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Graphing Examples of Starlike and Convex Functions of order β

Hameed Ur Rehman[1,2], *Maslina Darus*[1,*], *and Jamal Salah*[3]

[1] School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600 Selangor Darul Ehsan, Malaysia
[2] Center for Language and Foundation Studies, A' Sharqiyah University, Post Box No. 42, Post Code No. 400 Ibra, Sultanate of Oman
[3] College of Applied and Health Sciences, A' Sharqiyah University, Post Box No. 42, Post Code No. 400 Ibra, Sultanate of Oman

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Abstract: In this paper, we introduce and investigate two subclasses $H_\beta^*(n,b,\phi)$ and $K_\beta(n,b,\phi)$ of analytic functions with negative coefficients of order β in the unit disk D. Silverman [6] determined certain coefficient inequalities and distortion theorems for univalent functions with negative coefficients that are starlike $S(\alpha)$, and convex $K(\alpha)$ of order α respectively. We also obtain the same coefficient inequalities and distortion theorem for such univalent functions with negative coefficients. We point out that the coefficient estimates up to z^9 are enough to get the graphs of starlike and convex functions. Here in the sequel we give some examples, estimation and the graphical representation of such univalent functions by using the complex tool [8].

Keywords: Univalent Functions, Starlike functions, Convex functions, Distortion theorems.

1 Introduction

Let A denotes the class of univalent and analytic functions in the unit disk, $D=\{z\in\mathbb{C}:|z|<1\}$ which have the form:

$$f(z)=z+\sum_{k=n+2}^{\infty}a_kz^k,\quad (n\in N=\{1,2,3,...\}) \tag{1}$$

We also consider the analytic functions which is the subclass of A and its coefficients are real negative.

Let $A(n)$ denotes the subclass of A of the form:

$$f(z)=z-\sum_{k=n+1}^{\infty}a_kz^k,\quad (n\in N, a_k\geq 0). \tag{2}$$

A set $\mathbb{S}$ in the complex plane is said to be starlike if the linear segment joining $w_o=0$ to every other point $w\in\mathbb{S}$ lies inside D. Mathematical condition is given in equation (3).

Theorem 1.1. (Kobori [1]) A function $f(z)\in A$ is in S^* if and only if:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>0,\quad (z\in D). \tag{3}$$

A set $\mathbb{S}$ in the complex plane is said to be convex if the line segment joining any two points in $\mathbb{S}$ lies entirely in D. Mathematical condition is given in equation (4).

Theorem 1.2. (Kobori [1]) A function $f(z)\in A$ is in K if and only if:

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0,\quad (z\in D). \tag{4}$$

It is well known that $f\in K$ if and only if $zf'(z)\in S^*$.

Robertson [2] defined the subclasses $S^*(\alpha)$ and $K(\alpha)$ of A if the function $f(z)$ respectively satisfies the conditions,

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \tag{5}$$

for some α $(0\leq\alpha<1)$, and for all $z\in D$. The subclass of A consisting of all starlike functions of order α is denoted by $S^*(\alpha)$. And if the function $f(z)$ satisfies the condition,

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \tag{6}$$

* Corresponding author e-mail: maslina@ukm.edu.my

for some α $(0 \leq \alpha < 1)$, for all $z \in D$. The subclass of A consisting of all convex functions of order α is denoted by $K(\alpha)$.

Let $H(n)$ and $K(n)$ represents the subclass of $A(n)$ in equation (2) including the functions which are univalent in the unit disk D. Moreover $H_{\alpha}(n)$ the subclass of $A(n)$ is called to be starlike of order α $(0 \leq \alpha < 1)$ if and only if it satisfies the following condition,

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in D) \tag{7}$$

and $k_{\alpha}(n)$ the subclass of $A(n)$ is called to be convex of order α $(0 \leq \alpha < 1)$ if and only if,

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in D). \tag{8}$$

Theorem 1.3. (Chatterjea [3]) A function $f(z)$ in $A(n)$ also exists in $T_{\alpha}(n)$ if and only if,

$$f(z) = z - \sum_{k=n+1}^{\infty} (k-\alpha)a_k \leq 1-\alpha \tag{9}$$

for some $(0 \leq \alpha < 1)$, and for all $z \in D$.

Theorem 1.4. (Chatterjea [3]) A function $f(z)$ in $A(n)$ also exists in $k_{\alpha}(n)$ if and only if,

$$f(z) = z - \sum_{k=n+1}^{\infty} k(k-\alpha)a_k \leq 1-\alpha \tag{10}$$

for some $(0 \leq \alpha < 1)$, and for all $z \in D$.

Sekine and Owa [4] introduced the subclass $A(n,\theta)$ of A which is of the form (2) and the subclass $T^*(n,\theta)$ and $k_{\alpha}(n,\theta)$ of $A(n,\theta)$ as stated below.

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k z^k \tag{11}$$

where $(n \in N = \{1,2,3,...\}, a_k \geq 0)$.

They studied and noted that $A(1,0) = A(1)$ implies that $A(1,0)$ is a class of analytic functions with negative functions. It was also noted that $T^*(n,\theta)$ and $K_{\alpha}(n,\theta)$ the sub classes of $A(n,\theta)$ of starlike and convex functions of order α in D, respectively.

Furthermore, Sekine, Owa and Yamakawa [5] used the assertion (11) and found several examples of such univalent functions with its illustrated images as starlike and convex functions. They also determined the estimation of those univalent functions.

The main motivation of the paper is to give more examples of starlike and convex functions with its image illustrations.

Let $H^*_{\beta}(n,\theta)$ and $K_{\beta}(n,\theta)$ be the sub classes of $A(n,\theta)$ of starlike and convex functions of order β in D respectively. Therefore we state two theorems due to Silverman.

Theorem 1.5. (Silverman [6]) A function $f(z)$ in $A(n,\theta)$ is in $H^*_{\beta}(n,\theta)$ if and only if:

$$f(z) = \sum_{k=n+1}^{\infty} (k-\beta)a_k \leq 1-\beta. \tag{12}$$

Theorem 1.6. (Silverman [6]) A function $f(z)$ in $A(n,\theta)$ is in $K_{\beta}(n,\theta)$ if and only if:

$$f(z) = \sum_{k=n+1}^{\infty} k(k-\beta)a_k \leq 1-\beta. \tag{13}$$

Let $A(n,\phi)$ denote the subclass of A consisting of the functions in the form of,

$$f(z) = z - \sum_{k=n+1}^{\infty} \left(e^{i(k-1)\phi log(k-1)}\right) a_k z^k \tag{14}$$

for all $\phi \in \{\phi + 2p\pi, p = 1,2,3,...\}$.

Remark 1.1. We find that $A(n,\phi) = A(1,0) = A(1)$ that is the subclass of analytic functions with negative coefficients as in (2). Particularly if $n = 1$ we get the subclass of [4] that is $A(n,\theta)$ in the form of (11).

Remark 1.2. Let the extended form of equation (14) is given in (15) where b is the general base from common logarithm, say $A(n,b,\phi)$ denote the subclass of A and the subclass of $H^*_{\beta}(n,b,\phi)$ and $K_{\beta}(n,b,\phi)$ of $A(n,b,\phi)$ in the form of,

$$f(z) = z - \sum_{k=n+1}^{\infty} \left(e^{i(k-1)\phi log_b(k-1)}\right) a_k z^k \tag{15}$$

for all $\phi \in \{\phi + 2p\pi, p = 1,2,3,...\}$ and $(b \in R > 0, b \neq 1)$.

Remark 1.3. We note several interesting results that $A(n,b,0) = A(n)$ that is the subclass of analytic functions with negative coefficients as in example (3.2), and if the base, $b = n \neq 1$ we get the subclass of [4] that is $A(n,\theta)$ in (11).

We denote by $H^*_{\beta}(n,b,\phi)$ and $K_{\beta}(n,b,\phi)$ the subclasses of $A(n,b,\phi)$ that are, respectively starlike of order β and convex of order β. Hence,

$$H^*_{\beta}(n,b,\phi) = A(n,\phi) \cap S^*(\beta),$$

$$K_{\beta}(n,b,\phi) = A(n,\phi) \cap K(\beta).$$

So, $H^*_{\beta}(1,b,0) \sim H^*_{\beta}(1,0)$ and $K_{\beta}(1,b,0) \sim K_{\beta}(1,0)$.

We note that $f(z) \in K_{\beta}(n,b,\phi)$ if and only if $zf'(z) \in H^*_{\beta}(n,b,\phi)$ for some $\beta(0 \leq \beta < 1)$.

2 Main Results

Theorem 2.1. A function $f(z)$ in $A(n,\phi)$ is in $H^*_\beta(n,b,\phi)$ if and only if:

$$\sum_{k=n+1}^{\infty}(k-\beta)a_k \le 1-\beta. \tag{16}$$

Proof. By Theorem (1.3), if a function $f(z)\in A(n,\phi)$ then the coefficient inequality in (16) holds and let $|z|=1$. Therefore, we need to show that, $\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\beta$.

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}-1\right| &= \left|\frac{\sum_{k=n+1}^{\infty} e^{i(k-1)\phi log_b(k-1)}a_k z^{k-1}}{1-\sum\limits_{k=n+1}^{\infty} e^{i(k-1)\phi log_b(k-1)}a_k z^{k-1}}\right| \\ &\le \frac{\sum_{k=n+1}^{\infty}(k-1)a_k\left|z^{k-1}\right|}{1-\sum_{k=n+1}^{\infty}a_k\left|z^{k-1}\right|} \\ &< \frac{\sum_{k=n+1}^{\infty}(k-1)a_k}{1-\sum_{k=n+1}^{\infty}a_k} \\ &\le 1-\beta. \end{aligned} \tag{17}$$

Since the values of $\left|\frac{zf'(z)}{f(z)}\right|$ lie in a circle at $w=1$ with radius $1-\beta$ so we have, $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta$ that is the starlike of order β.

Conversely, assume that $f(z)\in H^*_\beta(n,b,\phi)$. Then

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} = Re\left\{\frac{z-\sum_{k=n+1}^{\infty} e^{i(k-1)\phi log_b(k-1)}ka_k z^k}{z-\sum_{k=n+1}^{\infty} e^{i(k-1)\phi log_b(k-1)}a_k z^k}\right\} > \beta, \quad \forall z\in D.$$

Choose the values of z on half line, $z=re^{-i\arg(z)}$ such that $(0\le r<1)$.

$$Re\left\{\frac{z-\sum_{k=n+1}^{\infty} e^{i(k-1)\phi log_b(k-1)}ka_k z^k}{z-\sum_{k=n+1}^{\infty} e^{i(k-1)\phi log_b(k-1)}a_k z^k}\right\} = \left\{\frac{1-\sum_{k=n+1}^{\infty}ka_k z^k}{1-\sum_{k=n+1}^{\infty}a_k z^k}\right\} > \beta \tag{18}$$

Since numerator and denominator on the right hand side of (18) are non-negative therefore the Inequality becomes,

$$1-\sum_{k=n+1}^{\infty}ka_k z^k > \beta\left(1-\sum_{k=n+1}^{\infty}a_k z^k\right)$$

Let $r\longrightarrow 1$ through the real values of $z=re^{-i\arg(z)}$; $(0\le r<1)$ in above, we get,

$$1-\sum_{k=n+1}^{\infty}ka_k > \beta\left(1-\sum_{k=n+1}^{\infty}a_k\right)$$

after simplification we get,

$$\sum_{k=n+1}^{\infty}(k-\beta)a_k \le 1-\beta$$

and hence the theorem is complete. □

Theorem 2.2. A function $f(z)\in A(n,\phi)$ is in $k_\beta(n,b,\phi)$ if and only if:

$$\sum_{k=n+1}^{\infty}k(k-\beta)a_k \le 1-\beta. \tag{19}$$

Proof. As We noted earlier in our discussion that $f(z)\in K_\beta(n,b,\phi)$ if and only if $zf'(z)\in H^*_\beta(n,b,\phi)$ for some β $(0\le\beta<1)$. As with simple calculations we see that,

$$zf'(z) = z-\sum_{k=n+1}^{\infty} e^{i(k-1)\phi log_b(k-1)}ka_k z^k$$

By replacing $ka_k=a_k$ in Theorem (2.1) we get the proof of Theorem (2.2). □

Now we enlist the following Distortion Results due to [4].

Lemma 2.1. (Sekine and Owa [4]) A function $f(z)\in A$ is in $H^*_\beta(n,\phi)$, then

$$\begin{aligned}|z|-\frac{1-\beta}{n+1-\beta}|z|^{n+1} &\le |f(z)| \\ \le |z|+\frac{1-\beta}{n+1-\beta}|z|^{n+1}\end{aligned} \tag{20}$$

Remark 2.1. The right-hand equality is true for the function

$$f(z) = |z|-e^{i(n)\phi log(n)}\frac{1-\beta}{n+1-\beta}|z|^{n+1}, \tag{21}$$

for $(z=re^{-i(\arg(z)+2p\pi)},\ r<1)$
and the left-hand equality is true for the function

$$f(z) = |z|-e^{i(n)\phi log(n)}\frac{1-\beta}{n+1-\beta}|z|^{n+1}, \tag{22}$$

for $(z=re^{-i\arg(z)},\ r<1)$.

Lemma 2.2. (Sekine and Owa [4]) A function $g(z)\in A$ is in $K_\beta(n,\phi)$, then

$$\begin{aligned}|z|-\frac{1-\beta}{(n+1)(n+1-\beta)}|z|^{n+1} &\le |g(z)| \\ \le |z|+\frac{1-\beta}{(n+1)(n+1-\beta)}|z|^{n+1}\end{aligned} \tag{23}$$

Remark 2.2. The right-hand equality is true for the function

$$g(z) = |z| - e^{i(n)\phi log(n)} \frac{1-\beta}{(n+1)(n+1-\beta)} |z|^{n+1}, \quad (24)$$

for $(z = re^{-i(\arg(z)+2p\pi)}, r < 1)$
and the left-hand equality is true for the function,

$$g(z) = |z| - e^{i(n)\phi log(n)} \frac{1-\beta}{(n+1)(n+1-\beta)} |z|^{n+1}, \quad (25)$$

for $(z = re^{-i\arg(z)}, r < 1)$.

We now show the examples followed by its images of $|z| \leqq 1$ by the approximate expressions using complex tool [8]. In view of these figures, we can image that the functions in Examples 3.1 and 3.2 of starlike functions and the functions in Examples 3.3 and 3.4 of convex functions.

Remark 2.3. We note that in case $k \longrightarrow \infty$ the coefficient of $z^k \longrightarrow 0$ so, the higher powers of the function does not effect the graphs of $f(z)$.

3 Examples

Let $H_\beta^{b*}(n,\phi,h)$ denote the subclass of $A(n,\phi)$ consisting of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\phi log_b(k-1)} a_{k,h} z^k \quad (n \geqq h), \quad (26)$$

where
$a_{k,h} = \dfrac{(1-\beta)^2}{(k+h-\beta)(k+1+h-\beta)(k-\beta)}; (0 \leq \beta \leq 1),$
and let $K_\beta^b(n,\phi,h)$ denote the subclass of $A(n,\phi)$ consisting of the form

$$g(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\phi log_b(k-1)} b_{k,h} z^k \quad (n \geqq h), \quad (27)$$

where
$b_{k,h} = \dfrac{(1-\beta)^2}{(k+h-\beta)(k+1+h-\beta)(k-\beta)k}; (0 \leq \beta \leq 1).$

Now to step in to our examples and graphs we need to state two theorems due to (Sekine, Owa and Yamakawa [5]).

Lemma 3.1. If $f(z) \in A_\beta^b(n,\phi,h)$, then $f(z) \in H_\beta^*(n,\phi)$.

Lemma 3.2. If $g(z) \in K_\beta^b(n,\phi,h)$, then $g(z) \in K_\beta(n,\phi)$.

Note: For $\phi = 0$, these theorems were proved by (Sekine and Yamanaka [7]).

Example 3.1. If $f(z) \in A_0^e(1, \frac{\pi}{4}, -1)$, then we have the following function and its graph in figure [1],

$$\begin{aligned} f(z) &= z - \sum_{k=2}^{\infty} e^{i(k-1)\frac{\pi}{4} ln(k-1)} \frac{1}{k^2(k-1)} z^k \\ &= z - \frac{z^2}{4} - \frac{2^{(-1+i\pi/2)}}{9} z^3 - \frac{3^{(-1+3i\pi/4)}}{16} z^4 \\ &\quad - \frac{4^{(-1+i\pi)}}{25} z^5 - \frac{5^{(-1+5i\pi/4)}}{36} z^6 - \frac{6^{(-1+3\imath\pi/2)}}{49} z^7 \\ &\quad - \frac{7^{(-1+7i\pi/4)}}{64} z^8 - \frac{8^{(-1+2i\pi)}}{81} z^9 - \ldots, \end{aligned} \quad (28)$$

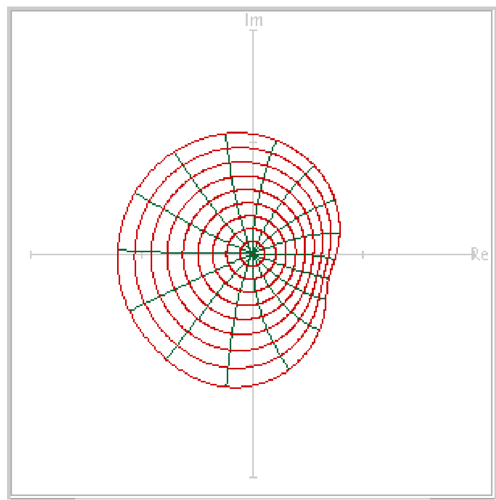

Fig. 1: Image of $|z| \leqq 1$ by $f(z) = z - \sum_{k=2}^{\infty} \frac{e^{i(k-1)\frac{\pi}{4} ln(k-1)}}{k^2(k-1)} z^k$

Example 3.2. If $f(z) \in A_0^{10}(1, \frac{\pi}{4}, -1)$, then we obtain a function and its graph in figure [2],

$$\begin{aligned} f(z) &= z - \sum_{k=2}^{\infty} e^{i(k-1)\dfrac{\pi}{4} log(k-1)} \frac{1}{k^2(k-1)} z^k \\ &= z - \frac{z^2}{4} - \frac{e^{(i\pi/2)log(2)}}{18} z^3 - \frac{e^{(3i\pi/4)log(3)}}{48} z^4 \\ &\quad - \frac{e^{(i\pi)log(4)}}{100} z^5 - \frac{e^{(5i\pi/4)log(5)}}{180} z^6 - \frac{e^{(3i\pi/2)log(6)}}{343} z^7 \\ &\quad - \frac{e^{(7i\pi/4)log(7)}}{448} z^8 - \frac{e^{(2i\pi)log(8)}}{648} z^9 - \ldots, \end{aligned} \quad (29)$$

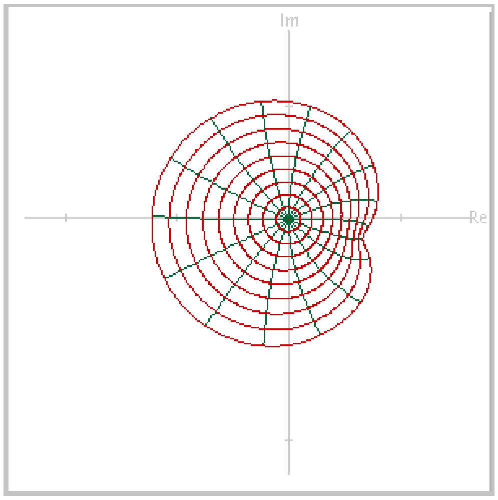

Fig. 2: Image of $|z| \leqq 1$ by $f(z) = z - \sum_{k=2}^{\infty} \frac{e^{i(k-1)\frac{\pi}{4} log(k-1)}}{k^2(k-1)} z^k$

Example 3.3. If $f(z) \in K_0^e(1, \frac{\pi}{4}, -1)$, then we obtain a function and its graph in figure [3],

$$\begin{aligned} f(z) &= z - \sum_{k=2}^{\infty} e^{i(k-1)\frac{\pi}{4}\ln(k-1)} \frac{1}{k^3(k-1)} z^k \\ &= z - \frac{z^2}{8} - \frac{2^{(-1+i\pi/2)}}{27} z^3 - \frac{3^{(-1+3i\pi/4)}}{64} z^4 \\ &- \frac{4^{(-1+i\pi)}}{125} z^5 - \frac{5^{(-1+5i\pi/4)}}{216} z^6 - \frac{6^{(-1+3i\pi/2)}}{343} z^7 \\ &- \frac{7^{(-1+7i\pi/4)}}{512} z^8 - \frac{8^{(-1+2i\pi)}}{729} z^9 - \dots, \end{aligned} \tag{30}$$

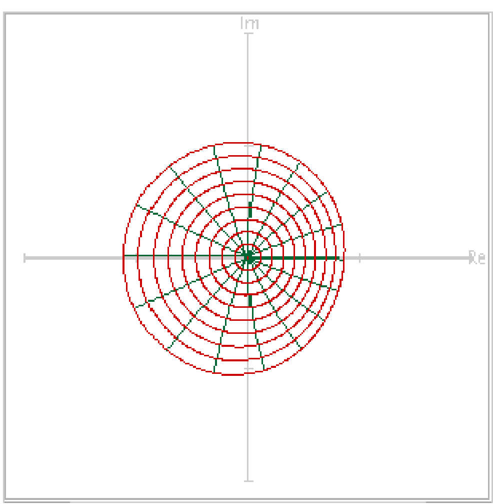

Fig. 3: Image of $|z| \leqq 1$ by $f(z) = z - \sum_{k=2}^{\infty} \frac{e^{i(k-1)\frac{\pi}{4} ln(k-1)}}{k^3(k-1)} z^k$

Example 3.4. If $f(z) \in K_0^{10}(1, \frac{\pi}{4}, -1)$, then we have the following function and graph in figure [4],

$$\begin{aligned} f(z) &= z - \sum_{k=2}^{\infty} e^{i(k-1)\frac{\pi}{4} log(k-1)} \frac{1}{k^3(k-1)} z^k \\ &= z - \frac{z^2}{8} - \frac{e^{(i\pi/2) log(2)}}{54} z^3 - \frac{e^{(-3i\pi/4) log(3)}}{192} z^4 \\ &- \frac{e^{(i\pi) log(4)}}{500} z^5 - \frac{e^{(5i\pi/4) log(5)}}{1080} z^6 - \frac{e^{(3i\pi/2) log(6)}}{2058} z^7 \\ &- \frac{e^{(7i\pi/4) log(7)}}{3584} z^8 - \frac{e^{(2i\pi) log(8)}}{5832} z^9 - \dots, \end{aligned} \tag{31}$$

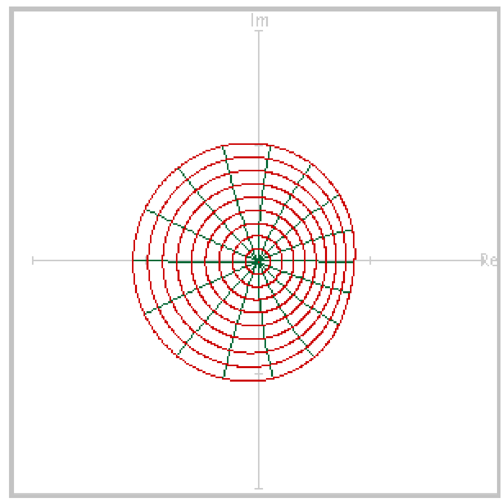

Fig. 4: Image of $|z| \leqq 1$ by $f(z) = z - \sum_{k=2}^{\infty} \frac{e^{i(k-1)\frac{\pi}{4} log(k-1)}}{k^3(k-1)} z^k$

Further we estimate the function in $H_\beta^*(n, \phi)$. Let $f(z) \in H_0^*(1, \frac{\pi}{4})$ then by Lemma (2.1), we have

$$|z| - \frac{1}{2}|z|^2 \leqslant |f(z)| \leqslant |z| + \frac{1}{2}|z|^2 \tag{32}$$

The right-hand equality is true for the function,
$f(z) = z - \frac{1-i}{2\sqrt{2}} z^2$, on the half line $z = re^{-i(\arg(z)+2p\pi)}$.
Also the Left-hand equality holds for $f(z) = z - \frac{1-i}{2\sqrt{2}} z^2$ on the half line $z = re^{-i\arg(z)}$ see figure [5].
By letting $|z| \longrightarrow 1$, we have $\frac{1}{2} \leq |f(z)| \leq \frac{3}{2}$ see figure [6].

Now to estimate the function in $K_\beta^b(n, \phi, h)$. Let $g(z) \in K_0^*(1, \frac{\pi}{4})$, we use Lemma (2.2) and get

$$|z| - \frac{1}{4}|z|^2 \leqslant |g(z)| \leqslant |z| + \frac{1}{4}|z|^2 \tag{33}$$

The right-hand equality is true for the function, $g(z) = z - \frac{1-i}{4\sqrt{2}} z^2$, on the half line $z = re^{-i(\arg(z)+2p\pi)}$ see figure [7].

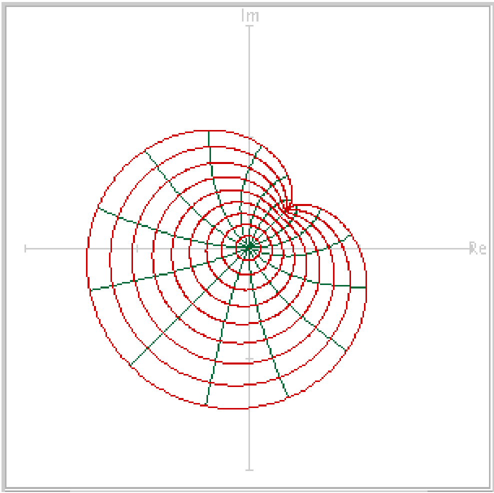

Fig. 5: Image of $|z| \leqq 1$ by $f(z) = z - \frac{1-i}{2\sqrt{2}}z^2$

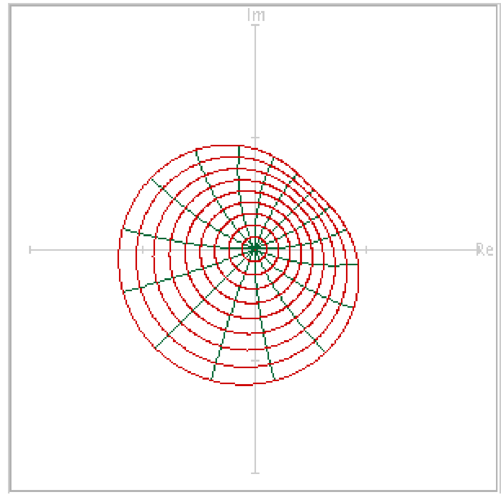

Fig. 7: Image of $|z| \leqq 1$ by $g(z) = z - \frac{1-i}{4\sqrt{2}}z^2$

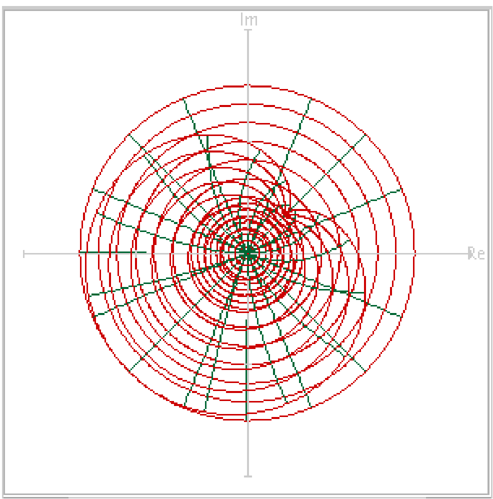

Fig. 6: Image of $|z| \leqq 1$ by $f(z) = z - \frac{1-i}{2\sqrt{2}}z^2$, $\frac{1}{2}z$ and $\frac{3}{2}z$

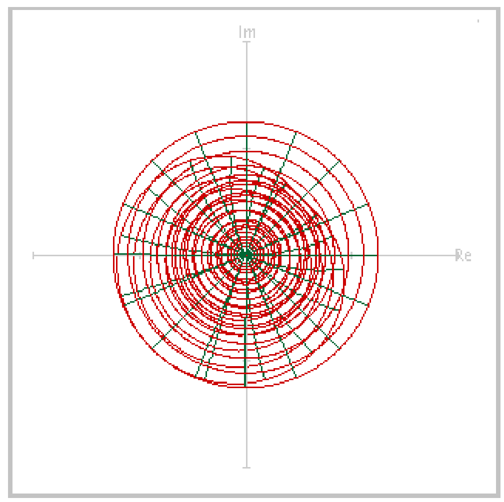

Fig. 8: Image of $|z| \leqq 1$ by $g(z) = z - \frac{1-i}{4\sqrt{2}}z^2$, $\frac{3}{4}z$ and $\frac{5}{4}z$

Also the Left-hand equality holds for, $g(z) = z - \dfrac{1-i}{4\sqrt{2}}z^2$ on the half line $z = re^{-i\arg(z)}$ and by letting $|z| \longrightarrow 1$ we have, $\dfrac{3}{4} \leqslant |g(z)| \leqslant \dfrac{5}{4}$ see figure [8].

4 Conclusion

In this paper we have introduced a new subclass of univalent function which is starlike and convex in the domain D. We offered several examples of it with its image illustration as well. As for function of real variables, we have defined general complex powers in terms of the complex logarithm and the complex exponential. It is evident from Examples 3.1 and 3.2 that if we put $b = e$ or $b = n \neq 1$ in the function $f(z) \in A^b_\beta(n, \phi, h)$, then we find the function in complex powers, $a^c = e^{c\ln a} = e^{c\ln|a| + i\arg(a)}$. We noted that c is complex in general, so $e^{c\ln|a|}$ is not necessarily real.

Conflict of Interest: The authors declare that there is no conflict of interest in terms of financial or anything else.

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References

[1] K. Kobori, On the necessary and sufficient condition for a power series to map the circular area onto the simple star-shaped or convex area, Mem. Coll. Sci. Kyoto Imp. Univ. (A), Vol. 15, pp. 279-291 (1932). (NOTE: Translated from German @Google Translator).

[2] M. S. Robertson, On the theory of univalent functions, Ann. of Math, Vol. 37, pp. 374–408 (1936).

[3] S. K. Chatterjea, On starlike functions, Journal of Pure Math, Vol. 1, pp. 23–26 (1981).

[4] T. Sekine and S. Owa, New problems of coefficients inequality, Publ. Res. Inst. Math. Sci, Vol. 1012, pp. 164–176 (1997).

[5] T. Sekine, S. Owa and R. Yamakawa, Examples of Starlike functions and convex Functions of order α, Research Institute of Number Analysis, Vol. 1062, pp. 120–127 (1998).

[6] H. Silverman, Univalent functions with negative coefficients, Proceedings of the American Mathematical Society, Vol. 51, pp. 109–116 (1975).

[7] T. Sekine and T. Yamanaka, Starlike functions and convex functions of order α with negative coefficients, Proceedings of the American Mathematical Society, Mathematical Sciences Research Hot-Line, Vol. 1, pp. 7–12 (1997).

[8] J. Rolf, Complex Analysis Project; Complex Tool (version 6.0.), U S Air Force Academy www.jimrolf.com/java/complexTool/bookComplexTool.html.

Hameed Ur Rehman is Lecturer in Mathematics. He received his Master of Science degree in Mathematics in 2002. Presently a student of PhD in Applied Mathematics at Universiti Kebangsaan Malaysia since 2016 under the supervision of Prof. Dr. Maslina Darus. His research interests are in the areas of integro-differential operator in fractional calculus and its applications, univalent functions, visualization and graphical analysis of complex functions. He has published research articles in reputed international journals of mathematics and mechanics, and in journals of geometry and Its applications.

Maslina Darus obtained her first degree in 1992 from Acadia University, Nova Scotia and obtained her PhD from University of Wales, Swansea, U. K. in 1996. She became a lecturer at Universiti Kebangsaan Malaysia and promoted as Associate Professor in 2001. In 2006, she was promoted to Full Professor. Presently she is working as a Deputy Dean, undergraduate of Faculty of Science and Technology. She is also holding a position as a Honorary Secertary of Malaysian Mathematical Sciences Society (PERSAMA) since 1997 till 2013. She has graduated more than 30 PhD students. She wrote more than 600 articles in refereed/indexed journals of various databases, be it in Scopus, ISI Thomson and other traditional journals globally.

Jamal Salah is Mathematics Assistant Professor. He obtained his PhD in Mathematics from National University Malaysia in 2012 in ”theory of Univalent Functions”. During his PhD research; he has introduced a new fractional calculus derivative operator called; The modified Caputo’s derivative operator by the mean of this operator he explored miscellaneous applications on the theory of univalent functions. Currently Jamal Salah expanded his field of research by attempting to relate the theory of univalent functions to the celebre Riemann Hypothesis together with fractional calculus applications. He is the reviewer and the member of editorial board for various Mathematical journals.