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On Real Quadratic Number Fields Related With A Specific Type of Continued Fraction Expansions

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Abstract: The present paper deals with classifying the real quadratic number fields $k = Q(\sqrt{d})$ having specific continued fraction expansion of the integral basis element where $d \equiv 2, 3(mod4)$ is a square free positive integer. Certain parametric representations are determined to calculate fundamental unit $\epsilon_d = \left(t_d + u_d\sqrt{d}\right)/2 \rangle 1$ of such real quadratic number fields as well as the parametrized forms of d. Moreover, Yokoi's d-invariants n_d and m_d in the relation to continued fraction expansion of w_d are mentioned by using coefficients of fundamental unit for such real quadratic fields. All results are concluded in the tables.

Keywords: Continued Fraction Expansion, Fundamental Unit, Quadratic Fields.

1 Introduction

The unit group of real quadratic fields has got nontrivial structure. The fundamental unit $\epsilon_d = (t_d + u_d\sqrt{d})/2\rangle 1$ of the ring of algebraic integers in a real quadratic number field $Q(\sqrt{d})$ is a generator of the group of units.
In [11], K.Tomita and K.Yamamuro gave some results for fundamental unit ϵ_d by using Fibonacci sequence and continued fraction, and he also determined the continued fraction expansion of w_d where $d \equiv 1(mod4)$ for $l(d) = 3$ in [10].The theorem of C. Friesen in [1] and F. Halter-Koch in [2] was examined a construction of infinite families of real quadratic fields with large fundamental units. In recent works, F.Kawamoto and K.Tomita determined minimal type of Continued fraction for certain real quadratic fields in [3]. Moreover, R.Sasaki [8] and R.A. Mollin [5] studied on lower bound of fundamental unit ϵ_d of $k = Q(\sqrt{d})$ and got certain important results. H.Yokoi defined two invariants important for class number problem and solutions of Pell equation by using coefficients of fundamental unit in [13]-[16]. You can also see Perron [7], Sierpinski [9] and Williams et al [12] references for getting more information about continued fraction expansions.
Let $k = Q(\sqrt{d})$ be a real quadratic number field where $d > 0$ is a positive square free integer. Integral basis element is denoted by $\omega_d = [a_0, \overline{a_1, ..., a_{l-1}, 2a_0}]$ where $a_1, a_2, \ldots, a_{l(d)-1}$ are partial quotients and $l(d)$ is the period length in simple continued fraction expansion of algebraic integer for $d \equiv 2, 3(mod4)$. Besides, Yokoi's invariants m_d and n_d are expressed by coefficients of fundamental unit and (S_n) is also a sequence which is defined in Section 2.

In this paper, the problem will be investigated of determining systematically the continued fraction expansions which have constant elements as $3s$ (except the last digit of the period) with a given period length. There are infinitely many values of d having all $3s$ in the symmetric part of period length. We will classify them with regard to arbitrary period length where $d \equiv 2, 3(mod4)$ is a square free positive integer.At the same time, another aim for this paper is to determine the general forms of fundamental units ϵ_d and t_d , u_d coefficents of fundamental units by using this new formulizations which have been unknown yet. The fundamental unit and Yokoi's invariants are calculated more easily for d square free integers by using these formulizations. Finally, some results such as Yokoi's invariants n_d and m_d, fundamental unit ϵ_d and continued fraction expansion of w_d will be obtained with the tables in Section 3.

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2 Preliminaries

We will begin by defining some fundamental concepts and lemmas as follow. First, we need to define a recursively sequence which will be useful in our main results for the next section.

Definition 1. *$\{S_i\}$ is said to be a sequence defined by recurence relation*

$$S_i = 3S_{i-1} + S_{i-2}$$

for $i \geq 2$, where $S_0 = 0$ and $S_1 = 1$.

Definition 2. *Let $c_n = ac_{n-1} + bc_{n-2}$ recurrence relation of $\{c_n\}$ sequence where a, b are real numbers. The polynomial is called as a characteristic equation written in the form:*

$$x^2 - ax - b = 0$$

The solutions will depend on the nature of the roots of the characteristic equation for recurence relation. By using the definition, we can find characteristic equation as

$$x^2 - 3x - 1 = 0$$

for $\{S_k\}$ sequence. So, we can write each element of sequence as follows:

$$S_k = \frac{1}{\sqrt{13}} \left[\left(\frac{3+\sqrt{13}}{2} \right)^k - \left(\frac{3-\sqrt{13}}{2} \right)^k \right]$$

for $k \geq 0$.

Remark. Let $\{S_n\}$ be the sequence defined as in Definition 1. Then, we state that:

$$S_n \equiv \begin{cases} 0\,(mod4)\,, \; n \equiv 0\,(mod6); \\ 1\,(mod4)\,, \; n \equiv 1,4,5\,(mod6); \\ 3\,(mod4)\,, \; n \equiv 2\,(mod6); \\ 2\,(mod4)\,, \; n \equiv 3\,(mod6). \end{cases}$$

where $n \geq 0$.

Lemma 1. *For a square-free positive integer d congruent to $2,3$ modulo 4, we put $\omega_d = \sqrt{d}$, $a_0 = \lfloor \omega_d \rfloor$ into the $\omega_R = a_0 + \omega_d$. Then $\omega_d \notin R(d)$, but $\omega_R \in R(d)$ holds. Moreover for the period $l = l(d)$ of ω_R, we get $\omega_R = \overline{[2a_0, a_1, ..., a_{l-1}]}$ and $\omega_d = [a_0, \overline{a_1, ..., a_{l-1}, 2a_0}]$. Furthermore, let $\omega_R = \frac{(P_l \omega_R + P_{l-1})}{(Q_l \omega_R + Q_{l-1})} = [2a_0, a_1, ..., a_{l-1}, \omega_R]$ be a modular automorphism of ω_R, then the fundamental unit ϵ_d of $Q\left(\sqrt{d}\right)$ is given by the following formula:*

$$\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{\ell(d)} + Q_{\ell(d)-1}\rangle 1$$

and

$$t_d = 2a_0.Q_{\ell(d)} + 2Q_{\ell(d)-1}, \quad u_d = 2Q_{\ell(d)}.$$

where Q_i is determined by $Q_0 = 0$, $Q_1 = 1$ and $Q_{i+1} = a_i Q_i + Q_{i-1}$, $(i \geq 1)$.

Proof. Proof is omitted in [10].

Lemma 2. *Let d be the square free positive integer congruent to $2,3$ modulo 4 and a_0 denote the $a_0 = \left[\left[\sqrt{d}\right]\right]$ the integer part of w_d. If we consider w_d which has got partial constant elements repeated 3s in the case of period $l = l(d)$, then we have continued fraction expansions*

$$w_d = \sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_{\ell(d)-1}, a_{\ell(d)}}] = [a_0; \overline{3, 3, \ldots, 3, 2a_0}]$$

for quadratic irrational numbers and $w_R = a_0 + \sqrt{d} = \overline{[2a_0, 3, 3, \ldots, 3]}$ for reduced quadratic irrational numbers.

Furthermore, $A_j = a_0 S_{j+1} + S_j$ and $B_j = S_{j+1}$ are determined in the continued fraction expansion of w_d where $\{A_j\}$ and $\{B_j\}$ are two sequences defined by:

$$A_{-2} = 0, \quad A_{-1} = 1, \quad A_j = a_j A_{j-1} + A_{j-2},$$

$$B_{-2} = 1, \quad B_{-1} = 0, \quad B_j = a_j B_{j-1} + B_{j-2},$$

for $0 \leq j < l(d)$ and $l(d)$ is period length of w_d. Also, $C_j = A_j/B_j$ is the j^{th} convergent in the continued fraction expansion of $\sqrt{d}$. Moreover, $A_l = 2a_0^2 S_l + 3a_0 S_{l-1} + S_{l-2}$ and $B_l = 2a_0 S_l + S_{l-1}$ for $j = l(d)$

Besides, in the continued fraction expansion of $w_R = a_0 + \sqrt{d} = [b_1, b_2, \ldots, b_n, ...] = [2a_0, 3, \ldots, 3, ...]$, we obtain $P_k = 2a_0 S_k + S_{k-1}$ and $Q_k = S_k$ where $\{P_k\}$ and $\{Q_k\}$ are two sequences defined by:

$$P_{-1} = 0, \quad P_0 = 1, \quad P_{k+1} = b_{k+1}.P_k + P_{k-1},$$

$$Q_{-1} = 1, \quad Q_0 = 0, \quad Q_{k+1} = b_{k+1}.Q_k + Q_{k-1},$$

for $k \geq 0$.

Proof. We can prove by using mathematical induction. Using the following table which includes values of A_k, B_k and a_k we can determine converge of

$$w_d = [a_0; \overline{3, 3, \ldots, 3, 2a_0}]$$

for $l(d) > 4$ .So, we can easily say that this is true for $k = 0$.

Table 2.1

k	-2	-1	0	1	2	3	4	5
a_k			a_0	3	3	3	3	....
A_k	0	1	(a_0) $a_0 S_1$	$(3a_0+1)$ $a_0 S_2 + S_1$	$(10a_0+3)$ $a_0 S_3 + S_2$	$(33a_0+10)$ $a_0 S_4 + S_3$	$(109a_0+33)$ $a_0 S_5 + S_4$	....
B_k	1	0	1 S_1	3 S_2	10 S_3	33 S_4	109 S_5	....

Now, we suppose that the result is true for $k < i$ and $0 < i \leq l - 1$ .Using the defined relations for $\{S_i\}$ sequence, we obtain

$$\begin{aligned} A_{k+1} &= a_{k+1}A_k + A_{k-1} = 3\left(a_0 S_{k+1} + S_k\right) + \left(a_0 S_k + S_{k-1}\right) \\ &= a_0\left(3S_{k+1} + S_k\right) + \left(3S_k + S_{k-1}\right) \\ &= a_0 S_{k+2} + S_{k+1} \end{aligned}$$

We can also get the following:

$$B_{k+1} = a_{k+1}B_k + B_{k-1} = 3S_{k+1} + S_k = S_{k+2}$$

Moreoever, since $a_l = 2a_0$, we obtain $A_l = 2a_0^2 S_l + 3a_0 S_{l-1} + S_{l-2}$ and $B_l = 2a_0.S_l + S_{l-1}$ for $k = l(d)$ in an easy way.

In a similar way in the continued fraction $a_0 + \sqrt{d} = [b_1, b_2, \ldots, b_n, ...] = [2a_0, 3, \ldots, 3, ...]$, we obtain $P_k = 2a_0 S_k + S_{k-1}$ and $Q_k = S_k$ for $k \geq 0$. This completes the proof.

3 Main Theorem and Results

First, we will give a main theorem allows us to determine real quadratic fields which include $w_d = \sqrt{d} = [\, a_0; \overline{3, 3, \ldots, 3,\ 2a_0}\,]$ where $l = \ell(d)$ is a period length and $d \equiv 2, 3(mod4)$ is square free integer.

Theorem 1. *Let d be the square free positive integer and $\ell > 1$ be a positive integer holding that ℓ is not congruent to $0(mod3)$. We assume that parametrization of d is*

$$d = \frac{(3 + (2\beta + 1)S_\ell)^2}{4} + ((2\beta + 1)S_{\ell-1}) + 1$$

for any $\beta \geq 0$ positive integer.Then following conditions hold:

(1) If $\ell \equiv 1(mod6)$ and $\beta \equiv 1(mod2)$ are positive integers then $d \equiv 2(mod4)$.
(2) If $\ell \equiv 2(mod6)$ and $\beta \equiv 0(mod2)$ are positive integers then $d \equiv 3(mod4)$.
(3) If $\ell \equiv 4(mod6)$ and $\beta \equiv 0(mod2)$ are positive integers then $d \equiv 3(mod4)$.
(4) If $\ell \equiv 5(mod6)$ and $\beta \equiv 0(mod2)$ are positive integers then $d \equiv 2(mod4)$.
In real quadratic fields, we obtain

$$w_d = \left[\frac{(2\beta + 1)S_\ell + 3}{2}; \overline{\underbrace{3, 3, \ldots, 3}_{\ell-1}, (2\beta + 1)S_\ell + 3} \right]$$

and $\ell = \ell(d)$.Moreover, we have following equalities:

$$\epsilon_d = \left(\frac{(2\beta + 1)S_\ell^2}{2} + \frac{3S_\ell}{2} + S_{\ell-1} \right) + S_\ell\sqrt{d}$$

$$t_d = (2\beta + 1)S_\ell^2 + 3S_\ell + 2S_{\ell-1} \quad \text{and} \quad u_d = 2S_\ell$$

for ϵ_d, t_d and u_d.

Proof. It is clear that d is not integer for $\ell \equiv 0(mod3)$ since Remark. That's why we assume that is not divided by 3 and $\ell \geq 2$. First of all, we should show that four conditions hold as the followings:

(1) if $\ell \equiv 1(mod6)$ and β is odd positive integer, then $S_\ell \equiv 1(mod4)$ and $S_{\ell-1} \equiv 0(mod4)$ hold. By substituting these values into parametrization of d, we obtain $d \equiv 2(mod4)$.
(2)If $\ell \equiv 2(mod6)$ and β is even positive integer, then $S_\ell \equiv 3(mod4)$ and $S_{\ell-1} \equiv 1(mod4)$. By substituting these values into parametrization of d and rearranging, we have $d \equiv 3(mod4)$.
(3)If $\ell \equiv 4(mod6)$ and β is even positive integer, then we have $S_\ell \equiv 1(mod4)$ and $S_{\ell-1} \equiv 2(mod4)$. By substituting these values into parametrization of d and rearranging, we have $d \equiv 3(mod4)$.
(4)If $\ell \equiv 5(mod6)$ and β is even positive integer then we get $S_\ell \equiv 1(mod4)$ and $S_{\ell-1} \equiv 1(mod4)$. By substituting these values into parametrization of d and rearranging, we have $d \equiv 2(mod4)$.
By using Lemma 1 , we get

$$w_R = \frac{(2\beta + 1)S_\ell + 3}{2} + \left[\frac{(\beta + 1)S_\ell + 3}{2}; \overline{\underbrace{3, 3, \ldots, 3}_{\ell-1}, (2\beta + 1)S_\ell + 3} \right],$$

so we get

$$w_R = ((2\beta + 1)S_\ell + 3) + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{\ddots + \cfrac{1}{3 + \cfrac{1}{w_R}}}}} = ((2\beta+1)S_\ell+3) + \frac{1}{3} + \cdots + \frac{1}{w_R}$$

Using Lemma 1, Lemma 2 and the properties of continued fraction expansion, we obtain

$$w_R = ((2\beta + 1)S_\ell + 3) + \frac{S_{\ell-1}w_R + S_{\ell-2}}{S_\ell w_R + S_{\ell-1}},$$

By rearranging and using the Definition 1 into the above equality, we have

$$w_R^2 - ((2\beta + 1)S_\ell + 3)\, w_R - (1 + (2\beta + 1)S_{\ell-1}) = 0.$$

This requires that $w_R = \frac{(2\beta+1)S_\ell+3}{2} + \sqrt{d}$ since $w_R > 0$. If we consider Lemma 1, we get

$$w_d = \sqrt{d} = \left[\frac{(2\beta + 1)S_\ell + 3}{2}; \overline{\underbrace{3, 3, \ldots, 3}_{\ell-1}, (2\beta + 1)S_\ell + 3} \right]$$

and $\ell = \ell(d)$. This shows that the part of first proof is completed.

Now, we should determine ϵ_d, t_d and u_d using Lemma 1, we can get easily

$$Q_1 = 1 = S_1, \quad Q_2 = a_1.Q_1 + Q_0 \Rightarrow Q_2 = 3 = S_2,$$

$$Q_3 = a_2 Q_2 + Q_1 = 3S_2 + S_1 = 3^2 + 1 = 10 = S_3,\ Q_4 = S_4, \ldots$$

So, this implies that $Q_i = S_i$ for $\forall i \geq 0$. If we substitute these values of sequence into the $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{l(d)} + Q_{l(d)-1}\rangle 1$ and rearranged, we have

$$\epsilon_d = \left(\frac{(2\beta + 1)S_\ell^2}{2} + \frac{3S_\ell}{2} + S_{\ell-1} \right) + S_\ell\sqrt{d}$$

$$t_d = (2\beta+1)S_\ell^2 + 3S_\ell + 2S_{\ell-1} \quad \text{and} \quad u_d = 2S_\ell$$

for ϵ_d, t_d and u_d which complete the proof of Theorem 1.

*Remark.*We should say that the present paper has got the most general results for such type real quadratic fields. Also, we can obtaine infinitely many values of d which correspond to new real quadratic fields $Q(\sqrt{d})$ by using our results.

Corollary 1.*Let d be the square free positive integer, $\ell > 1$ be a positive integer not divided by 3 and holding ℓ is not congruent to $1(mod 6)$. Suppose that the parametrization of d is*

$$d = \frac{(3+S_\ell)^2}{4} + S_{\ell-1} + 1$$

then we obtain $d \equiv 2, 3(mod 4)$ and

$$w_d = \left[\frac{S_\ell + 3}{2}; \overline{\underbrace{3, 3, \ldots, 3}_{\ell-1}, S_\ell + 3}\right]$$

and $\ell = \ell(d)$.Additionaly, we get the fundamental unit and ts coefficients as follows:

$$\epsilon_d = \left(\frac{S_\ell^2}{2} + \frac{3S_\ell}{2} + S_{\ell-1}\right) + S_\ell\sqrt{d}$$

$$t_d = S_\ell^2 + S_{\ell+1} + S_{\ell-1} \quad \text{and} \quad u_d = 2S_\ell$$

as well as Yokoi's invariant

$$m_d = \begin{cases} 1, \text{ if } \ell = 2; \\ 3, \text{ if } \ell \geq 4. \end{cases}$$

.

Furthermore, we state the following Table 3.1 where fundamental unit is ϵ_d, integral basis element is w_d and Yokoi's invariant is m_d for $2 \leq \ell(d) \leq 14$.

*Proof.*This corollary is obtained if we substitute $\beta = 0$ into Theorem 1. Now, we have to determine the value of m_d. We know that

$$m_d = \left[\left[\frac{u_d^2}{t_d}\right]\right]$$

since H.Yokoi references. If we substitue t_d and u_d into the m_d and rearranged, then we get

$$m_d = \left[\left[\frac{u_d^2}{t_d}\right]\right] = \left[\left[\frac{4S_\ell^2}{S_\ell^2 + 3S_\ell + 2S_{\ell-1}}\right]\right]$$

Using above equality, we have $m_d = 1$, for $\ell = 2$. From assumption and S_ℓ is increasing sequence, we obtain

$$4 > 4.\left(1 + \frac{3}{S_\ell} + \frac{2S_{\ell-1}}{S_\ell^2}\right)^{-1} > 3,605$$

for $\ell \geq 4$.Therefore, we have

$$m_d = \begin{cases} 1, \text{ if } \ell = 2; \\ 3, \text{ if } \ell \geq 4. \end{cases}$$

Besides, Table 3.1 is obtained as an illustrate of the corollary.

Corollary 2.*Let d be the square free positive integer and $\ell > 1$ be a positive integer such that $\ell \equiv 1(mod 6)$.If we suppose that parametrization of d is*

$$d = \frac{(1+3S_\ell)^2}{4} + S_{\ell+1} + 2S_{\ell-1} + 3$$

then we obtain $d \equiv 2(mod 4)$ and

$$w_d = \left[\frac{3S_\ell + 3}{2}; \overline{\underbrace{3, 3, \ldots, 3}_{\ell-1}, 3S_\ell + 3}\right]$$

and $\ell = \ell(d)$.Moreover, we have following equalities:

$$\epsilon_d = \left(\frac{3S_\ell^2}{2} + \frac{3S_\ell}{2} + S_{\ell-1}\right) + S_\ell\sqrt{d}$$

$$t_d = 3S_\ell^2 + S_{\ell+1} + S_{\ell-1} \quad \text{and} \quad u_d = 2S_\ell$$

$$m_d = 1$$

for ϵ_d, t_d, u_d and Yokoi's invariant m_d. Additionaly, we state the following Table 3.2. where fundamental unit is ϵ_d, integral basis element is w_d and Yokoi's invariant is m_d for $2 < \ell(d) \leq 13$.

*Proof.*The corollary is obtained for $\beta = 1$ using Theorem 1. We should prove that $m_d = 1$ where $\ell \equiv 1(mod 6)$ for such parametrization of d. If we put t_d and u_d into the m_d and rearranged, then we obtain

$$2 > 4.\left(3 + \frac{3}{S_\ell} + \frac{2S_{\ell-1}}{S_\ell^2}\right)^{-1} > 1,331$$

for $\ell \equiv 1(mod 6)$ since S_ℓ is increasing sequence.By using the above equality, we have $m_d = 1$. Also, Table 3.2. is given as an illustrate of this corollary.

Corollary 3.*We assume that d is square free integer and ℓ is different from $1(mod 6)$ defined as in Theorem 1. If we choose the parametrization of d as*

$$d = \frac{(1+5S_\ell)^2}{4} + 5S_\ell + 5S_{\ell-1} + 3$$

then $d \equiv 2, 3(mod 4)$ and

$$w_d = \left[\frac{5S_\ell + 3}{2}; \overline{\underbrace{3, 3, \ldots, 3}_{\ell-1}, 5S_\ell + 3}\right]$$

with $\ell = \ell(d)$.Moreover, we get following equalities:

Table 3.1

d	$\ell(d)$	m_d	w_d	ϵ_d
11	2	1	$[3;\overline{3,6}]$	$10+3\sqrt{11}$
335	4	3	$[18;\overline{3,3,3,36}]$	$604+33\sqrt{335}$
3170	5	3	$[56;\overline{3,3,3,3,112}]$	$6137+109\sqrt{3170}$
3862415	8	3	$[1965;\overline{3,3,...3,3930}]$	$7717744+3927\sqrt{3862415}$
458829371	10	3	$[21420;\overline{3,3,...,3,42480}]$	$917581510+42837\sqrt{458829371}$
5004473402	11	3	$[70742;\overline{3,3,...,3,141484}]$	$10008691739+141481\sqrt{5004473402}$
6495480739451	14	3	$[2548623;\overline{3,3,...,3,5097246}]$	$12990952289710+5097243\sqrt{6495480739451}$

Table 3.2

d	$\ell(d)$	m_d	w_d	ϵ_d
3187306	7	1	$[1785;\overline{3,3,...,3,3570}]$	$2122725+1189\sqrt{3187306}$
5359147692130	13	1	$[2314982;\overline{3,3,...,3,4629966}]$	$3572762345823+1543321\sqrt{5359147692130}$

Table 3.3

d	$\ell(d)$	n_d	w_d	ϵ_d
87	2	1	$[9;\overline{3,18}]$	$28+3\sqrt{87}$
7107	4	1	$[84;\overline{3,3,3,168}]$	$2782+33\sqrt{7107}$
75242	5	1	$[274;\overline{3,3,3,3,548}]$	$29899+109\sqrt{75242}$
96418707	8	1	$[9819;\overline{3,3,...,3,19638}]$	$38560402+3927\sqrt{964187007}$
11469189687	10	1	$[107094;\overline{3,3,...,3,214188}]$	$4587598648+42837\sqrt{11469189687}$
125106733802	11	1	$[353704;\overline{3,3,...,3,707408}]$	$50042438461+141481\sqrt{125106733802}$

$$\epsilon_d = \left(\frac{5S_\ell^2}{2} + \frac{3S_\ell}{2} + S_{\ell-1}\right) + S_\ell\sqrt{d}$$

$$t_d = 5S_\ell^2 + 3S_\ell + 2S_{\ell-1} \quad \textit{and} \quad u_d = 2S_\ell$$

$$n_d = 1$$

Also, we state the following Table 3.3. where fundamental unit is ϵ_d, integral basis element is w_d and and Yokoi's invariant is n_d for $2 \leq \ell(d) \leq 11$.

*Proof.*This claim is obtained if we substitute $\beta = 2$ into Theorem 1. Now we have to prove that Yokoi's d- invariant value is $n_d = 1$ for $\ell \geq 2$.
We know from H. Yokoi's references [13],[14], [15], [16] that $n_d = \left[\left[\frac{t_d}{u_d^2}\right]\right]$. If we substitute t_d and u_d into n_d, then we get

$$n_d = \left[\left[\frac{t_d}{u_d^2}\right]\right] = \left[\left[\frac{5S_\ell^2 + 3S_\ell + 2S_{\ell-1}}{4S_\ell^2}\right]\right] = 1,$$

since S_ℓ is increasing and $0 < \frac{1}{4} + \frac{3}{4S_\ell} + \frac{S_{\ell-1}}{2S_\ell^2} < 0,56$ for $\ell \geq 2$. Therefore, we obtain $n_d = 1$ for $\ell \geq 2$ which completes the proof of the corollary. For the numerical examples, we create Table 3.3.

4 Conclusion

Quadratic fields have applications in different areas of mathematics such as quadratic forms, algebraic geometry,analytic number theory, computer science, diophantine equations, discrete mathematic, algebraic number theory, and even cryptography.

In this paper, we interested in the concept of real quadratic field structures such as continued fraction expansions, fundamental unit and Yokoi invariants.So, we obtained general interesting and important results for that.

The results provide us a practical method so as to rapidly determine continued fraction expansion of w_d, fundamental unit ϵ_d, Yokoi's invariants n_d and m_d for such real quadratic number fields.

We are sure that these results will help the researchers to enhance and promote their studies on quadratic fields to carry out a general framework for their applications in life.

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