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A Spectral Collocation Method for Coupled System of Two Dimensional Abel Integral Equations of the Second Kind

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Abstract: While the usual polynomial-based spectral collocation methods are capable of providing highly accurate simulation results for nonlinear integral equations with smooth solutions, its accuracy can be deleteriously degraded when the solutions exhibit locally singular behaviours. Recovering non-smooth solutions of nonlinear systems of weakly singular integral equations is a highly non-trivial task. Nonetheless, such problems arise in many applications of practical interest. The present letter develops high-order spectral collocation method for a coupled system of Abel integral equations of the second kind with non-smooth solutions on two dimensional domains. The proposed solver builds an approximation to the solution via a smoothing transformation for the integral equation. The principal advantage of the proposed method is that it is *direct* (as opposed to *iterative*) and efficiently regains high accuracy without excessively increasing the number of degrees of freedom used.

Keywords: Spectral collocation method, nonsmooth solutions, Abel integral equations. coupled system

1 Introduction

In this paper, we consider the following coupled system of two dimensional Abel integral equations of the second kind:

$$\begin{aligned} u(t_1,t_2) &= g_1(t_1,t_2) + \int_0^{t_1}\int_0^{t_2} \frac{1}{\sqrt{(t_1-r_1)(t_2-r_2)}} \\ &\quad \times f_1(t_1,r_1,t_2,r_2,u(r_1,r_2),v(r_1,r_2))\,dr_2 dr_1, \\ v(t_1,t_2) &= g_2(t_1,t_2) + \int_0^{t_1}\int_0^{t_2} \frac{1}{\sqrt{(t_1-r_1)(t_2-r_2)}} \\ &\quad \times f_2(t_1,r_1,t_2,r_2,u(r_1,r_2),v(r_1,r_2))\,dr_2 dr_1, \end{aligned} \tag{1}$$

where $f_i : D\times\mathbb{R}^2 \to \mathbb{R},\ i=\{1,2\}$ are given functions and $D := \{(t_1,r_1,t_2,r_2) : 0 \le r_i \le t_i \le 1\}$. Equations of the typre (1) arise in the study of various problems in many disciplines, such as insurance mathematics, electromagnetic scattering, demography, viscoelastic materials [1]. The numerical treatment of (1) is not a simple task, this is due to the fact that its solution usually have a weak singularity at the beginning time, even when the nonhomogeneous term $g_i(t_1,t_2)$ is smooth. It was pointed out in [2, 3] that near $t=0$ the n-th derivative of the solutions behaves like $t^{\frac{1}{2}-n}$, which indicates that $y \notin \mathbb{C}^n[0,1]$. Therefore, it is difficult to employ high order numerical schemes for solving (1).

In recent years, numerous works have been focusing on the development of more advanced and efficient numerical schemes for integral equations and integro-differential equations. Such equations have been subject of many theoretical and numerical investigations. Spectral method has high accuracy, and so has been successfully used for computations in science and engineering [4–9]. Mokhtary [10] developed and analyzed a Jacobi spectral tau method for fractional weakly singular integro-differential equations. Gu and Chen [11] proposed a Legendre spectral collocation method for solving linear Volterra integral equations of the second kind with non-vanishing delay. Tang et al. [12] developed a Legendre spectral collocation method for for

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solving linear Volterra integral equations of the second kind assuming that the kernel and the solutions are sufficiently smooth. Khan and Gondal [13] constructed a two-step Laplace decomposition scheme for the solution of Abel's type singular integral equations. Chen et al. [14–16] developed Jacobi spectral collocation schemes for linear Volterra integral equations of the second kind with weakly singular kernels provided that the solutions are sufficiently smooth. Talaei et al. [17] proposed Muntz-Legendre recursive tau method for the numerical solution of Abel-Volterra type integral equations with non-smooth solutions. Doha et al. [18, 19] proposed Jacobi-Gauss-collocation schemes for solving Volterra, Fredholm and systems of Volterra-Fredholm fractional integro-differential equations with smooth solutions. Zaky and Ameen [20] developed a Legendre-Jacobi collocation approach for nonlinear systems of fractional boundary value problems and related Volterra-Fredholm integral equations with smooth solutions. Zaky et al. [21–24] proposed spectral-collocation methods to solve various types of integral and related fractional differential equations.

The main purpose of this letter is to provide a novel numerical scheme based on a spectral approach for the coupled system of two dimensional Abel integral equations of the second kind with non-smooth solutions. The rest of the paper is organized as follows. In the next section we provide the necessary background and notation. In Section 3, the spectral collocation method is constructed. In Section 4, a numerical example to illustrate the effectiveness of the proposed method is presented. The conclusion of this work is given in Section 5.

2 Jacobi interpolation approximation

We begin establishing our notational conventions.

– Let $\mathbb{R}$ (resp. $\mathbb{N}$) be the set of all real numbers (resp. non-negative integers), and let $\mathbb{N}_0 = \mathbb{N} \mathbb{U} \cup 0$.
– We use boldface lowercase letters to denote 2-dimensional multi-indexes and vectors, e.g., $\mathbf{i} = (i_1, i_2) \in \mathbb{N}_0^2$, and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2) \in \mathbb{R}^2$. Also, $\mathbf{1} = (1,1) \in \mathbb{N}^2$. For a scalar $a \in \mathbb{R}$, we define the following component-wise operations:

$$\begin{aligned} \boldsymbol{\gamma} + \mathbf{k} &= (\gamma_1 + k_1, \gamma_2 + k_2), \\ \boldsymbol{\gamma} + a &:= \boldsymbol{\gamma} + a\mathbf{1} = (\gamma_1 + a, \gamma_2 + a), \end{aligned} \tag{2}$$

and use the following conventions:

$$\begin{aligned} &\boldsymbol{\gamma} \geq \boldsymbol{k} \Leftrightarrow \forall_{1 \leq i \leq 2}\ \gamma_i \geq k_i \\ &\boldsymbol{\gamma} \geq r \Leftrightarrow \boldsymbol{\gamma} \geq r\mathbf{1} \Leftrightarrow \forall_{1 \leq i \leq 2}\ \gamma_i \geq r. \end{aligned} \tag{3}$$

– We denote

$$|\mathbf{k}|_1 = k_1 + k_2,\ |\mathbf{k}|_\infty = \max_{1 \leq i \leq 2} k_i,$$
$$\prod \mathbf{z}\mathbf{y}^{\mathbf{z}} = \prod_{i=1}^{2} z_i y_i^{z_i},\ \int_{\mathbf{a}}^{\mathbf{b}} \cdot\, d\boldsymbol{\rho} = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdot\, d\rho_2 d\rho_1. \tag{4}$$

– Let $\Lambda := (-1,1)$. Given a multivariate function $u(\mathbf{z})$, we denote the $|\mathbf{k}|_1$-th (mixed) partial derivative by

$$\partial_{\boldsymbol{z}}^{\boldsymbol{k}} u = \frac{\partial^{|\boldsymbol{k}|_1} u}{\partial_{z_1}^{k_1} \partial_{z_2}^{k_2}} = \partial_{z_1}^{k_1} \partial_{z_2}^{k_2} u.$$

– Let $\mathscr{P}_N$ be the space of all algebraic polynomials of degree up to N in Λ.

Now, we recall some relevant properties of the one-dimensional Jacobi polynomials (cf. [25]). Let $\omega^{\alpha,\gamma}(z) = (1-z)^\alpha (1+z)^\gamma$ be the Jacobi weight function defined in Λ. The set of Jacobi polynomials, denoted by $J_n^{\alpha,\gamma}(z)$, $(\alpha,\ \gamma > -1)$ forms a complete orthogonal system in $L^2_{\omega^{\alpha,\gamma}}(\Lambda)$,

$$\int_\Lambda J_n^{\alpha,\gamma}(z) J_m^{\alpha,\gamma}(z) \omega^{\alpha,\gamma}(z) dz = \beta_n^{\alpha,\gamma} \delta_{n,m}, \tag{5}$$

where $\delta_{m,n}$ is the Kronecker delta and

$$\beta_n^{\alpha,\gamma} = \frac{2^{(\alpha+\gamma+1)} \Gamma(\alpha+n+1) \Gamma(n+\gamma+1)}{(\alpha+\gamma+2n+1) \Gamma(n+1+\alpha+\gamma) n!}. \tag{6}$$

The Jacobi polynomials are generated by the three-term recurrence relation:

$$\begin{aligned} &J_{n+1}^{\alpha,\gamma}(z) = (A_n^{\alpha,\gamma} z - B_n^{\alpha,\gamma}) J_n^{\alpha,\gamma}(z) - C_n^{\alpha,\gamma} J_{n-1}^{\alpha,\gamma}(z),\ n \geq 1, \\ &J_0^{\alpha,\gamma}(z) = 1, J_1^{\alpha,\gamma}(z) = \frac{1}{2}(\alpha+\gamma+2)z + \frac{1}{2}(\alpha-\gamma), \end{aligned} \tag{7}$$

where

$$\begin{aligned} A_n^{\alpha,\gamma} &= \frac{(2n+\alpha+\gamma+1)(2n+\alpha+\gamma+2)}{2(n+1)(n+\alpha+\gamma+1)}, \\ B_n^{\alpha,\gamma} &= \frac{(2n+\alpha+\gamma+1)(\gamma^2-\alpha^2)}{2(n+1)(n+\alpha+\gamma+1)(2n+\alpha+\gamma)}, \\ C_n^{\alpha,\gamma} &= \frac{(2n+\alpha+\gamma+2)(n+\alpha)(n+\gamma)}{(n+1)(n+\alpha+\gamma+1)(2n+\alpha+\gamma)}. \end{aligned}$$

This relation allows us to evaluate the Jacobi polynomials at any given abscissa $z \in \Lambda = [-1,1]$, and it is the starting point to derive other properties. In the following, we denote by $P_n(z) := J_n^{-\frac{1}{2},0}(z)$ and $\omega(z) := \omega^{-\frac{1}{2},0}(z)$. The two-dimensional Jacobi polynomial and Jacobi weight function are given as

$$\begin{aligned} \boldsymbol{P_n(z)} &= P_{n_1}(z_1) P_{n_2}(z_2), \\ \boldsymbol{\omega(z)} &= \omega_1(z_1)\omega_2(z_2), \quad \forall z_1, z_2 \in \Lambda. \end{aligned} \tag{8}$$

We deduce that

$$\begin{aligned} &\int_{\Lambda^2} \boldsymbol{P_n(z) P_m(z) \omega(z)} dz = \boldsymbol{\beta_n \delta}_{n,m} \\ &= \beta_{n_1} \beta_{n_2} \delta_{n_1,m_1} \delta_{n_2,m_2}, \quad \boldsymbol{n}, \boldsymbol{m} \geq 0. \end{aligned} \tag{9}$$

Let $\{\xi_{j_k}, \varpi_{j_k}\}_{j_k=0}^N$ be the one-dimensional Jacobi-Gauss points and weights related to the Jacobi parameters $\alpha=-\frac{1}{2}$, $\gamma=0$, and let $\mathscr{I}_{z_k,N}$ be its associated interpolation operator in z_k direction.

The two-dimensional nodes and weights $\left\{\boldsymbol{\xi}_{\boldsymbol{j}},\boldsymbol{\varpi}_{\boldsymbol{j}}\right\}_{|\boldsymbol{j}|_\infty\le N}$ in Λ^2 are given by

$$\boldsymbol{\xi}_{\boldsymbol{j}}=(\xi_{j_1},\xi_{j_2}),\boldsymbol{\varpi}_{\boldsymbol{j}}=(\varpi_{j_1},\varpi_{j_2}).$$

The two-dimensional Jacobi-Gauss quadrature enjoys the exactness

$$\int_{\Lambda^2}\varphi(z)\boldsymbol{\omega}(z)dz=\sum_{|\mathbf{j}|_\infty\le N}\varphi\left(\boldsymbol{\xi}_{\boldsymbol{j}}\right)\boldsymbol{\varpi}_{\boldsymbol{j}},\quad\forall\varphi(z)\in\mathscr{P}^2_{2N+1}. \tag{10}$$

Hence

$$\sum_{|\mathbf{k}|_\infty\le N}\boldsymbol{P_n}(\boldsymbol{\xi_k})\boldsymbol{P_m}(\boldsymbol{\xi_k})\boldsymbol{\varpi_k}=\boldsymbol{\beta_n}\boldsymbol{\delta}_{n,m},\quad\forall\,0\le\boldsymbol{n}+\boldsymbol{m}\le 2N+1. \tag{11}$$

For any $u\in\mathbb{C}(\Lambda^2)$, the Jacobi-Gauss interpolation operator $\mathscr{I}_{\boldsymbol{z},N}:\mathbb{C}(\Lambda^2)\longrightarrow\mathscr{P}^2_N$ is determined uniquely by

$$(\mathscr{I}_{\boldsymbol{z},N}u)(\boldsymbol{\xi_j})=u(\boldsymbol{\xi_j})\;\forall\boldsymbol{j}\in\mathbb{N}^2,\;|\mathbf{k}|_\infty\le N. \tag{12}$$

For simplicity, we assume that the number of nodes in each direction is the same (i.e., $N+1$ points). One verifies that

$$\mathscr{I}_{\boldsymbol{z},N}=\mathscr{I}_{z_1,N}\circ\mathscr{I}_{z_2,N}. \tag{13}$$

The interpolation condition (12) implies that $\mathscr{I}_{\boldsymbol{z},N}u=u$ for all $u\in\mathscr{P}^2_N$. On the other hand, since $\mathscr{I}_{\boldsymbol{z},N}u\in\mathscr{P}^2_N$, we can write

$$\mathscr{I}_{\boldsymbol{z},N}u(\boldsymbol{z})=\sum_{|\mathbf{n}|_\infty\le N}\widehat{u}_{\boldsymbol{n}}\boldsymbol{P}_n(\boldsymbol{z}), \tag{14}$$

where

$$\widehat{u}_{\boldsymbol{n}}=\frac{1}{\boldsymbol{\beta_n}}\sum_{|\mathbf{j}|_\infty\le N}u\left(\boldsymbol{\xi_j}\right)\boldsymbol{P}_n(\boldsymbol{\xi_j})\boldsymbol{\varpi_j}. \tag{15}$$

3 The collocation scheme

Taking the change of variable $t_i\to\phi_i(z_i)=\left(\frac{z_i+1}{2}\right)^2$ in equation (1), and letting $r_i\to\phi_i(\rho_i)=\left(\frac{\rho_i+1}{2}\right)^2$, $\rho_i\in\Lambda, i=1,2$, we obtain the following equivalent system to (1)

$$\begin{aligned}&u(\phi_1(z_1),\phi_2(z_2))=g_1(\phi_1(z_1),\phi_d(z_2))\\&+\int_{-1}^{z_1}\int_{-1}^{z_2}\frac{\phi_i'(\rho_i)}{\sqrt{(\phi_1(z_1)-\phi_1(\rho_1))(\phi_2(z_2)-\phi_2(\rho_2))}}\\&\times f_1(\phi_1(z_1),\phi_1(\rho_1),\phi_2(z_2),\phi_2(\rho_2),\\&u(\phi_1(\rho_1),\phi_2(\rho_2)),v(\phi_1(\rho_1),\phi_2(\rho_2)))d\rho_2d\rho_1,\end{aligned} \tag{16}$$

$$\begin{aligned}&v(\phi_1(z_1),\phi_2(z_2))=g_2(\phi_1(z_1),\phi_d(z_2))\\&+\int_{-1}^{z_1}\int_{-1}^{z_2}\frac{\phi_i'(\rho_i)}{\sqrt{(\phi_1(z_1)-\phi_1(\rho_1))(\phi_2(z_2)-\phi_2(\rho_2))}}\\&\times f_2(\phi_1(z_1),\phi_1(\rho_1),\phi_2(z_2),\phi_2(\rho_2),\\&u(\phi_1(\rho_1),\phi_2(\rho_2)),v(\phi_1(\rho_1),\phi_2(\rho_2)))d\rho_2d\rho_1.\end{aligned}$$

This system is still weakly singular but has a smooth solution. It can be simplified to be

$$\begin{aligned}&U(\mathbf{z})=G_1(\mathbf{z})\\&+\frac{1}{2}\int_{-\mathbf{1}}^{\mathbf{z}}F_1(\mathbf{z},\boldsymbol{\rho},U(\boldsymbol{\rho}),v(\boldsymbol{\rho}))\prod(\mathbf{z}-\boldsymbol{\rho})^{-\frac{1}{2}}d\boldsymbol{\rho},\\&V(\mathbf{z})=G_2(\mathbf{z})\\&+\frac{1}{2}\int_{-\mathbf{1}}^{\mathbf{z}}F_2(\mathbf{z},\boldsymbol{\rho},U(\boldsymbol{\rho}),V(\boldsymbol{\rho}))\prod(\mathbf{z}-\boldsymbol{\rho})^{-\frac{1}{2}}d\boldsymbol{\rho},\end{aligned} \tag{17}$$

where,

$$\begin{aligned}&U(\mathbf{z})=U(z_1,z_2)=u(\phi_1(z_1),\phi_2(z_2)),\\&G_1(\mathbf{z})=G_1(z_1,z_2)=g_1(\phi_1(z_1),\phi_2(z_2)),\\&F_1(\mathbf{z},\boldsymbol{\rho},U(\boldsymbol{\rho}),V(\boldsymbol{\rho}))=f_1(\phi_1(z_1),\phi_1(\rho_1),\phi_2(z_2),\phi_2(\rho_2)\\&\quad,u(\phi_1(\rho_1),\phi_2(\rho_2)),v(\phi_1(\rho_1),\phi_2(\rho_2)))k_1(z_1,\rho_1,z_2,\rho_2),\\&k_1(z_1,\rho_1,z_2,\rho_2)=k_2(z_1,\rho_1,z_2,\rho_2)\\&=\prod_{i=1}^{2}(\rho_i+1)\left(\sum_{j=0}^{1}\left(\frac{z_i+1}{2}\right)^j\left(\frac{\rho_i+1}{2}\right)^{1-j}\right)^{-\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}&V(\mathbf{z})=V(z_1,z_2)=u_2(\phi_1(z_1),\phi_2(z_2)),\\&G_2(\mathbf{z})=G_2(z_1,z_2)=g_2(\phi_1(z_1),\phi_2(z_2)),\\&F_2(\mathbf{z},\boldsymbol{\rho},U(\boldsymbol{\rho}),V(\boldsymbol{\rho}))=f_2(\phi_1(z_1),\phi_1(\rho_1),\phi_2(z_2),\phi_2(\rho_2)\\&\quad,u(\phi_1(\rho_1),\phi_2(\rho_2)),v(\phi_1(\rho_1),\phi_2(\rho_2)))k_2(z_1,\rho_1,z_2,\rho_2).\end{aligned}$$

In order to evaluate integral term with high accuracy using Jacobi-Gauss quadrature formula, equation (17) can be simplified to be

$$\begin{aligned}U(z)&=\int_{-1}^{\mathbf{1}}F_1(z,\boldsymbol{\rho}_{z,\boldsymbol{\tau}},U(\boldsymbol{\rho}_{z,\boldsymbol{\tau}}),V(\boldsymbol{\rho}_{z,\boldsymbol{\tau}}))\\&\quad\times\prod\frac{1}{2}\sqrt{\frac{z+1}{1-\boldsymbol{\tau}}}d\boldsymbol{\tau}+G_1(z),\\V(z)&=\int_{-1}^{\mathbf{1}}F_2(z,\boldsymbol{\rho}_{z,\boldsymbol{\tau}},U(\boldsymbol{\rho}_{z,\boldsymbol{\tau}}),V(\boldsymbol{\rho}_{z,\boldsymbol{\tau}}))\\&\quad\times\prod\frac{1}{2}\sqrt{\frac{z+1}{1-\boldsymbol{\tau}}}d\boldsymbol{\tau}+G_2(z)\end{aligned} \tag{18}$$

where

$$\begin{aligned}&\boldsymbol{\rho}_{z,\tau}=\left(\rho_1(z_1,\tau_1),\rho_2(z_2,\tau_2)\right),\\&\rho_i(z_i,\tau_i)=\frac{1+z_i}{2}\tau_i-\frac{1-z_i}{2},\ z_i,\ \tau_i\in\Lambda,\ i=1,2.\end{aligned}\tag{19}$$

The Jacobi spectral-collocation method to (18) is to seek the approximate solution in the form

$$\begin{aligned}U_N(\boldsymbol{z})&=\sum_{|\boldsymbol{i}|_\infty\leqslant N}\hat{u}_{\boldsymbol{i}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z})\in\mathscr{P}_N^2,\\V_N(\boldsymbol{z})&=\sum_{|\boldsymbol{j}|_\infty\leqslant N}\hat{h}_{\boldsymbol{j}}\boldsymbol{P}_{\boldsymbol{j}}(\boldsymbol{z})\in\mathscr{P}_N^2.\end{aligned}\tag{20}$$

Hence, inserting (20) into (18) leads to the following

$$\begin{aligned}U_N(\boldsymbol{z})&=\mathscr{I}_{z,N}G_1(\boldsymbol{z})+\mathscr{I}_{z,N}\left[\int_{-1}^{1}\prod\frac{1}{2}\sqrt{\frac{z+1}{1-\tau}}\right.\\&\left.\quad\mathscr{I}_{\tau,N}F_1\left(\boldsymbol{z},\boldsymbol{\rho}_{z,\tau},U_N\left(\boldsymbol{\rho}_{z,\tau}\right),V_N\left(\boldsymbol{\rho}_{z,\tau}\right)\right)d\boldsymbol{\tau}\right],\\V_N(\boldsymbol{z})&=\mathscr{I}_{z,N}G_2(\boldsymbol{z})+\mathscr{I}_{z,N}\left[\int_{-1}^{1}\prod\frac{1}{2}\sqrt{\frac{z+1}{1-\tau}}\right.\\&\left.\quad\mathscr{I}_{\tau,N}F_2\left(\boldsymbol{z},\boldsymbol{\rho}_{z,\tau},U_N\left(\boldsymbol{\rho}_{z,\tau}\right),V_N\left(\boldsymbol{\rho}_{z,\tau}\right)\right)d\boldsymbol{\tau}\right].\end{aligned}\tag{21}$$

We now describe the implementation our formulation (21) using the Jacobi-Gauss interpolation, which serves as a base for the scheme. Setting

$$\begin{aligned}&\mathscr{I}_{z,N}\mathscr{I}_{\tau,N}\left(F_1\left(\boldsymbol{z},\boldsymbol{\rho}_{z,\tau},U_N\left(\boldsymbol{\rho}_{z,\tau}\right),V_N\left(\boldsymbol{\rho}_{z,\tau}\right)\right)\right.\\&\left.\times\prod\left(\frac{1+z}{4}\right)^{\frac{1}{2}}\right)=\sum_{|\boldsymbol{i}|_\infty\leq N}\sum_{|\boldsymbol{j}|_\infty\leq N}\hat{v}_{\boldsymbol{i},\boldsymbol{j}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z})\boldsymbol{P}_{\boldsymbol{j}}(\boldsymbol{\tau}),\\&\text{and}\\&\mathscr{I}_{z,N}\mathscr{I}_{\tau,N}\left(F_2\left(\boldsymbol{z},\boldsymbol{\rho}_{z,\tau},U_N\left(\boldsymbol{\rho}_{z,\tau}\right),V_N\left(\boldsymbol{\rho}_{z,\tau}\right)\right)\right.\\&\left.\times\prod\left(\frac{1+z}{4}\right)^{\frac{1}{2}}\right)=\sum_{|\boldsymbol{i}|_\infty\leq N}\sum_{|\boldsymbol{j}|_\infty\leq N}\hat{d}_{\boldsymbol{i},\boldsymbol{j}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z})\boldsymbol{P}_{\boldsymbol{j}}(\boldsymbol{\tau}).\end{aligned}\tag{22}$$

With this, we obtain

$$\begin{aligned}&\int_{-1}^{1}\prod(1-\boldsymbol{\tau})^{-\frac{1}{2}}\mathscr{I}_{z,N}\mathscr{I}_{\tau,N}\\&\times\left[F_1\left(\boldsymbol{z},\boldsymbol{\rho}_{z,\tau},U_N\left(\boldsymbol{\rho}_{z,\tau}\right),V_N\left(\boldsymbol{\rho}_{z,\tau}\right)\right)\prod\left(\frac{1+z}{4}\right)^{\frac{1}{2}}\right]d\boldsymbol{\tau}\\&=\sum_{|\boldsymbol{i}|_\infty\leq N}\sum_{|\boldsymbol{j}|_\infty\leq N}\hat{v}_{\boldsymbol{i},\boldsymbol{j}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z})\int_{-1}^{1}\boldsymbol{J}_{\boldsymbol{j}}(\boldsymbol{\tau})\prod(1-\boldsymbol{\tau})^{-\frac{1}{2}}d\boldsymbol{\tau}\\&=8\sum_{|\boldsymbol{i}|_\infty\leq N}\hat{v}_{\boldsymbol{i},\boldsymbol{0}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z}),\end{aligned}\tag{23}$$

also,

$$\begin{aligned}&\int_{-1}^{1}\prod(1-\boldsymbol{\tau})^{-\frac{1}{2}}\mathscr{I}_{z,N}\mathscr{I}_{\tau,N}\\&\times\left[F_2\left(\boldsymbol{z},\boldsymbol{\rho}_{z,\tau},U_N\left(\boldsymbol{\rho}_{z,\tau}\right),V_N\left(\boldsymbol{\rho}_{z,\tau}\right)\right)\prod\left(\frac{1+z}{4}\right)^{\frac{1}{2}}\right]d\boldsymbol{\tau}\\&=\sum_{|\boldsymbol{i}|_\infty\leq N}\sum_{|\boldsymbol{j}|_\infty\leq N}\hat{d}_{\boldsymbol{i},\boldsymbol{j}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z})\int_{-1}^{1}\boldsymbol{J}_{\boldsymbol{j}}(\boldsymbol{\tau})\prod(1-\boldsymbol{\tau})^{-\frac{1}{2}}d\boldsymbol{\tau}\\&=8\sum_{|\boldsymbol{i}|_\infty\leq N}\hat{d}_{\boldsymbol{i},\boldsymbol{0}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z}).\end{aligned}\tag{24}$$

Where, from (11) and (22) yields

$$\begin{aligned}\hat{v}_{\boldsymbol{i},\boldsymbol{0}}&=\left(\frac{4i+1}{8}\right)^2\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z_r})\boldsymbol{\varpi}_{\boldsymbol{r}}\boldsymbol{\varpi}_{\boldsymbol{s}}\\&\times F_1\left(\boldsymbol{z_r},\boldsymbol{\rho}_{z_r,\tau_r},U_N\left(\boldsymbol{\rho}_{z_r,\tau_r}\right),V_N(\boldsymbol{z_r},\boldsymbol{\tau_r})\right).\end{aligned}\tag{25}$$

And,

$$\begin{aligned}\hat{d}_{\boldsymbol{i},\boldsymbol{0}}&=\left(\frac{4i+1}{8}\right)^2\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z_r})\boldsymbol{\varpi}_{\boldsymbol{r}}\boldsymbol{\varpi}_{\boldsymbol{s}}\\&\times F_2\left(\boldsymbol{z_r},\boldsymbol{\rho}_{z_r,\tau_r},U_N\left(\boldsymbol{\rho}_{z_r,\tau_r}\right),V_N(\boldsymbol{z_r},\boldsymbol{\tau_r})\right).\end{aligned}\tag{26}$$

By using (20)-(26) we deduce that

$$\begin{aligned}&\sum_{|\boldsymbol{i}|_\infty\leqslant N}\hat{u}_{\boldsymbol{i}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z})\\&=8\sum_{|\boldsymbol{i}|_\infty\leq N}\hat{v}_{\boldsymbol{i},\boldsymbol{0}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z})+\sum_{|\boldsymbol{i}|_\infty\leq N}\hat{\boldsymbol{w}}_{\boldsymbol{i}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z}),\\&\text{and}\\&\sum_{|\boldsymbol{i}|_\infty\leqslant N}\hat{h}_{\boldsymbol{i}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z})\\&=8\sum_{|\boldsymbol{i}|_\infty\leq N}\hat{d}_{\boldsymbol{i},0}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z})+\sum_{|\boldsymbol{i}|_\infty\leq N}\hat{\boldsymbol{\mu}}_{\boldsymbol{i}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z}).\end{aligned}\tag{27}$$

where

$$\hat{\boldsymbol{w}}_{\boldsymbol{i}}=\frac{4i+1}{4}\sum_{|\mathbf{j}|_\infty\leq N}G_1\left(\boldsymbol{z_j}\right)\boldsymbol{\varpi}_{\boldsymbol{j}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z_j}),$$

$$\hat{\boldsymbol{\mu}}_{\boldsymbol{i}}=\frac{4i+1}{4}\sum_{|\mathbf{j}|_\infty\leq N}G_2\left(\boldsymbol{z_j}\right)\boldsymbol{\varpi}_{\boldsymbol{j}}\boldsymbol{P}_{\boldsymbol{i}}(\boldsymbol{z_j}).$$

Comparing the expansion coefficients of (27) yields the following system of algebraic equations

$$\begin{aligned}\hat{u}_{\boldsymbol{i}}&=8\,\hat{v}_{\boldsymbol{i},\boldsymbol{0}}+\hat{\boldsymbol{w}}_{\boldsymbol{i}},\\\hat{h}_{\boldsymbol{i}}&=8\,\hat{d}_{\boldsymbol{i},\boldsymbol{0}}+\hat{\boldsymbol{\mu}}_{\boldsymbol{i}}\end{aligned}\tag{28}$$

which can be solved by the Newton's iterative method.

4 Numerical results

In this section, we demonstrate the effectiveness of the spectral collocation method for the coupled system of two dimensional Abel integral equation (1) with $f_1 = u^2 + v^2$, $f_2 = v^2 + u^2$ and g_1 and g_2 are chosen such that the exact solution $u = v = \frac{\sqrt{t_1 t_2}}{2}$. In Table 1, the numerical errors are listed.

Table 1: The errors versus N.

N	Error
3	1.22553×10^{-6}
4	8.13822×10^{-9}
5	7.02514×10^{-10}
6	1.82314×10^{-11}
7	4.32456×10^{-13}
8	1.75223×10^{-15}

5 Conclusion

When solving an integral equation like (1), it is common to have a localized loss of regularity due to the weakly singular kernel. To efficiently regain high accuracy without excessively increasing the number of degrees of freedom used, we developed a direct solver relying on an efficient spectral discretizations. The spectral method is a direct scheme for computing a higher-order collocation solution by performing a series of smoothing transformations. The scheme proposed differs from traditional spectral schemes in a notable aspect. It is capable of providing high order accuracy for non-smooth solutions. Preliminary results suggest that the spectral algorithm can be successfully applied to fractional differential equations with non-smooth solutions. A detailed description of the scheme is currently in preparation.

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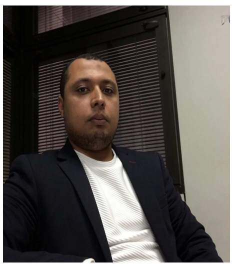

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