{p,q}x \\
&\quad -\frac{p\mu}{b-a}f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right)-\frac{p(1-\mu)}{b-a}f(a) \\
&\quad -\left[\frac{p(1-\mu)-p/2}{b-a}f\left(\left(1-\frac{1}{2p}\right)a+\frac{b}{2p}\right)\right. \\
&\quad -\frac{p(1-\mu)}{b-a}f(a)+\frac{p(p-q)}{b-a}\sum_{n=0}^{\infty}\frac{q^n}{2p^{n+1}} \\
&\quad \left.\times f\left(\left(1-\frac{q^n}{2p^{n+1}}\right)a+\frac{q^nb}{2p^{n+1}}\right)\right].
\end{aligned} \tag{20}$$

By subsysting (19) and (20), we have

$$\begin{aligned}
&I_1+I_2 \\
&= \frac{p}{(b-a)^2}\int_a^b f(x)\ {}_ad_{p,q}x \\
&\quad -\frac{p\left[(\lambda-\mu)f\left(\frac{b}{2p}+\left(1-\frac{1}{2p}\right)a\right)+(1-\lambda)f(a)+\mu f\left(\frac{b}{p}+\left(1-\frac{1}{p}\right)a\right)\right]}{b-a}.
\end{aligned}$$

Multiplying both sides with $b-a$, we obtain required result in (18), this completes the proof.

Corollary 1.*[25]Under the assumption of Lemma 5, if we take $p=1$, then the equality (18) reduces to equality*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx \\
&-\left[(\lambda-\mu)f\left(\frac{a+b}{2}\right)+(1-\lambda)f(a)+\mu f(b)\right] \\
&= (b-a)\int_0^{\frac{1}{2}}(1-\lambda-qt)\ {}_aD_qf(tb+(1-t)a)\ {}_0d_{p,q}t \\
&\quad +(b-a)\int_{\frac{1}{2}}^1(1-\mu-qt) \\
&\quad \times {}_bD_qf\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)b\right)\ {}_0d_{p,q}t\ .
\end{aligned} \tag{21}$$

This inequality can be named as q-Hadamard-Simpson type equality.

Remark.[25]If $p=1$ and $q\to 1^-$, then the equality (18) reduces to

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\,dx \\
&-\left[(\lambda-\mu)f\left(\frac{a+b}{2}\right)+(1-\lambda)f(a)+\mu f(b)\right] \\
&= (b-a)\int_0^{\frac{1}{2}}(1-\lambda-t)f'(tb+(1-t)a)\,dt \\
&\quad +(b-a)\int_{\frac{1}{2}}^1(1-\mu-t)f'(tb+(1-t)a)\,dt.
\end{aligned} \tag{22}$$

See also Lemma 2.1 in [26] with $\eta(a,b)=b-a$ and $s=m=1$.

Corollary 2.*[25]Under the assumption of Lemma 5,*

1.If we take $\lambda=\frac{5}{6}$, $\mu=\frac{1}{6}$ in (18), then we have

$$\begin{aligned}
&\frac{1}{6}\left[pf(a)+4pf\left(\frac{b}{2p}+\left(1-\frac{1}{2p}\right)a\right)\right. \\
&\left.+pf\left(\frac{b}{p}+\left(1-\frac{1}{p}\right)a\right)\right] \\
&-\frac{p}{b-a}\int_a^b f(x)\ {}_ad_{p,q}x \\
&= (b-a)\int_0^{\frac{1}{2}}\left(qt-\frac{1}{6}\right) \\
&\quad \times {}_aD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)a\right)\ {}_0d_{p,q}t \\
&\quad +(b-a)\int_{\frac{1}{2}}^1\left(qt-\frac{5}{6}\right) \\
&\quad \times {}_bD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)b\right)\ {}_0d_{p,q}t
\end{aligned} \tag{23}$$

2.If we take $\lambda=\frac{1}{2}$, $\mu=\frac{1}{2}$ in (18), then we have

$$\begin{aligned}
&\frac{p}{b-a}\int_a^b f(x)\ {}_ad_{p,q}x \\
&-p\left[\frac{f(a)+f\left(\frac{b}{p}+\left(1-\frac{1}{p}\right)a\right)}{2}\right] \\
&= (b-a)\int_0^{\frac{1}{2}}\left(\frac{1}{2}-qt\right) \\
&\quad \times {}_aD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)a\right)\ {}_0d_{p,q}t \\
&\quad +(b-a)\int_{\frac{1}{2}}^1\left(\frac{1}{2}-qt\right) \\
&\quad \times {}_bD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)b\right)\ {}_0d_{p,q}t
\end{aligned} \tag{24}$$

3.If we take $\lambda = 1$, $\mu = 0$ in (18), then we have

$$\begin{aligned}&\frac{p}{b-a}\int_a^b f(x)\ {}_ad_{p,q}x\\&-p\left[f\left(\frac{b}{2p}+\left(1-\frac{1}{2p}\right)a\right)\right]\end{aligned} \tag{25}$$

$$\begin{aligned}&=(b-a)\int_0^{\frac{1}{2}}(-qt)\\&\times {}_aD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)a\right)\ {}_0d_{p,q}t\\&+(b-a)\int_{\frac{1}{2}}^1(1-qt)\\&\times {}_bD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)b\right)\ {}_0d_{p,q}t\ .\end{aligned}$$

Corollary 3.*[25] In Corollary 2;*

1.If we take $p = 1$ in (24), we have

$$\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx-\frac{f(a)+f(b)}{2} \tag{26}$$

$$\begin{aligned}&=(b-a)\int_0^{\frac{1}{2}}\left(\frac{1}{2}-qt\right)\\&\times {}_aD_qf(tb+(1-t)a)\ {}_0d_qt\\&+(b-a)\int_{\frac{1}{2}}^1\left(\frac{1}{2}-qt\right)\\&\times {}_aD_qf(tb+(1-t)a)\ {}_0d_qt\ .\end{aligned}$$

2.If we take $p = 1$ in (25), we have

$$\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx-f\left(\frac{a+b}{2}\right) \tag{27}$$

$$\begin{aligned}&=(b-a)\int_0^{\frac{1}{2}}(-qt)\\&\times {}_aD_qf(tb+(1-t)a)\ {}_0d_qt\\&+(b-a)\int_{\frac{1}{2}}^1(1-qt)\\&\times {}_aD_qf(tb+(1-t)a)\ {}_0d_qt\ .\end{aligned}$$

Remark.[25] In Corollary 2;

a. If we take $p = 1$ in (23), (23) reduces to equality (9).

b. If we take $p = 1$ and $q \to 1^-$ in (23), (23) reduces to (7).

c. If we take $p = 1$ and $q \to 1^-$ in (24), (24) reduces to (4).

d. If we take $p = 1$ and $q \to 1^-$ in (25), (25) reduces to (5).

Lemma 6.*[25]For $0 < q < 1$ and $0 < t < p \le 1$, the following equality holds:*

$$\int_0^a t^u(nt-m)\ {}_0d_{p,q}t \tag{28}$$

$$=(p-q)\left[\frac{na^{u+2}}{p^{u+2}-q^{u+2}}-\frac{na^{u+1}}{p^{u+1}-q^{u+1}}\right].$$

The following calculations will be used in the sequel.

$$\begin{aligned}[A]_\lambda&=\int_0^{\frac{1}{2}}|1-\lambda-qt|\ {}_0d_{p,q}t\\&=\int_0^{\frac{1-\lambda}{q}}(1-\lambda-qt)\ {}_0d_{p,q}t\\&\quad+\int_{\frac{1-\lambda}{q}}^{\frac{1}{2}}(qt-(1-\lambda))\ {}_0d_{p,q}t\\&=\frac{2(1-\lambda)^2}{q}-\frac{1-\lambda}{2}+\frac{\frac{q}{4}-\frac{2(1-\lambda)^2}{q}}{p+q}\\&=\frac{1}{4}\frac{8p+8q+16\lambda-16p\lambda-16q\lambda-8\lambda^2+8p\lambda^2}{q(p+q)}\\&\quad+\frac{1}{4}\frac{8q\lambda^2+2q^2\lambda-2pq-q^2+2pq\lambda-8}{q(p+q)}\end{aligned} \tag{29}$$

$$\begin{aligned}[N]_\lambda&=\int_0^{\frac{1}{2}}\frac{t}{p}|1-\lambda-qt|\ {}_0d_{p,q}t\\&=\int_0^{\frac{1-\lambda}{q}}\frac{t}{p}(1-\lambda-qt)\ {}_0d_{p,q}t\\&\quad+\int_{\frac{1-\lambda}{q}}^{\frac{1}{2}}\frac{t}{p}(qt-(1-\lambda))\ {}_0d_{p,q}t\\&=\frac{1}{p}\left(\frac{\frac{2(1-\lambda)^3}{q^2}-\frac{1-\lambda}{4}}{p+q}+\frac{\frac{q}{8}-\frac{2(1-\lambda)^3}{q^2}}{p^2+pq+q^2}\right)\end{aligned} \tag{30}$$

$$\begin{aligned}[A]_\mu&=\int_{\frac{1}{2}}^1|1-\mu-qt|\ {}_0d_{p,q}t\\&=\int_{\frac{1}{2}}^{\frac{1-\mu}{q}}(1-\mu-qt)\ {}_0d_{p,q}t\\&\quad+\int_{\frac{1-\mu}{q}}^1(qt-(1-\mu))\ {}_0d_{p,q}t\\&=\frac{2(1-\mu)^2}{q}-\frac{3(1-\mu)}{2}+\frac{\frac{5q}{4}-\frac{2(1-\mu)^2}{q}}{p+q}\\&=\frac{8p+8q+16\mu-16p\mu-16q\mu-8\mu^2+8p\mu^2}{4q(p+q)}\\&\quad+\frac{1}{4}\frac{8q\mu^2+6q^2\mu-6pq-q^2+6pq\mu-8}{q(p+q)}\end{aligned} \tag{31}$$

$$
\begin{aligned}
[N]_\mu &= \int_{\frac{1}{2}}^{1} \frac{t}{p}\left|1-\mu-qt\right| {}_0d_{p,q}t \\
&= \int_{\frac{1}{2}}^{\frac{1-\mu}{q}} \frac{t}{p}\left(1-\mu-qt\right) {}_0d_{p,q}t \\
&\quad + \int_{\frac{1-\mu}{q}}^{1} \frac{t}{p}\left(qt-(1-\mu)\right) {}_0d_{p,q}t \\
&= \frac{1}{p}\left(\frac{\frac{2(1-\mu)^3}{q^2}-\frac{5(1-\mu)}{4}}{p+q}+\frac{\frac{9q}{8}-\frac{2(1-\mu)^3}{q^2}}{p^2+pq+q^2}\right) \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
[K]_\lambda &= \int_0^{\frac{1}{2}} \frac{t^2}{p^2}\left|1-\lambda-qt\right| {}_0d_{p,q}t \\
&= \frac{1}{p^2}\left[\int_0^{\frac{1-\lambda}{q}} t^2\left(1-\lambda-qt\right) {}_0d_{p,q}t\right. \\
&\quad \left. + \int_{\frac{1-\lambda}{q}}^{\frac{1}{2}} t^2\left(qt-(1-\lambda)\right) {}_0d_{p,q}t\right] \\
&= \frac{p-q}{p^2}\left(\frac{\frac{q}{16}-\frac{2q(1-\lambda)^4}{q^4}}{p^4-q^4}-\frac{\frac{1-\lambda}{48}-\frac{2(1-\lambda)^4}{q^3}}{p^3-q^3}\right) \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
[K]_\mu &= \int_{\frac{1}{2}}^{1} \frac{t^2}{p^2}\left|1-\mu-qt\right| {}_0d_{p,q}t \\
&= \frac{1}{p^2}\left[\int_{\frac{1}{2}}^{\frac{1-\mu}{q}} t^2\left(1-\mu-qt\right) {}_0d_{p,q}t\right. \\
&\quad \left. + \int_{\frac{1-\mu}{q}}^{1} t^2\left(qt-(1-\mu)\right) {}_0d_{p,q}t\right] \\
&= \frac{p-q}{p^2}\left(\frac{\frac{17q}{16}-\frac{2q(1-\mu)^4}{q^4}}{p^4-q^4}-\frac{\frac{1-\mu}{4}-\frac{(1-\mu)^4}{2q^3}}{p^3-q^3}\right). \qquad (34)
\end{aligned}
$$

3 RESULTS AND DISCUSSION

In this section, we prove $(p,q)-$Hadamard-Simpson inequalities for tgs-convex functions and derive some consequences via Hermite-Hadamard and Simpson integral inequalities.

Theorem 2.*Let $f: J \to \mathbb{R}$ be a (p,q)-differantiable function on J° (the interior of J) and ${}_aD_{p,q}f$ be continuous and integrable function on J for $0<q<1$ and $0<q<p\leq 1$ and $0<t<p\leq 1$. If $\left|{}_aD_{p,q}f\right|$ is tgs-convex on J, then for $0\leq\lambda,\ \mu<1$ the following inequality holds :*

$$
\begin{aligned}
&\left|\frac{p}{b-a}\int_a^b f(x)\, {}_ad_{p,q}x\right. \\
&-p\left[(\lambda-\mu)f\left(\frac{b}{2p}+\left(1-\frac{1}{2p}\right)a\right)\right. \\
&\left.\left.+(1-\lambda)f(a)+\mu f\left(\frac{b}{p}+\left(1-\frac{1}{p}\right)a\right)\right]\right| \\
&\leq (b-a)\left(\left|{}_aD_{p,q}f(a)+\left|{}_aD_{p,q}f(b)\right|\right|\right)\left([N]_\lambda-[K]_\lambda\right) \\
&\times\left([N]_\mu-[K]_\mu\right).
\end{aligned} \qquad (35)
$$

*Proof.*Taking the absolute value on both sides of (18) and by using the tgs-convexity of $\left|{}_aD_{p,q}f\right|$, we have

$$
\begin{aligned}
&\left|\frac{p}{b-a}\int_a^b f(x)\, {}_ad_{p,q}x\right. \\
&-p\left[(\lambda-\mu)f\left(\frac{b}{2p}+\left(1-\frac{1}{2p}\right)a\right)\right. \\
&\left.\left.+(1-\lambda)f(a)+\mu f\left(\frac{b}{p}+\left(1-\frac{1}{p}\right)a\right)\right]\right| \\
&\leq (b-a)\left|\int_0^{\frac{1}{2}}(1-\lambda-qt)\right. \\
&\times {}_aD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)a\right) {}_0d_{p,q}t \\
&\left.+\int_{\frac{1}{2}}^{1}(1-\mu-qt)\, {}_bD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)b\right) {}_0d_{p,q}t\right| \\
&\leq (b-a)\int_0^{\frac{1}{2}}\left|1-\lambda-qt\right| \\
&\times\left|{}_aD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)a\right)\right| {}_0d_{p,q}t \\
&+(b-a)\int_{\frac{1}{2}}^{1}\left|1-\mu-qt\right| \\
&\times\left|D_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)b\right)\right| {}_0d_{p,q}t \\
&\leq (b-a)\left(\left|{}_aD_{p,q}f(a)\right|+\left|{}_aD_{p,q}f(b)\right|\right) \\
&\times\int_0^{\frac{1}{2}}\left|1-\lambda-qt\right|\left(1-\frac{t}{p}\right)\frac{t}{p}\, {}_0d_{p,q}t \\
&+(b-a)\left(\left|{}_aD_{p,q}f(a)\right|+\left|{}_aD_{p,q}f(b)\right|\right) \\
&\times\int_{\frac{1}{2}}^{1}\left|1-\mu-qt\right|\left(1-\frac{t}{p}\right)\frac{t}{p}\, {}_0d_{p,q}t
\end{aligned}
$$

Making the use of (30)-(34), we get

$$\left|\frac{p}{b-a}\int_a^b f(x)\,{}_a d_{p,q}x \right.$$
$$-p\left[(\lambda-\mu) f\left(\frac{b}{2p}+\left(1-\frac{1}{2p}\right)a\right)\right.$$
$$\left.\left.+(1-\lambda) f(a)+\mu f\left(\frac{b}{p}+\left(1-\frac{1}{p}\right)a\right)\right]\right|$$
$$\leq (b-a)\left(\left|{}_aD_{p,q}f(a)\right|+\left|{}_aD_{p,q}f(b)\right|\right)\left([N]_\lambda-[K]_\lambda\right)$$
$$\left([N]_\mu-[K]_\mu\right).$$

Thus, the proof is accomplished.

Corollary 4. *Under the assumption of Theorem 2*

1. *If we take $p=1$, then the inequality (35) reduces to*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,{}_a d_{p,q}x \right.$$
$$\left.-\left[(\lambda-\mu) f\left(\frac{a+b}{2}\right)+(1-\lambda) f(a)+\mu f(b)\right]\right|$$
$$\leq (b-a)\left(\left|{}_aD_{p,q}f(a)\right|+\left|{}_aD_{p,q}f(b)\right|\right)$$
$$\times\left(\left([N]_{\lambda,1}-[K]_{\lambda,1}\right)\left([N]_{\mu,1}-[K]_{\mu,1}\right)\right).$$

One can easily evaluate formulas by taking $p=1$ in (30)-(34) as follows:

$$[K]_{\lambda,1}=\frac{2q-383\lambda+q\lambda+576\lambda^2-384\lambda^3}{48(q+1)(q+q^2+1)(q^2+1)} \quad (36)$$
$$+\frac{96\lambda^4+q^2\lambda+q^3\lambda+2q^2+2q^3+95}{48(q+1)(q+q^2+1)(q^2+1)} \quad (37)$$

$$[K]_{\mu,1}=(1-q)\left[\frac{\frac{17q}{16}-\frac{2(1-\mu)^4}{q^3}}{1-q^4}-\frac{\frac{1-\mu}{4}-\frac{2(1-\mu)^4}{2q^3}}{1-q^3}\right], \quad (38)$$

$$[N]_{\lambda,1}=\frac{A}{8(q+1)(q+q^2+1)}, \quad (39)$$

$$[N]_{\mu,1}=\frac{B}{8(q+1)(q+q^2+1)} \quad (40)$$

where

$$A=-q-46\lambda+2q\lambda+48\lambda^2-16\lambda^3+2q^2\lambda-q^2+14 \quad (41)$$

$$B=-q-38\mu+10q\mu+48\mu^2-16\mu^3+10q^2\mu-q^2+6. \quad (42)$$

2. *If we take $p=1$ and $q\to 1^-$ then the inequality (35) reduces to inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,{}_a d_{q}x \right. \quad (43)$$
$$\left.-\left[(\lambda-\mu) f\left(\frac{a+b}{2}\right)+(1-\lambda) f(a)+\mu f(b)\right]\right|$$
$$\leq (b-a)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)$$
$$\times\left(\frac{-96\lambda^4+193\lambda^3-124\lambda+43}{576}\right)$$
$$+\left(\frac{64\mu^4-320\mu^3+576\mu^2-344\mu+45}{192}\right).$$

Corollary 5. *Under the assumption of Theorem 2*

1. If we take $\lambda=\frac{5}{6}$, $\mu=\frac{1}{6}$ in (35), then we have (p,q)-Simpson type integral inequality:

$$\left|\frac{p}{b-a}\int_a^b f(x)\,{}_a d_{p,q}x \right. \quad (44)$$
$$-\frac{1}{6}\left[pf(a)+4pf\left(\frac{b}{2p}+\left(1-\frac{1}{2p}\right)a\right)\right.$$
$$\left.\left.+pf\left(\frac{b}{p}+\left(1-\frac{1}{p}\right)a\right)\right]\right|$$
$$\leq (b-a)\left(\left|{}_aD_{p,q}f(a)\right|+\left|{}_aD_{p,q}f(b)\right|\right)$$
$$\times\left([N]_{\lambda=\frac{5}{6}}-[K]_{\lambda=\frac{5}{6}}\right)\left([N]_{\mu=\frac{1}{6}}-[K]_{\mu=\frac{1}{6}}\right)$$

One can easily evaluate formulas by taking $\lambda=\frac{5}{6}$ and $\mu=\frac{1}{6}$ in (30)-(34) as follows:

$$[K]_{\lambda=\frac{5}{6}}$$
$$=\frac{-4pq^2-4p^2q-153pq^5-153p^2q^4+9p^3q^3}{-2592p^2q^3(p+q)(p^2+q^2)(pq+p^2+q^2)}$$
$$-\frac{4pq+4p^2-4p^3+4q^2-4q^3-153q^6}{2592p^2q^3(p+q)(p^2+q^2)(pq+p^2+q^2)},$$

$$[K]_{\mu=\frac{1}{6}}$$
$$=\frac{-625(pq^2+p^2q)-2214(pq^5+p^2q^4+540p^3q^3}{-2592p^2q^3(p+q)(pq+p^2+q^2)(p^2+q^2)}$$
$$-\frac{2500(pq+p^2+q^2)-625(p^3+q^3)-2214q^6}{2592p^2q^3(p+q)(pq+p^2+q^2)(p^2+q^2)},$$

$$[N]_{\lambda=\frac{5}{6}}$$
$$=\frac{2(p+q-pq-p^2-q^2)-9(2pq^3+p^2q^2-2q^4)}{-216pq^2(p+q)(pq+p^2+q^2)},$$

$$[N]_{\mu=\frac{1}{6}}=\frac{G+H}{-216pq^2(p+q)(pq+p^2+q^2)} \quad (45)$$

where
$$G=250(p+q-pq-p^2-q^2) \tag{46}$$
and
$$H=-18(pq^3+q^4)+225p^2q^2. \tag{47}$$

2.If we take $\lambda=\frac{1}{2}$, $\mu=\frac{1}{2}$ in (35), then we have (p,q)-Hadamard type integral inequality:
$$(48)$$
$$\left|\frac{p}{b-a}\int_a^b f(x)\,{}_ad_{p,q}x \right. \tag{49}$$
$$\left. -p\left[\frac{f(a)+f\left(\frac{b}{p}+\left(1-\frac{1}{p}\right)a\right)}{2}\right]\right|$$
$$\leq (b-a)\left(\left|{}_aD_{p,q}f(a)\right|+\left|{}_aD_{p,q}f(b)\right|\right)$$
$$\times\left([N]_{\lambda=\frac{1}{2}}-[K]_{\lambda=\frac{1}{2}}\right)\left([N]_{\mu=\frac{1}{2}}-[K]_{\mu=\frac{1}{2}}\right).$$

One can easily evaluate formulas by taking $\lambda=\frac{1}{2}$ and $\mu=\frac{1}{2}$ in (30)-(34) as follows:
$$[K]_{\lambda=\frac{1}{2}} = \frac{-12pq^2-12p^2q-5pq^5-5p^2q^4+p^3q^3}{-96p^2q^3(p+q)(pq+p^2+q^2)(p^2+q^2)} \tag{50}$$
$$-\frac{12pq+12p^2-12p^3+12q^2-12q^3-5q^6}{96p^2q^3(p+q)(pq+p^2+q^2)(p^2+q^2)} \tag{51}$$
$$[K]_{\mu=\frac{1}{2}} = \frac{-pq^2-p^2q-30pq^5-30p^2q^4+4p^3q^3}{-32p^2q^3(p+q)(pq+p^2+q^2)(p^2+q^2)} \tag{52}$$
$$-\frac{4pq+4p^2-p^3+4q^2-q^3-30q^6}{32p^2q^3(p+q)(pq+p^2+q^2)(p^2+q^2)} \tag{53}$$
$$[N]_{\lambda=\frac{1}{2}} = \frac{2p+2q+p^2q^2-2pq-2p^2-2q^2}{-8pq^2(p+q)(pq+p^2+q^2)}, \tag{54}$$
$$[N]_{\mu=\frac{1}{2}} = \frac{C}{-8pq^2(p+q)(pq+p^2+q^2)}. \tag{55}$$
where
$$C=2(p+q)-4pq^3+5p^2q^2-2pq-2p^2-2q^2-4q^4. \tag{56}$$

3.If we take $\lambda=1$, $\mu=0$ in (35), then we have (p,q)-Hadamard type integral inequality:
$$\left|\frac{p}{b-a}\int_a^b f(x)\,{}_ad_{p,q}x \right. \tag{57}$$
$$\left. -p\left[f\left(\frac{b}{2p}+\left(1-\frac{1}{2p}\right)a\right)\right]\right|$$
$$\leq (b-a)\left(\left|{}_aD_{p,q}f(a)\right|+\left|{}_aD_{p,q}f(b)\right|\right) \tag{58}$$
$$\times\left([N]_{\lambda=1}-[K]_{\lambda=1}\right)\left([N]_{\mu=0}-[K]_{\mu=0}\right).$$

One can easily evaluate formulas by taking $\lambda=1$ and $\mu=0$ in (30)-(34) as follows:
$$[K]_{\lambda=1} = \frac{1}{16}\frac{q}{p^2(p+q)(p^2+q^2)}, \tag{59}$$
$$[K]_{\mu=0} = \frac{-8pq^2-8p^2q-13pq^5-13p^2q^4+4p^3q^3}{-16p^2q^3(p+q)(pq+p^2+q^2)(p^2+q^2)} \tag{60}$$
$$-\frac{32pq+32p^2-8p^3+32q^2-8q^3-13q^6}{16p^2q^3(p+q)(pq+p^2+q^2)(p^2+q^2)} \tag{61}$$
$$[N]_{\lambda=1} = \frac{1}{8}\frac{q}{p(pq+p^2+q^2)}, \tag{62}$$
$$[N]_{\mu=0} = \frac{E}{-8pq^2(p+q)(pq+p^2+q^2)} \tag{63}$$
where
$$E=16p+16q+pq^3+10p^2q^2-16pq-16p^2-16q^2+q^4. \tag{64}$$

Corollary 6. *In Corollary 5;*

1.If we take $p=1$ in (44), then we have q-Simpson type integral inequality:
$$\left|\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx \right. \tag{65}$$
$$\left. -\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\leq (b-a)\left(\left|{}_aD_qf(a)\right|+\left|{}_aD_qf(b)\right|\right) \tag{66}$$
$$\times\left([N]_{\lambda=\frac{5}{6},1}-[K]_{\lambda=\frac{5}{6},1}\right)\left([N]_{\mu=\frac{1}{6},1}-[K]_{\mu=\frac{1}{6},1}\right)$$

2.If we take $p=1$ in (48), then we have q-Hadamard type integral inequality:
$$\left|\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx-\frac{f(a)+f(b)}{2}\right|$$
$$\leq (b-a)\left(\left|{}_aD_qf(a)\right|+\left|{}_aD_qf(b)\right|\right) \tag{67}$$
$$\times\left([N]_{\lambda=\frac{1}{2},1}-[K]_{\lambda=\frac{1}{2},1}\right)\left([N]_{\mu=\frac{1}{2},1}-[K]_{\mu=\frac{1}{2},1}\right).$$

3.If we take $p=1$ in (58), then we have q-Hadamard type integral inequality:
$$\left|\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq (b-a)\left(\left|{}_aD_qf(a)\right|+\left|{}_aD_qf(b)\right|\right) \tag{68}$$
$$\times\left([N]_{\lambda=1,1}-[K]_{\lambda=1,1}\right)\left([N]_{\mu=0,1}-[K]_{\mu=0,1}\right).$$

One can easily evaluate the following formulas by using (30)-(34);
$$[K]_{\lambda=\frac{5}{6},1} = \frac{1}{2592}\frac{153q+153q^2+153q^3-5}{(q^2+1)(q+1)(q+q^2+1)}, \tag{69}$$
$$[K]_{\mu=\frac{1}{6},1} = \frac{1}{2592}\frac{-1875q-1875q^2+85q^31875}{q^3(q^2+1)(q+1)(q+q^2+1)} \tag{70}$$
$$+\frac{2214q^4+2214q^5+2214q^6-1875}{2592q^3(q^2+1)(q+1)(q+q^2+1)}, \tag{71}$$

$$[N]_{\lambda=\frac{5}{6},1} = \frac{1}{216}\frac{18q+18q^2-7}{(q+1)(q+q^2+1)}, \tag{72}$$

$$[N]_{\mu=\frac{1}{6},1} = \frac{1}{216}\frac{18q+18q^2+25}{(q+1)(q+q^2+1)}, \tag{73}$$

$$[K]_{\lambda=\frac{1}{2},1} = \frac{1}{96}\frac{5q+5q^2+5q^3+11}{(q^2+1)(q+1)(q+q^2+1)}, \tag{74}$$

$$[K]_{\mu=\frac{1}{2},1} = \frac{3(-q-q^2-q^3+10(q^4+q^5+q^6)-1)}{32q^3(q^2+1)(q+1)(q+q^2+1)}, \tag{75}$$

$$[N]_{\lambda=\frac{1}{2},1} = \frac{1}{8(q+1)(q^2+q+1)}, \tag{76}$$

$$[N]_{\mu=\frac{1}{2},1} = \frac{1}{8}(2q+3)\frac{2q-1}{(q+1)(q+q^2+1)}, \tag{77}$$

$$[K]_{\lambda=1,1} = \frac{1}{16}\frac{q}{(q^2+1)(q+1)}, \tag{78}$$

$$[K]_{\mu=0,1} = \frac{F}{16q^3(q^2+1)(q+1)(q+q^2+1)}, \tag{79}$$

$$[N]_{\lambda=1,1} = \frac{1}{8}\frac{q}{q^2+q+1}, \tag{80}$$

$$[N]_{\mu=0,1} = -\frac{q^4+q^3-6q^2}{8q^2(q+1)(q^2+q+1)} \tag{81}$$

where

$$F = -24q-24q^2+4q^3+13q^4+13q^5+13q^6-24. \tag{82}$$

Corollary 7. *If we take $p=1$ and $q\to 1^-$ then the inequality (35) reduces to inequality*

1) For $\lambda=\frac{5}{6}$, $\mu=\frac{1}{6}$, then we have Simpson type integral inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \tag{83}$$

$$\leq \frac{151}{7776}(b-a)\left(f'(a)+f'(b)\right) \tag{84}$$

2) For $\lambda=\frac{1}{2}$, $\mu=\frac{1}{2}$, then we have Hadamard type integral inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \tag{85}$$

$$\leq \frac{25}{96}(b-a)\left(f'(a)+f'(b)\right). \tag{86}$$

Theorem 3. *Let $f: J\to\mathbb{R}$ be a (p,q)-differantiable function on J° (the interior of J) and ${}_aD_{p,q}f$ be continuous and integrable function on J for $0<q<1$ and $0<q<p\leq 1$ and $0<t<p\leq 1$. If $\left|{}_aD_{p,q}f\right|^r$ is tgs-convex on J for $r\geqslant 1$, then for $0\leq\lambda,\ \mu<1$ the following inequality holds:*

$$\left|\frac{p}{b-a}\int_a^b f(x)\,{}_ad_{p,q}x \right. \tag{87}$$

$$-p\left[(\lambda-\mu)f\left(\frac{b}{2p}+\left(1-\frac{1}{2p}\right)a\right)\right.$$

$$\left.\left.+(1-\lambda)f(a)+\mu f\left(\frac{b}{p}+\left(1-\frac{1}{p}\right)a\right)\right]\right|$$

$$\leq (b-a)\left(\left|{}_aD_{p,q}f(a)+{}_aD_{p,q}f(b)\right|\right)$$

$$\times\left[\left([A]_\lambda\right)^{1-\frac{1}{r}}\left([N]_\lambda-[K]_\lambda\right)^{\frac{1}{r}}+\left([A]_\mu\right)^{1-\frac{1}{r}} \right. \tag{88}$$

$$\left.\times\left([N]_\mu-[K]_\mu\right)^{\frac{1}{r}}\right] \tag{89}$$

Proof. Taking absolute value on both sides of (18), applying the power mean inequality and by using the tgs-convexity of $\left|{}_aD_{p,q}f\right|^r$ for $r\geqslant 1$, we have

$$\left|\frac{p}{b-a}\int_a^b f(x)\,{}_ad_{p,q}x \right. \tag{90}$$

$$-p\left[(\lambda-\mu)f\left(\frac{b}{2p}+\left(1-\frac{1}{2p}\right)a\right)\right. \tag{91}$$

$$\left.\left.+(1-\lambda)f(a)+\mu f\left(\frac{b}{p}+\left(1-\frac{1}{p}\right)a\right)\right]\right| \tag{92}$$

$$\leq (b-a)\left|\int_0^{\frac{1}{2}}(1-\lambda-qt)\right. \tag{93}$$

$$\times\,{}_aD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)a\right)\,{}_0d_{p,q}t \tag{94}$$

$$+\int_{\frac{1}{2}}^{1}(1-\mu-qt) \tag{95}$$

$$\left.\times\,{}_bD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)b\right)\,{}_0d_{p,q}t\right| \tag{96}$$

$$\tag{97}$$

$$\leq (b-a)\left(\int_0^{\frac{1}{2}} |1-\lambda-qt|\ {}_0d_{p,q}t\right)^{1-\frac{1}{r}} \tag{98}$$

$$\times\left(\int_0^{\frac{1}{2}} |1-\lambda-qt|\right.$$

$$\left.\times\left|{}_aD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)a\right)\right|^r {}_0d_{p,q}t\right)^{\frac{1}{r}} \tag{99}$$

$$+(b-a)\left(\int_0^{\frac{1}{2}} |1-\mu-qt|\ {}_0d_{p,q}t\right)^{1-\frac{1}{r}} \tag{100}$$

$$\times\left(\int_0^{\frac{1}{2}} |1-\mu-qt|\right.$$

$$\left.\times\left|{}_aD_{p,q}f\left(\frac{tb}{p}+\left(1-\frac{t}{p}\right)a\right)\right|^r {}_0d_{p,q}t\right)^{\frac{1}{r}} \tag{101}$$

$$\leq (b-a)\left(\int_0^{\frac{1}{2}} |1-\lambda-qt|\ {}_0d_{p,q}t\right)^{1-\frac{1}{r}} \tag{102}$$

$$\times\left(\left|{}_aD_{p,q}f(a)+{}_aD_{p,q}f(b)\right|^r\right.$$

$$\left.\times\int_0^{\frac{1}{2}} |1-\lambda-qt|\left(1-\frac{t}{p}\right)\frac{t}{p}\ {}_0d_{p,q}t\right)^{\frac{1}{r}}$$

$$+(b-a)\left(\int_0^{\frac{1}{2}} |1-\mu-qt|\ {}_0d_{p,q}t\right)^{1-\frac{1}{r}} \tag{103}$$

$$\times\left(\left|{}_aD_{p,q}f(a)+{}_aD_{p,q}f(b)\right|^r\right.$$

$$\left.\times\int_0^{\frac{1}{2}} |1-\mu-qt|\left(1-\frac{t}{p}\right)\frac{t}{p}\ {}_0d_{p,q}t\right)^{\frac{1}{r}} \tag{104}$$

Making the use of (29)-(34), we get

$$\left|\frac{p}{b-a}\int_a^b f(x)\ {}_ad_{p,q}x\right. \tag{105}$$

$$-p\left[(\lambda-\mu)f\left(\frac{b}{2p}+\left(1-\frac{1}{2p}\right)a\right)\right. \tag{106}$$

$$\left.\left.(1-\lambda)f(a)+\mu f\left(\frac{b}{p}+\left(1-\frac{1}{p}\right)a\right)\right]\right| \tag{107}$$

$$\leq (b-a)\left|{}_aD_{p,q}f(a)+{}_aD_{p,q}f(b)\right| \tag{108}$$

$$\times\left[([A]_\lambda)^{1-\frac{1}{r}}([N]_\lambda-[K]_\lambda)^{\frac{1}{r}}+\left([A]_\mu\right)^{1-\frac{1}{r}}\right. \tag{109}$$

$$\left.\times\left([N]_\mu-[K]_\mu\right)^{\frac{1}{r}}\right]. \tag{110}$$

Thus, the proof is accomplished.

Corollary 8. *Under the assumption of Theorem 3*

1. *If we take $p=1$, then the inequality (89) reduces to*

$$\left|\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\right. \tag{111}$$

$$\left.-\left[(\lambda-\mu)f\left(\frac{a+b}{2}\right)+(1-\lambda)f(a)+\mu f(b)\right]\right| \tag{112}$$

$$\leq (b-a)\left|{}_aD_qf(a)+{}_aD_qf(b)\right| \tag{113}$$

$$\times\left[\left([A]_{\lambda,1}\right)^{1-\frac{1}{r}}\left([N]_{\lambda,1}-[K]_{\lambda,1}\right)^{\frac{1}{r}}+\left([A]_{\mu,1}\right)^{1-\frac{1}{r}}\right. \tag{114}$$

$$\left.\left([N]_{\mu,1}-[K]_{\mu,1}\right)^{\frac{1}{r}}\right]. \tag{115}$$

One can easily evaluate formulas by taking $p=1$ in (29) and (31) as follows:

$$[A]_{\lambda,1}=\frac{1}{4}\frac{-q-14\lambda+2q\lambda+8\lambda^2+6}{q+1} \tag{116}$$

$$[A]_{\mu,1}=\frac{1}{4}\frac{-q-10\mu+6q\mu+8\mu^2+2}{q+1} \tag{117}$$

2. *If we take $p=1$ and $q\to 1^-$, we have*

(118)

$$\left|\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\right. \tag{119}$$

$$\left.-\left[(\lambda-\mu)f\left(\frac{a+b}{2}\right)+(1-\lambda)f(a)+\mu f(b)\right]\right|$$

$$\leq (b-a)\left|f'(a)+f'(b)\right|\left(\tfrac{8\lambda^2-12\lambda+5}{2}\right)^{1-\frac{1}{r}} \tag{120}$$

$$\times\left(\tfrac{-96\lambda^4-193\lambda^3-124\lambda+43}{576}\right)^{\frac{1}{r}}$$

$$+(b-a)\left(\tfrac{8\mu^2-4\mu+1}{8}\right)^{1-\frac{1}{r}} \tag{121}$$

$$\times\left(\tfrac{64\mu^4-320\mu^3+576\mu^2-344\mu+45}{192}\right)^{\frac{1}{r}}.$$

Corollary 9. *Under the assumption of Theorem 3*

1. If we take $\lambda=\frac{5}{6}$, $\mu=\frac{1}{6}$ in (89), then we have (p,q)-Simpson type integral inequality:

(122)

$$\left|\frac{p}{b-a}\int_a^b f(x)\ {}_ad_{p,q}x\right.$$

$$-\frac{1}{6}\left[pf(a)+4pf\left(\frac{b+a(2p-1)}{2p}\right)\right. \tag{123}$$

$$\left.\left.+pf\left(\frac{b+a(p-1)}{p}\right)\right]\right|$$

$$\leq (b-a)\left|{}_aD_{p,q}f(a)+{}_aD_{p,q}f(b)\right|\left[\left([A]_{\lambda=\frac{5}{6}}\right)^{1-\frac{1}{r}}\right.$$

$$\times\left([N]_{\lambda=\frac{5}{6}}-[K]_{\lambda=\frac{5}{6}}\right)^{\frac{1}{r}}$$

$$\left.+\left([A]_{\mu=\frac{5}{6}}\right)^{1-\frac{1}{r}}\left([N]_{\mu=\frac{5}{6}}-[K]_{\mu}\right)^{\frac{1}{r}}\right]$$

One can easily evaluate formulas by taking $\lambda = \frac{5}{6}$ and $\mu = \frac{1}{6}$ in (29) and (31) as follows:

$$[A]_{\lambda=\frac{5}{6}} = -\frac{1}{36}\frac{-2p-2q+3pq-6q^2+2}{q(p+q)} \tag{124}$$

$$[A]_{\mu=\frac{1}{6}} = -\frac{5}{36}\frac{-10p-10q+9pq+10}{q(p+q)} \tag{125}$$

2.If we take $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$ in (89), then we have (p,q)-Hadamard type integral inequality:

$$\begin{aligned} &\left|\frac{p}{b-a}\int_a^b f(x)\,{}_ad_{p,q}x \right. \\ &\left. -p\left[\frac{f(a)+f\left(\frac{b+a(p-1)}{p}\right)}{2}\right]\right| \\ &\le (b-a)\left|{}_aD_{p,q}f(a)+{}_aD_{p,q}f(b)\right|\left[\left([A]_{\lambda=\frac{1}{2}}\right)^{1-\frac{1}{r}}\right. \\ &\times\left([N]_{\lambda=\frac{1}{2}}-[K]_{\lambda=\frac{1}{2}}\right)^{\frac{1}{r}} \\ &\left.+\left([A]_{\mu=\frac{1}{2}}\right)^{1-\frac{1}{r}}\left([N]_{\mu=\frac{1}{2}}-[K]_{\mu=\frac{1}{2}}\right)^{\frac{1}{r}}\right]. \end{aligned} \tag{126}$$

One can easily evaluate formulas by taking $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{2}$ in (29) and (31) as follows:

$$[A]_{\lambda=\frac{1}{2}} = -\frac{1}{4}\frac{-2p-2q+pq+2}{q(p+q)} \tag{127}$$

$$[A]_{\mu=\frac{1}{2}} = -\frac{1}{4}\frac{-2p-2q+3pq-2q^2+2}{q(p+q)} \tag{128}$$

3.If we take $\lambda = 1$, $\mu = 0$ in (89), then we have (p,q)-Hadamard type integral inequality:

$$\begin{aligned} &\left|\frac{p}{b-a}\int_a^b f(x)\,{}_ad_{p,q}x - p\left[f\left(\frac{b+(2p-1)a}{2p}\right)\right]\right| \\ &\le (b-a)\left|{}_aD_{p,q}f(a)+{}_aD_{p,q}f(b)\right|\left[([A]_{\lambda=1})^{1-\frac{1}{r}}\right. \\ &\times([N]_{\lambda=1}-[K]_{\lambda=1})^{\frac{1}{r}} \\ &\left.+\left([A]_{\mu=0}\right)^{1-\frac{1}{r}}\left([N]_{\mu=0}-[K]_{\mu=0}\right)^{\frac{1}{r}}\right] \end{aligned} \tag{129}$$

One can easily evaluate formulas by taking $\lambda = 1$ and $\mu = 0$ in (29) and (31) as follows:

$$[A]_{\lambda=1} = \frac{1}{4}\frac{q}{p+q} \tag{130}$$

$$[A]_{\mu=0} = -\frac{6pq-8q-8p+q^2+8}{4q(p+q)} \tag{131}$$

Corollary 10. *In Corollary 9;*

1.If we take $p=1$ in (122), then we have q-Simpson type integral inequality:

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx - \frac{f(a)+f(b)}{2}\right| \\ &\le (b-a)\left|{}_aD_qf(a)+{}_aD_qf(b)\right|\left[\left([A]_{\lambda=\frac{5}{6},1}\right)^{1-\frac{1}{r}}\right. \\ &\times\left([N]_{\lambda=\frac{5}{6},1}-[K]_{\lambda=\frac{5}{6},1}\right)^{\frac{1}{r}} \\ &\left.+\left([A]_{\mu=\frac{5}{6},1}\right)^{1-\frac{1}{r}}\left([N]_{\mu=\frac{5}{6},1}-[K]_{\mu,1}\right)^{\frac{1}{r}}\right] \end{aligned} \tag{132}$$

2.If we take $p=1$ in (126), then we have q-Hadamard type integral inequality:

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx - \left[\frac{f(a)+f(b)}{2}\right]\right| \\ &\le (b-a)\left|{}_aD_qf(a)+{}_aD_qf(b)\right|\left[\left([A]_{\lambda=\frac{1}{2},1}\right)^{1-\frac{1}{r}}\right. \\ &\times\left([N]_{\lambda=\frac{1}{2},1}-[K]_{\lambda=\frac{1}{2},1}\right)^{\frac{1}{r}} \\ &\left.+\left([A]_{\mu=\frac{1}{2},1}\right)^{1-\frac{1}{r}}\left([N]_{\mu=\frac{1}{2},1}-[K]_{\mu=\frac{1}{2},1}\right)^{\frac{1}{r}}\right]. \end{aligned} \tag{133}$$

3.If we take $p=1$ in (129), then we have q-Hadamard type integral inequality:

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx - f\left(\frac{a+b}{2}\right)\right| \\ &\le (b-a)\left|{}_aD_qf(a)+{}_aD_qf(b)\right|\left[\left([A]_{\lambda=1,1}\right)^{1-\frac{1}{r}}\right. \\ &\times\left([N]_{\lambda=1,1}-[K]_{\lambda=1,1}\right)^{\frac{1}{r}} \\ &\left.+\left([A]_{\mu=0,1}\right)^{1-\frac{1}{r}}\left([N]_{\mu=0,1}-[K]_{\mu=0,1}\right)^{\frac{1}{r}}\right]. \end{aligned} \tag{134}$$

Corollary 11. *If we take $p=1$ and $q\to 1^-$ then the inequality (35) reduces to inequality*

1) For $\lambda = \frac{5}{6}$, $\mu = \frac{1}{6}$, then we have Simpson type integral inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \tag{135}$$

$$\le (b-a)\left|f'(a)+f'(b)\right|\left(\frac{5}{72}\right)^{1-\frac{1}{r}} \tag{136}$$

$$\times\left[\left(\frac{121}{15552}\right)^{\frac{1}{r}}+\left(\frac{181}{15552}\right)^{\frac{1}{r}}\right] \tag{137}$$

2) For $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$, then we have Hadamard type integral inequality:

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2} \right| \tag{138}$$

$$\leq (b-a) \left| f'(a) + f'(b) \right| \left(\frac{1}{8} \right)^{1-\frac{1}{r}} \tag{139}$$

$$\times \left[\left(\frac{5}{192} \right)^{\frac{1}{r}} + \left(\frac{15}{64} \right)^{\frac{1}{r}} \right]. \tag{140}$$

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