

Reliable Numerical Algorithm for Handling Differential-Algebraic System Involving Integral-Initial Conditions

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Abstract: The aim of the present analysis is to implement a relatively recent computational method, the reproducing kernel Hilbert space, for obtaining numerical solutions for differential algebraic system of integral-initial conditions. Two extended inner product spaces $W[0, 1]$ and $H[0, 1]$ are constructed in which the integral-initial conditions of the systems are satisfied. Whilst, two smooth kernel functions $R_t(s)$ and $r_t(s)$ are used throughout the evolution of the algorithm in order to obtain the required grid points. An efficient construction is given to obtain the numerical solutions for the systems together with an existence proof of the exact solutions based upon the reproducing kernel theory. In this approach, computational results of some numerical examples are presented to illustrate the viability, simplicity, and applicability of the algorithm developed. Finally, the utilized results show that the present algorithm and simulated annealing provide a good scheduling methodology to such systems.

Keywords: Differential algebraic systems, Reproducing kernel theory, Gram-Schmidt process

1 Introduction

Differential algebraic systems (DASs) of integral-initial conditions (IICs), which consist of systems of first-order differential equations coupled with purely systems of algebraic equations subject to given integral conditions and initial conditions, respectively, are important branch of modern mathematics that arises naturally from the mathematical models, or indirectly from converting the partial differential equations and the optimal control problems into ordinary differential equations [1, 2, 3, 4, 5, 6, 7]. There are a range of physical phenomena for which DASs of IICs provide the model examples, can be found in many areas of engineering and science ranging, from simple beam bending problems in mechanics to the chemical engineering areas of absorption phenomena, chemical reactions, radiation effects and problems connected with heat transfer, fluid flow, dissipation of energy, and control theory [8, 9, 10, 11, 12, 13, 14]. Generally, it is difficult to obtain the closed form solutions for DASs of IICs in terms of elementary functions, especially, for nonlinear, non-constant coefficients, and non-homogeneous cases. Factually, in most cases, only approximate solutions or numerical solutions can be expected; therefore, it has attracted much attention and has been studied by many authors. In this

regards, there are many iterative methods have been proposed to be one of the suitable and successful classes of numerical techniques for obtaining the solutions of numerous types of DASs in general (see, for instance, [15, 16, 17, 18, 19, 20, 21] and the references therein).

The reproducing kernel Hilbert space (RKHS) method is a numerical, as well as, analytical technique for solving a large variety of ordinary and partial differential equations associated to different kind of constraint conditions, and usually provides the solutions in term of rapidly convergent series with components that can be elegantly computed. In this study, a general technique based on the reproducing kernel theory is proposed for solving DASs of IICs in the appropriate inner product spaces. The main idea is to construct the direct sum of the RKHSs that satisfying the IICs of the given systems in order to determining their exact and their numerical solutions. The exact and the numerical solutions are represented in the form of series through the functions value at the right-hand side of the corresponding differential and algebraic equations. The advantages of the utilized approach lie in the following main advantages; firstly, it can produce good globally smooth numerical solutions, and with ability to solve many differential systems with complex constraint conditions,

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which are difficult to solve; secondly, the numerical solutions and their derivatives are converge uniformly to the exact solutions and their derivatives, respectively; thirdly, the method is mesh-free, easily implemented and capable in treating various differential systems and various constraint conditions.

Anyhow, DASs of IICs have been investigated systematically in this analysis for the development, analysis, and implementation of an accurate algorithm which allows for the use of some form of concurrent processing technique. More precisely, we consider the following set of differential equations:

$$\begin{cases} a_1(t) u'_1(t) = f_1(t, u_1(t), u_2(t), \dots, u_n(t)), \\ a_2(t) u'_2(t) = f_2(t, u_1(t), u_2(t), \dots, u_n(t)), \\ \vdots \\ a_p(t) u'_p(t) = f_p(t, u_1(t), u_2(t), \dots, u_n(t)), \end{cases} \quad (1)$$

subject to the following set of integral conditions:

$$\begin{cases} u_1(0) = \int_0^1 k_1(s) u_1(s) + \eta_1, \\ u_2(0) = \int_0^1 k_2(s) u_2(s) + \eta_2, \\ \vdots \\ u_p(0) = \int_0^1 k_p(s) u_p(s) + \eta_p, \end{cases} \quad (2)$$

simultaneously with the following set of purely systems of algebraic equations:

$$\begin{cases} a_{p+1}(t) u_{p+1}(t) = f_{p+1}(t, u_1(t), u_2(t), \dots, u_n(t)), \\ a_{p+2}(t) u_{p+2}(t) = f_{p+2}(t, u_1(t), u_2(t), \dots, u_n(t)), \\ \vdots \\ a_n(t) u_n(t) = f_n(t, u_1(t), u_2(t), \dots, u_n(t)), \end{cases} \quad (3)$$

subject to one of the following set of initial conditions:

$$\begin{cases} \{u_{p+1}(0) = \eta_{p+1}, u_{p+2}(0) = \eta_{p+2}, \dots, u_n(0) = \eta_n, \\ \{u_{p+1}(1) = \eta_{p+1}, u_{p+2}(1) = \eta_{p+2}, \dots, u_n(1) = \eta_n, \end{cases} \quad (4)$$

where $\{u_j\}_{j=1}^p \subset W_2^2[0, 1]$ and $\{u_j\}_{j=p+1}^n \subset \underline{W}_2^1[0, 1]$

or $\{u_j\}_{j=p+1}^n \subset \overline{W}_2^1[0, 1]$ are unknown functions to be determined, $\{f_j(t, v_1, v_2, \dots, v_n)\}_{j=1}^n$ are continuous terms in $W_2^1[0, 1]$ as $\{v_j\}_{j=1}^p \subset W_2^2[0, 1]$ and $\{v_j\}_{j=p+1}^n \subset \underline{W}_2^1[0, 1]$ or $\{v_j\}_{j=p+1}^n \subset \overline{W}_2^1[0, 1]$, $-\infty < v_j < \infty$, and $W_2^1[0, 1], \underline{W}_2^1[0, 1], \overline{W}_2^1[0, 1], W_2^2[0, 1]$ are four RKHSs. Here, the set $\{a_j\}_{j=1}^n$ are analytical real-valued functions on $[0, 1]$ and may take the values $a_j(t_\lambda) = 0$ for some $t_\lambda \in [0, 1]$ and some $j \in \{1, 2, \dots, n\}$ which make Eqs. (1) and (3) to be singular at $t = t_\lambda$. Through this paper, we assume that Eqs. (1) - (4) have unique analytical solutions on $[0, 1]$. The theory of reproducing

kernel was used for the first time at the beginning of the 20th century as a novel solver for the boundary value problems of harmonic and biharmonic functions types. This theory, which is representative in the RKHS method, has been successfully applied to various important application in numerical analysis, computational mathematics, image processing, machine learning, probability and statistics, and finance [22, 23, 24, 25, 26, 27]. The RKHS method is a useful framework for constructing numerical solutions of great interest to applied sciences. In the recent years, based on this theory, extensive work has been proposed and discussed for the numerical solutions of several integral and differential operators side by side with their theories. The reader is kindly requested to go through [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] in order to know more details about the RKHS method, including its modification and scientific applications, its characteristics and symmetric kernel functions, and others.

This article is organized as follows. In the next section, two extended inner product spaces needed in the analysis are constructed, and two extended reproducing kernel functions are obtained. After that, in Section 3, the solutions and the essential theoretical results are presented based upon the reproducing kernel theory. In Section 4, an efficient iterative technique for the solutions is described, whilst, convergent theorem and error behavior are also presented. Numerical algorithm and numerical outcomes are discussed to demonstrate the accuracy and the applicability of the presented method as utilized in Section 5. Finally, in Section 6 some concluding remarks and brief conclusions are utilized.

2 Building appropriate inner product spaces

The reproducing kernel approach builds on a Hilbert space H and requires that all Dirac evaluation functional in H are bounded and continuous [39, 40, 41, 42, 43, 44, 45, 46, 47]. In this section, two extended inner product spaces $H[0, 1]$ and $W[0, 1]$ are constructed. Then, we utilize the reproducing kernel concept to obtain two extended reproducing kernel functions $R_t(s)$ and $r_t(s)$ in order to formulate the solutions in the mentioned spaces.

Let H be a Hilbert space of function $\theta : \Omega \rightarrow H$ on a set Ω . A function $R : \Omega \times \Omega \rightarrow \mathbb{C}$ is a reproducing kernel of H if the following conditions are met. Firstly, $R(\cdot, t) \in H$ for each $t \in \Omega$. Secondly, $\langle \theta(\cdot), R(\cdot, t) \rangle = \theta(t)$ for each $\theta \in H$ and each $t \in \Omega$. The condition $\langle \theta(\cdot), R(\cdot, t) \rangle = \theta(t)$ is called the reproducing property, which means that, the value of θ at the point t is reproducing by the inner product of θ with $R(\cdot, t)$. Indeed, a Hilbert space which possesses a reproducing kernel is called a RKHS. Through the remainder sections, the symbol $R_t(\cdot)$ whenever used means $R(t, \cdot)$.

Definition 2.1 [26] The space $W_2^1[0, 1]$ is defined as $W_2^1[0, 1] = \{z : z \text{ is absolutely continuous function on } [0, 1] \text{ and } z' \in L^2[0, 1]\}$. On the other hand, the inner product and the norm in $W_2^1[0, 1]$ are defined, respectively, by

$$\langle z_1(t), z_2(t) \rangle_{W_2^1} = z_1(0)z_2(0) + \int_0^1 z_1'(t)z_2'(t)dt, \quad (5)$$

and $\|z_1\|_{W_2^1} = \sqrt{\langle z_1(t), z_1(t) \rangle_{W_2^1}}$, where $z_1, z_2 \in W_2^1[0, 1]$.

Definition 2.2 [26] The space $\underline{W}_2^1[0, 1]$ is defined as $\underline{W}_2^1[0, 1] = \{z : z \text{ is absolutely continuous function on } [0, 1], z(0) = 0, \text{ and } z' \in L^2[0, 1]\}$. On the other hand, the inner product and the norm in $\underline{W}_2^1[0, 1]$ are defined, respectively, by

$$\langle z_1(t), z_2(t) \rangle_{\underline{W}_2^1} = z_1(0)z_2(0) + \int_0^1 z_1'(t)z_2'(t)dt, \quad (6)$$

and $\|z_1\|_{\underline{W}_2^1} = \sqrt{\langle z_1(t), z_1(t) \rangle_{\underline{W}_2^1}}$, where $z_1, z_2 \in \underline{W}_2^1[0, 1]$.

Definition 2.3 The space $\overline{W}_2^1[0, 1]$ is defined as $\overline{W}_2^1[0, 1] = \{z : z \text{ is absolutely continuous function on } [0, 1], z(1) = 0, \text{ and } z' \in L^2[0, 1]\}$. On the other hand, the inner product and the norm in $\overline{W}_2^1[0, 1]$ are defined, respectively, by

$$\langle z_1(t), z_2(t) \rangle_{\overline{W}_2^1} = z_1(1)z_2(1) + \int_0^1 z_1'(t)z_2'(t)dt, \quad (7)$$

and $\|z_1\|_{\overline{W}_2^1} = \sqrt{\langle z_1(t), z_1(t) \rangle_{\overline{W}_2^1}}$, where $z_1, z_2 \in \overline{W}_2^1[0, 1]$.

Definition 2.4 The space $W_2^{2k}[0, 1]$ is defined as $W_2^{2k}[0, 1] = \{z : z, z' \text{ are absolutely continuous functions on } [0, 1], z(0) - \int_0^1 k(s)z(s)ds = 0, z'' \in L^2[0, 1]\}$. On the other hand, the inner product and the norm in $W_2^{2k}[0, 1]$ are defined, respectively, by

$$\begin{aligned} \langle z_1(t), z_2(t) \rangle_{W_2^{2k}} &= z_1(0)z_2(0) + z_1'(0)z_2'(0) \\ &+ \int_0^1 z_1''(t)z_2''(t)dt, \end{aligned} \quad (8)$$

and $\|z_1\|_{W_2^{2k}} = \sqrt{\langle z_1(t), z_1(t) \rangle_{W_2^{2k}}}$, where $z_1, z_2 \in W_2^{2k}[0, 1]$.

Here, $\langle z_1(t), z_2(t) \rangle_{W_2^{2k}}$ satisfies all the requirements of the inner product as follows; firstly, $\langle z_1(t), z_1(t) \rangle_{W_2^{2k}} \geq 0$; secondly,

$\langle z_1(t), z_2(t) \rangle_{W_2^{2k}} = \langle z_2(t), z_1(t) \rangle_{W_2^{2k}}$; thirdly, $\langle \gamma z_1(t), z_2(t) \rangle_{W_2^{2k}} = \gamma \langle z_1(t), z_2(t) \rangle_{W_2^{2k}}$; fourthly, $\langle z_1(t) + z_2(t), z_3(t) \rangle_{W_2^{2k}} = \langle z_1(t), z_3(t) \rangle_{W_2^{2k}} + \langle z_2(t), z_3(t) \rangle_{W_2^{2k}}$. Indeed, $\langle z_1(t), z_1(t) \rangle_{W_2^{2k}} = 0$ if and only if $z_1(t) = 0$. To see this, when $z_1(t) = 0$, then $\langle z_1(t), z_1(t) \rangle_{W_2^{2k}} = 0$, whilst, if $\langle z_1(t), z_1(t) \rangle_{W_2^{2k}} = 0$, then $(z_1(0))^2 + (z_1'(0))^2 + \int_0^1 (z_1''(t))^2 dt = 0$, therefore $z_1(0) = z_1'(0) = z_1''(t) = 0$ or $z_1(t) = 0$.

An important subsets of RKHSs are those associated to continuous kernels. These spaces have wide applications, including complex analysis, harmonic analysis, quantum mechanics, statistics, and machine learning [22, 23, 24, 25]. Next, before any further discussion, we need to obtain the reproducing kernel functions of the spaces $W_2^1[0, 1]$, $\underline{W}_2^1[0, 1]$, $\overline{W}_2^1[0, 1]$, and $W_2^{2k}[0, 1]$, respectively, as follows:

Theorem 2.1 [26] The Hilbert space $W_2^1[0, 1]$ is a complete reproducing kernel with the reproducing kernel function

$$R_t^{\{1\}}(s) = \begin{cases} R_{t,1}^{\{1\}}(s) = 1 + s, & s \leq t, \\ R_{t,2}^{\{1\}}(s) = 1 + t, & s > t. \end{cases} \quad (9)$$

Theorem 2.2 The Hilbert space $\underline{W}_2^1[0, 1]$ is a complete reproducing kernel with the reproducing kernel function

$$\underline{R}_t^{\{1\}}(s) = \begin{cases} \underline{R}_{t,1}^{\{1\}}(s) = s, & s \leq t, \\ \underline{R}_{t,2}^{\{1\}}(s) = t, & s > t. \end{cases} \quad (10)$$

Proof. By applying the tabular integration formula on $u'(s)\partial_s^1 \underline{R}_t^{\{1\}}(s)$ and considering Eq. (6), it easy to see that

$$\begin{aligned} \langle u(s), \underline{R}_t^{\{1\}}(s) \rangle_{\underline{W}_2^1} &= u(0) \left[\underline{R}_t^{\{1\}}(0) - \partial_s^1 \underline{R}_t^{\{1\}}(0) \right] \\ &+ u(1) \partial_s^1 \underline{R}_t^{\{1\}}(1) - \int_0^1 u(s) \partial_s^2 \underline{R}_t^{\{1\}}(s) ds. \end{aligned} \quad (11)$$

But since $u(s), \underline{R}_t^{\{1\}}(s) \in \underline{W}_2^1[0, 1]$, it follows that $u(0) = 0$ and $\underline{R}_t^{\{1\}}(0) = 0$. Thus, using the properties $\underline{R}_t^{\{1\}}(t+0) - \underline{R}_t^{\{1\}}(t-0) = 0$ and $\partial_s^1 \underline{R}_t^{\{1\}}(t+0) - \partial_s^1 \underline{R}_t^{\{1\}}(t-0) = -1$, the rules of $\underline{R}_t^{\{1\}}(s)$ can be obtained directly.

Theorem 2.3 The Hilbert space $\overline{W}_2^1[0, 1]$ is a complete reproducing kernel with the reproducing kernel function

$$\overline{R}_t^{\{1\}}(s) = \begin{cases} \overline{R}_{t,1}^{\{1\}}(s) = 1 - s, & s \leq t, \\ \overline{R}_{t,2}^{\{1\}}(s) = 1 - t, & s > t. \end{cases} \quad (12)$$

Proof. By applying the tabular integration formula on $u'(s)\partial_s^1 \bar{R}_t^{\{1\}}(s)$ and considering Eq. (7), it easy to see that

$$\begin{aligned} \left\langle u(s), \bar{R}_t^{\{1\}}(s) \right\rangle_{\bar{W}_2^1} &= u(1) \left[\bar{R}_t^{\{1\}}(1) + \partial_s^1 \bar{R}_t^{\{1\}}(1) \right] \\ &\quad - u(0) \partial_s^1 \bar{R}_t^{\{1\}}(0) - \int_0^1 u(s) \partial_s^2 \bar{R}_t^{\{1\}}(s) ds. \end{aligned} \quad (13)$$

But since $u(s), \bar{R}_t^{\{1\}}(s) \in \bar{W}_2^1[0, 1]$, it follows that $u(1) = 0$ and $\bar{R}_t^{\{1\}}(1) = 0$. Thus, using the properties $\bar{R}_t^{\{1\}}(t+0) - \bar{R}_t^{\{1\}}(t-0) = 0$ and $\partial_s^1 \bar{R}_t^{\{1\}}(t+0) - \partial_s^1 \bar{R}_t^{\{1\}}(t-0) = -1$, the rules of $\bar{R}_t^{\{1\}}(s)$ can be obtained directly.

Theorem 2.4 The Hilbert space $W_2^{2k}[0, 1]$ is a complete reproducing kernel with the reproducing kernel function

$$R_t^{\{2,k\}}(s) = \begin{cases} R_{t,1}^{\{2,k\}}(s), & s \leq t, \\ R_{t,2}^{\{2,k\}}(s), & s > t. \end{cases} \quad (14)$$

where $R_{t,1}^{\{2,k\}}(s) = \sum_{i=1}^4 a_i(t) s^{i-1} + C(t, s),$

$$R_{t,2}^{\{2,k\}}(s) = \sum_{i=1}^4 b_i(t) s^{i-1} + C(t, s) \quad \text{and}$$

$$C(t, s) = c(t) \int_0^s \int_0^s \int_0^s \int_0^s k(s) ds ds ds ds.$$

Proof. The proof of the completeness and the reproducing property of $W_2^{2k}[0, 1]$ is similar to the proof in [28]. Let us find out the expression form of $R_t^{\{2,k\}}(s)$ in $W_2^{2k}[0, 1]$. Since

$$\begin{aligned} \left\langle u(s), R_t^{\{2,k\}}(s) \right\rangle_{W_2^{2k}} &= \sum_{i=0}^1 u^{(i)}(0) \partial_s^i R_t^{\{2,k\}}(0) \\ &\quad + \int_0^1 u''(s) \partial_s^2 R_t^{\{2,k\}}(s) ds \\ &\quad + c(t) \left[u(0) - \int_0^1 k(s) u(s) ds \right], \end{aligned} \quad (15)$$

where $u(s), R_t^{\{2,k\}}(s) \in W_2^{2k}[0, 1]$. By applying the tabular integration formula on $u''(s)\partial_s^2 R_t^{\{2,k\}}(s)$, we get

$$\begin{aligned} \int_0^1 u''(s) \partial_s^2 R_t^{\{2,k\}}(s) ds &= \\ \sum_{i=0}^1 (-1)^{1-i} u^{(i)}(s) \partial_s^{3-i} R_t^{\{2,k\}}(s) \Big|_{s=0}^{s=1} \end{aligned}$$

$$+ \int_0^1 u(s) \partial_s^4 R_t^{\{2,k\}}(s) ds. \quad (16)$$

According to Eq. (8), one can write

$$\begin{aligned} \left\langle u(s), R_t^{\{2,k\}}(s) \right\rangle_{W_2^{2k}} &= \\ u(0) \left[R_t^{\{2,k\}}(0) + \partial_s^3 R_t^{\{2,k\}}(0) + c(t) \right] \\ &\quad + u'(0) \left[\partial_s^1 R_t^{\{2,k\}}(0) - \partial_s^2 R_t^{\{2,k\}}(0) \right] \\ &\quad - u(1) \partial_s^3 R_t^{\{2,k\}}(1) + u'(1) \partial_s^2 R_t^{\{2,k\}}(1) \\ &\quad + \int_0^1 u(s) \left[\partial_s^4 R_t^{\{2,k\}}(s) - c(t) k(s) \right] ds. \end{aligned} \quad (17)$$

Since $R_t^{\{2,k\}}(s) \in W_2^{2k}[0, 1]$, it follows that $R_t^{\{2,k\}}(0) = \int_0^1 k(s) R_t^{\{2,k\}}(s) ds$. Thus, if $R_t^{\{2,k\}}(0) + \partial_s^3 R_t^{\{2,k\}}(0) + c(t) = 0$, $\partial_s^1 R_t^{\{2,k\}}(0) - \partial_s^2 R_t^{\{2,k\}}(0) = 0$, $\partial_s^3 R_t^{\{2,k\}}(1) = 0$, and $\partial_s^2 R_t^{\{2,k\}}(1) = 0$. then Eq. (17) implies that $\left\langle u(s), R_t^{\{2,k\}}(s) \right\rangle_{W_2^{2k}} = \int_0^1 u(s) \left[\partial_s^4 R_t^{\{2,k\}}(s) - c(t) k(s) \right] ds$. Now, for each $t \in [0, 1]$, if $R_t^{\{2,k\}}(s)$ satisfies

$$\partial_s^4 R_t^{\{2,k\}}(s) - c(t) k(s) = \delta(s - t), \quad (18)$$

then $\left\langle u(s), R_t^{\{2,k\}}(s) \right\rangle_{W_2^{2k}} = u(t)$. Obviously, $R_t^{\{2,k\}}(s)$ is the reproducing kernel function of $W_2^{2k}[0, 1]$. For the conduct of proceedings in the proof, it requires the expression form of $R_t^{\{2,k\}}(s)$. For $t \neq s$, the auxiliary formula of Eq. (18) is $\lambda^4 = 0$, and its auxiliary values are $\lambda = 0$ with multiplicity 4. So, let the expression form of $R_t^{\{2,k\}}(s)$ be as defined in Eq. (14). But on the other aspect as well, for Eq. (18), let $R_t^{\{2,k\}}(s)$ satisfies $\partial_s^q R_t^{\{2,k\}}(t+0) - \partial_s^q R_t^{\{2,k\}}(t-0) = 0, q = 0, 1, 2$. Integrating Eq. (18) from $x - \varepsilon$ to $t + \varepsilon$ with respect to y and letting $\varepsilon \rightarrow 0$, we have the jump degree of $\partial_s^3 R_t^{\{2,k\}}(s)$ at $s = t$ such that $\partial_s^3 R_t^{\{2,k\}}(t+0) - \partial_s^3 R_t^{\{2,k\}}(t-0) = -1$. Through the last descriptions and by using Maple 13 software package, the unknown coefficients of $R_t^{\{2,k\}}(s)$ can be obtained.

Throughout this paper and without the loss of generality we are focusing on the construction proof by using $W_2^{2k}[0, 1]$ and $\underline{W}_2^1[0, 1]$ as the domain space. Actually, in the same manner, we can employ our

construction if $W_2^{2k} [0, 1]$ and $\overline{W}_2^1 [0, 1]$ are the domain space.

Remark 2.1 Henceforth and not to conflict unless stated otherwise, we denote the following symbols:

$$H [0, 1] = \bigoplus_{j=1}^n W_2^1 [0, 1],$$

$$W [0, 1] = \left(\bigoplus_{j=1}^p W_2^{2k_j} [0, 1] \right) \oplus \left(\bigoplus_{j=p+1}^n \underline{W}_2^1 [0, 1] \right). \quad (19)$$

$$R_t (s) = \left(R_t^{\{2, k_1\}} (s), \dots, \left(R_t^{\{2, k_p\}} \right)_{p\text{th}} (s) \right),$$

$$\left(\underline{R}_t^{\{1\}} \right)_{(p+1)\text{th}} (s), \dots, \left(\underline{R}_t^{\{1\}} \right)_{n\text{th}} (s)^T,$$

$$r_t (s) = \left(R_t^{\{1\}} (s), R_t^{\{1\}} (s), \dots, \left(R_t^{\{1\}} \right)_{n\text{th}} (s) \right)^T. \quad (20)$$

Definition 2.5 [26] The inner product Hilbert space $H [0, 1]$ can be defined as

$$H [0, 1] = \{ (z_1, z_2, \dots, z_n)^T : \{z_j\}_{j=1}^n \subset W_2^1 [0, 1] \}. \quad (21)$$

The inner product and the norm in $H [0, 1]$ are building as

$$\langle z(t), w(t) \rangle_H = \sum_{j=1}^n \langle z_j(t), w_j(t) \rangle_{W_2^1}, \quad (22)$$

and $\|z\|_H = \sqrt{\sum_{j=1}^n \|z_j\|_{W_2^1}^2}$, respectively, where $z, w \in H [0, 1]$.

Definition 2.6 The inner product Hilbert space $W [0, 1]$ can be defined as

$$W [0, 1] = \{ (z_1, z_2, \dots, z_p, z_{p+1}, \dots, z_n)^T$$

such that

$$z_j \in \begin{cases} W_2^{2k_j} [0, 1], & j = 1, 2, \dots, p, \\ \underline{W}_2^1 [0, 1], & j = p+1, p+2, \dots, n. \end{cases} \quad (23)$$

The inner product and the norm in $W [0, 1]$ are building as

$$\langle z(t), w(t) \rangle_W = \sum_{j=1}^p \langle z_j(t), w_j(t) \rangle_{W_2^{2k_j}} + \sum_{j=p+1}^n \langle z_j(t), w_j(t) \rangle_{\underline{W}_2^1}, \quad (24)$$

and $\|z\|_W = \sqrt{\sum_{j=1}^p \|z_j\|_{W_2^{2k_j}}^2 + \sum_{j=p+1}^n \|z_j\|_{\underline{W}_2^1}^2}$, respectively, where $z, w \in W [0, 1]$.

The spaces $H [0, 1]$ and $W [0, 1]$ are complete Hilbert with some special properties. So, all the properties of the Hilbert space will be hold. Further, these spaces possesses some special and better properties which could make some systems be solved easier. For instance, many systems studied in $L^2 [0, 1]$ space, which is a complete Hilbert, requires large amount of integral-differential computations and such computations may be very difficult in some cases. Thus, the numerical integrals-differentials have to be calculated in the cost of losing some accuracy. However, the properties of the spaces $H [0, 1]$ and $W [0, 1]$ require no more integral-differential computations for some functions, instead of computing some values of a function at some grid points. In fact, this simplification of computations not only improves the computational speed, but also improves the computational accuracy.

3 Representation of analytical and numerical solutions

In this section, we will show how to solve the DAS of IICs of Eqs. (1) - (4) by using the RKHS method in detail and we will see what the influence choice of the continuous linear operators. Anyhow, the formulation and the implementation method of the exact and the numerical solutions are given in the extended RKHSs $W [0, 1]$ and $H [0, 1]$. Meanwhile, we construct an orthogonal function systems of the space $W [0, 1]$ based on the use of the Gram-Schmidt process.

Now, to apply the RKHS method on $H [0, 1]$ and $W [0, 1]$, we will define the following linear operators:

$$L_j : \begin{cases} W_2^{2k_j} [0, 1] \rightarrow W_2^1 [0, 1], & j = 1, 2, \dots, p, \\ \underline{W}_2^1 [0, 1] \rightarrow W_2^1 [0, 1], & j = p+1, p+2, \dots, n, \end{cases} \quad (25)$$

such that

$$L_j u_j(t) = \begin{cases} a_j u_j'(t), & j = 1, 2, \dots, p, \\ a_j u_j(t), & j = p+1, p+2, \dots, n. \end{cases} \quad (26)$$

For the conduct of proceedings in the algorithm construction, we put

$$F = \text{Column} (f_1, f_2, \dots, f_p, f_{p+1}, \dots, f_n),$$

$$u = \text{Column} (u_1, u_2, \dots, u_p, u_{p+1}, \dots, u_n),$$

$$u' = \text{Column} (u'_1, u'_2, \dots, u'_p, u'_{p+1}, \dots, u'_n),$$

$$\alpha = \left(\int_0^1 k_1(s) u_1 + \eta_1, \dots, \int_0^1 k_p(s) u_p + \eta_p, \dots, 0 \right)_{n \times 1},$$

$$e_j = \text{Column} (0, 0, \dots, 0, 1_{j\text{th}}, 0, \dots, 0)_{n \times 1},$$

$$L = \text{Diagonal} (L_1, L_2, \dots, L_p, L_{p+1}, \dots, L_n). \quad (27)$$

Thus, based on this, the DAS of IICs of Eqs. (1) - (4) can be converted into the following equivalent form:

$$L : W[0, 1] \rightarrow H[0, 1] \text{ such that } Lu(t) = F(t, u(t)), \quad (28)$$

in which $u \in W[0, 1]$ and $F \in H[0, 1]$;

subject to the following sets of constraint conditions:

$$e_j^T u(0) = e_j^T \alpha, j = 1, 2, \dots, p$$

and

$$e_j^T u(0) = e_j^T \eta, j = p+1, p+2, \dots, n. \quad (29)$$

Lemma 3.1 The operators $\{L_j\}_{j=1}^p : W_2^{2k_j}[0, 1] \rightarrow W_2^1[0, 1]$ and $\{L_j\}_{j=p+1}^n : W_2^1[0, 1] \rightarrow W_2^1[0, 1]$ are bounded and linear.

Proof. In this proof, we are focusing on $\{L_j\}_{j=1}^p : W_2^{2k_j}[0, 1] \rightarrow W_2^1[0, 1]$. The linearity part is obvious, for the boundedness part, we need to prove that $\|L_j u_j\|_{W_2^1} \leq M \|u_j\|_{W_2^{2k_j}}$, where $M > 0$. By the Schwarz inequality, one can write

$$|u_j(t)| = \left| \left\langle u_j(\cdot), R_t^{\{2,k\}}(\cdot) \right\rangle_{W_2^{2k_j}} \right| \leq \|u_j\|_{W_2^{2k_j}} \|R_t^{\{2,k\}}(\cdot)\|_{W_2^{2k_j}},$$

$$|u_j'(t)| = \left| \left\langle u_j(\cdot), \partial_t^1 R_t^{\{2,k\}}(\cdot) \right\rangle_{W_2^{2k_j}} \right| \leq \|u_j\|_{W_2^{2k_j}} \|\partial_t^1 R_t^{\{2,k\}}(\cdot)\|_{W_2^{2k_j}},$$

and

$$|u_j''(t)| \leq \|u_j\|_{W_2^{2k_j}} \|\partial_t^2 R_t^{\{2,k\}}(\cdot)\|_{W_2^{2k_j}}.$$

From the definition of the inner product and the norm of $W_2^1[0, 1]$, it follows that

$$\begin{aligned} & \|a_j u_j'\|_{W_2^1} \\ &= \left((a_j(0) u_j'(0))^2 + \int_0^1 (a_j'(t) u_j''(t))^2 dt \right)^{\frac{1}{2}} \\ &\leq \left(a_j^2(0) (u_j'(0))^2 + \left(\int_0^1 (a_j'(t))^4 (u_j''(t))^4 dt \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ &\leq [a_j^2(0) \|u_j\|_{W_2^{2k_j}}^2 \|\partial_t^1 R_0^{\{2,k\}}(\cdot)\|_{W_2^{2k_j}}^2 \\ &\quad + \left(\int_0^1 (a_j'(t))^4 \|u_j\|_{W_2^{2k_j}}^4 \|\partial_t^2 R_t^{\{2,k\}}(\cdot)\|_{W_2^{2k_j}}^4 dt \right)^{\frac{1}{2}}]^{\frac{1}{2}} \\ &\leq \|u_j\|_{W_2^{2k_j}} [a_j^2(0) \|\partial_t^1 R_0^{\{2,k\}}(\cdot)\|_{W_2^{2k_j}}^2 \\ &\quad + \left(\int_0^1 \max((a_j')^4 [0, 1]) \|\partial_t^2 R_t^{\{2,k\}}(\cdot)\|_{W_2^{2k_j}}^4 dt \right)^{\frac{1}{2}}]^{\frac{1}{2}} \\ &= \|u_j\|_{W_2^{2k_j}} [a_j^2(0) \|\partial_t^1 R_0^{\{2,k\}}(\cdot)\|_{W_2^{2k_j}}^2 \\ &\quad + \max((a_j')^4 [0, 1]) \|\partial_t^2 R_t^{\{2,k\}}(\cdot)\|_{W_2^{2k_j}}^2]^{\frac{1}{2}} \\ &\leq M \|u_j\|_{W_2^{2k_j}}, M > 0, \end{aligned} \quad (30)$$

where

$$M = [a_j^2(0) \|\partial_t^1 R_0^{\{2,k\}}(\cdot)\|_{W_2^{2k_j}}^2 + \max((a_j')^4 [0, 1]) \|\partial_t^2 R_t^{\{2,k\}}(\cdot)\|_{W_2^{2k_j}}^2]^{\frac{1}{2}}.$$

Theorem 3.1 [26] The operator $L : W[0, 1] \rightarrow H[0, 1]$ is bounded and linear.

Proof. Clearly, L is linear operator from $W[0, 1]$ into $H[0, 1]$. The boundedness part is shown as follows: for each $u \in W[0, 1]$, we have

$$\begin{aligned} \|Lu\|_H &= \sqrt{\sum_{j=1}^n \|L_j u_j\|_{W_2^1}^2} \\ &= \sqrt{\sum_{j=1}^p \|L_j u_j\|_{W_2^1}^2 + \sum_{j=p+1}^n \|L_j u_j\|_{W_2^1}^2} \\ &\leq \sqrt{\sum_{j=1}^p \|L_j\|^2 \|u_j\|_{W_2^{2k_j}}^2 + \sum_{j=p+1}^n \|L_j\|^2 \|u_j\|_{W_2^1}^2} \\ &\leq \sqrt{\sum_{j=1}^p \|L_j\|^2 \sum_{j=1}^p \|u_j\|_{W_2^{2k_j}}^2 + \sum_{j=p+1}^n \|L_j\|^2 \sum_{j=p+1}^n \|u_j\|_{W_2^1}^2} \\ &\leq \sqrt{\left(\sum_{j=1}^p \|L_j\|^2 + \sum_{j=p+1}^n \|L_j\|^2 \right) \left(\sum_{j=1}^p \|u_j\|_{W_2^{2k_j}}^2 + \sum_{j=p+1}^n \|u_j\|_{W_2^1}^2 \right)} \\ &= \|L\| \|u\|_W. \end{aligned} \quad (31)$$

Considering Lemma 3.1 and the boundedness of $\{L_j\}_{j=1}^p$ and $\{L_j\}_{j=p+1}^n$ implies that L is bounded.

Next, we construct an orthogonal function systems of the space $W[0, 1]$ as follows: put

$$\varphi_{ij}(t) = (r_{ti}(t))_j e_j =$$

$$\begin{cases} (R_{t_i}^{\{1\}}(t), 0, 0, \dots, 0)^T, j = 1, \\ (0, R_{t_i}^{\{1\}}(t), 0, \dots, 0)^T, j = 2, \\ \vdots \\ (0, 0, \dots, 0, R_{t_i}^{\{1\}}(t))^T, j = n, \end{cases} \quad (32)$$

and $\psi_{ij}(t) = L^* \varphi_{ij}(t)$, $i = 1, 2, \dots, j = 1, 2, \dots, n$, where $L^* = \text{Diagonal}(L_1^*, L_2^*, \dots, L_p^*, L_{p+1}^*, \dots, L_n^*)$ is the adjoint operator of L , $R_t^{\{1\}}(s)$ is the reproducing

kernel function of $W_2^1[0, 1]$, and $\{t_i\}_{i=1}^\infty$ is dense on $[0, 1]$. where

Algorithm 3.1 The orthonormal function systems $\{\bar{\psi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,n)}$ of the space $W[0, 1]$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,n)}$ as follows:

Step 1: For $l = 1, 2, \dots$ and $k = 1, 2, \dots, l$ do the following:

$$\begin{aligned} &\text{If } l = k = 1, \text{ then set } \mu_{lk}^{ij} = \frac{1}{\|\psi_{11}\|_W}; \\ &\text{If } l = k \neq 1, \text{ then set } \mu_{lk}^{ij} = \frac{1}{\sqrt{\|\psi_{lk}\|_W^2 - \sum_{p=1}^{l-1} \langle \psi_{lk}(t), \bar{\psi}_{lp}(t) \rangle_W^2}}; \\ &\text{If } l > k, \text{ then set } \mu_{lk}^{ij} = \frac{-1}{\sqrt{\|\psi_{lk}\|_W^2 - \sum_{p=1}^{l-1} \langle \psi_{lk}(t), \bar{\psi}_{lp}(t) \rangle_W^2}} \sum_{p=k}^{l-1} \langle \psi_{lk}(t), \bar{\psi}_{lp}(t) \rangle \mu_{pk}^{ij}; \end{aligned}$$

Output: the orthogonalization coefficients μ_{lk}^{ij} of the orthonormal systems $\bar{\psi}_{ij}(t)$.

Step 2: For $i = 1, 2, \dots$ and $j = 1, 2, \dots, n$ set

$$\bar{\psi}_{ij}(t) = \sum_{l=1}^i \sum_{k=1}^j \mu_{lk}^{ij} \psi_{lk}(t); \quad (33)$$

Output: the orthonormal function systems $\{\bar{\psi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,n)}$.

Step 3: Stop.

The subscript s by the operator L , denoted by L_s , indicates that the operator L applies to the function of s . Indeed, it is easy to see that, $\psi_{ij}(t) = L^* \varphi_{ij}(t) = \langle L^* \varphi_{ij}(s), R_t(s) \rangle_W = \langle \varphi_{ij}(s), L_s R_t(s) \rangle_H = L_s R_t(s)|_{s=t_i} \in W[0, 1]$. Thus, $\psi_{ij}(t)$ can be expressed in the form of $\psi_{ij}(t) = L_s R_t(s)|_{s=t_i}$.

Theorem 3.2 For Eqs. (28) and (29), if $\{t_i\}_{i=1}^\infty$ is dense on $[0, 1]$, then $\{\psi_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,n)}$ is the complete function systems of the space $W[0, 1]$.

Proof. For each fixed $u \in W[0, 1]$, let $\langle u(t), \psi_{ij}(t) \rangle_W = 0$. Then, $\langle u(t), \psi_{ij}(t) \rangle_W = \langle u(t), L^* \varphi_{ij}(t) \rangle_W = \langle Lu(t), \varphi_{ij}(t) \rangle_H = Lu(t_i) = 0$. Whilst on the other hand,

$$u(t) = \sum_{j=1}^n u_j(t) e_j = \sum_{j=1}^n \langle u(\cdot), (R_t(\cdot))_j e_j \rangle_W e_j,$$

$$(R_t(\cdot))_j e_j = \begin{cases} \left(R_t^{\{2,k_1\}}(s), 0, 0, \dots, 0 \right)^T, & j = 1, \\ \left(0, R_t^{\{2,k_2\}}(s), 0, 0, \dots, 0 \right)^T, & j = 2, \\ \vdots \\ \left(0, \dots, 0, \left(R_t^{\{2,k_p\}}(s) \right)_{p\text{th}}, 0, \dots, 0 \right)^T, & j = p, \\ \left(0, \dots, 0, \left(R_t^{\{1\}}(s) \right)_{(p+1)\text{th}}, 0, \dots, 0 \right)^T, & j = p+1, \\ \vdots \\ \left(0, 0, \dots, 0, \left(R_t^{\{1\}}(s) \right)_{(n)\text{th}} \right)^T, & j = n. \end{cases} \quad (34)$$

Hence, $Lu(t_i) = \sum_{j=1}^n \langle Lu(t), \varphi_{ij}(t) \rangle_H e_j = 0$. But since $\{t_i\}_{i=1}^\infty$ is dense on $[0, 1]$, we must have $Lu(t) = 0$. It follows that $u(t) = 0$ from the existence of L^{-1} .

Theorem 3.3 If $\{t_i\}_{i=1}^\infty$ is dense on $[0, 1]$ and the solution of Eqs. (28) and (29) is unique, then their exact solution satisfies the infinite expansion form

$$\begin{aligned} u(t) &= \sum_{i=1}^\infty \sum_{j=1}^n A_{ij} \bar{\psi}_{ij}(t), \\ A_{ij} &= \sum_{l=1}^i \sum_{k=1}^j \mu_{lk}^{ij} f_k(t_l, u(t_l)). \end{aligned} \quad (35)$$

Proof. Applying Theorem 3.2, it is easy to see that $\{\bar{\psi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,n)}$ is the complete orthonormal basis of $W[0, 1]$. Since, $\langle u(t), \varphi_{ij}(t) \rangle = u_j(t_i)$ for each $u \in W[0, 1]$, while on the other hand, $\sum_{i=1}^\infty \sum_{j=1}^n \langle u(t), \bar{\psi}_{ij}(t) \rangle_W \bar{\psi}_{ij}(t)$ is the Fourier series expansion about $\{\bar{\psi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,n)}$. Then the series $\sum_{i=1}^\infty \sum_{j=1}^n \langle u(t), \bar{\psi}_{ij}(t) \rangle_W \bar{\psi}_{ij}(t)$ is convergent in the

sense of $\|\cdot\|_W$. Thus, using Eq. (33), we have

$$\begin{aligned}
 u(t) &= \sum_{i=1}^{\infty} \sum_{j=1}^n \langle u(t), \bar{\psi}_{ij}(t) \rangle_W \bar{\psi}_{ij}(t) \\
 &= \sum_{i=1}^{\infty} \sum_{j=1}^n \langle u(t), \sum_{l=1}^i \sum_{k=1}^j \mu_{lk}^{ij} \psi_{lk}(t) \rangle_W \bar{\psi}_{ij}(t) \\
 &= \sum_{i=1}^{\infty} \sum_{j=1}^n \sum_{l=1}^i \sum_{k=1}^j \mu_{lk}^{ij} \langle u(t), L^* \varphi_{lk}(t) \rangle_W \bar{\psi}_{ij}(t) \\
 &= \sum_{i=1}^{\infty} \sum_{j=1}^n \sum_{l=1}^i \sum_{k=1}^j \mu_{lk}^{ij} \langle Lu(t), \varphi_{lk}(t) \rangle_H \bar{\psi}_{ij}(t) \\
 &= \sum_{i=1}^{\infty} \sum_{j=1}^n \sum_{l=1}^i \sum_{k=1}^j \mu_{lk}^{ij} \langle f_k(t, u(t)), \varphi_{lk}(t) \rangle_H \bar{\psi}_{ij}(t) \\
 &= \sum_{i=1}^{\infty} \sum_{j=1}^n A_{ij} \bar{\psi}_{ij}(t).
 \end{aligned} \tag{36}$$

Therefore, the form of Eq. (35) is the exact solution of Eqs. (28) and (29).

Anyhow, since $W[0, 1]$ is a Hilbert space, it is clear that $u(t) = \sum_{i=1}^{\infty} \sum_{j=1}^n A_{ij} \bar{\psi}_{ij}(t) < \infty$. Therefore, the finite sequence of functions

$$u_{\eta}(t) = \sum_{i=1}^{\eta} \sum_{j=1}^n A_{ij} \bar{\psi}_{ij}(t), \tag{37}$$

is convergent in the sense of the norm of $W[0, 1]$ and the numerical solution $u_{\eta}(t)$ of $u(t)$ for Eqs. (28) and (29) can be obtained and calculated directly by Eq. (37).

4 Construct and implement the iterative technique

In this section, we consider the given DAS of IICs and construct an iterative technique to find their solutions in the space $W[0, 1]$ for linear and nonlinear case simultaneously. Also, numerical solutions of the same system, obtained using proposed method with existing IICs are proved to converge to the exact solutions with decreasing absolute difference between the exact values and the values obtained using RKHS method.

The basis of our RKHS method for solving Eqs. (28) and (29) is summarized below for the exact and the numerical solutions. Firstly, we shall make use of the following facts about the linear and the nonlinear case depending on the internal structure of the function F .

Case 1: If Eq. (28) is linear, then the exact and the numerical solutions can be obtained directly from Eqs. (35) and (37), respectively.

Case 2: If Eq. (28) is nonlinear, then the exact and the numerical solutions can be obtained by using the following iterative process.

According to Eq. (35), the representation form of the exact solution of Eqs. (28) - (29) can be written as

$$u(t) = \sum_{i=1}^{\infty} \sum_{j=1}^n A_{ij} \bar{\psi}_{ij}(t). \tag{38}$$

For numerical computations, we define the η -term numerical solution of $u(t)$ by

$$u_{\eta}(t) = \sum_{i=1}^{\eta} \sum_{j=1}^n B_{ij} \bar{\psi}_{ij}(t), \tag{39}$$

where the coefficients B_{ij} and the successive approximations $u_i(t)$, $i = 1, 2, \dots, \eta$ are given as

$$B_{ij} = \sum_{l=1}^i \sum_{k=1}^j \mu_{lk}^{ij} f_k(t_l, u_{l-1}(t_l)). \tag{40}$$

In the iterative process of Eqs. (39) and (40), we can guarantee that the numerical solution $u_{\eta}(t)$ satisfies the constraints conditions of Eq. (29). Now, we will proof that $u_{\eta}(t)$ in the iterative formula of Eqs. (39) and (40) is converge to the exact solution $u(t)$ of Eqs. (28) and (29). **Lemma 4.1** If $u \in W[0, 1]$, then the numerical solution $u_{\eta}(t)$ and its derivative $u'_{\eta}(t)$ are converging uniformly to the exact solution $u(t)$ and its derivative $u'(t)$ as $\eta \rightarrow \infty$, respectively.

Proof. For each $t \in [0, 1]$, one can write

$$\begin{aligned}
 &|u_{\eta}^{(i)}(t) - u^{(i)}(t)| \\
 &= \left| \sum_{j=1}^n \langle u_{\eta}(t) - u(t), \partial_t^i (R_t(t))_j e_j \rangle_W e_j \right| \\
 &\leq \sum_{j=1}^n |\langle u_{\eta}(t) - u(t), \partial_t^i (R_t(t))_j e_j \rangle_W e_j| \\
 &\leq \sum_{j=1}^n \|u_{\eta} - u\|_W \left\| \partial_t^i (R_t(t))_j e_j \right\|_W |e_j| \\
 &= \|u_{\eta} - u\|_W \sum_{j=1}^n \left\| \partial_t^i (R_t(t))_j e_j \right\|_W |e_j| \\
 &\leq M_i \|u_{\eta} - u\|_W, \quad M_i > 0, i = 0, 1,
 \end{aligned} \tag{41}$$

where $M_i = \sum_{j=1}^n \left\| \partial_t^i (R_t(t))_j e_j \right\|_W |e_j|$. Hence, if $\|u_{\eta} - u\|_W \rightarrow 0$ as $\eta \rightarrow \infty$, then $u_{\eta}(t)$ and $u'_{\eta}(t)$ are converge uniformly to $u(t)$ and $u'(t)$, respectively.

Theorem 4.1 If $\|u_{\eta-1} - u\|_W \rightarrow 0$, $t_{\eta} \rightarrow s$ as $\eta \rightarrow \infty$, $\|u_{\eta-1}\|_W$ is bounded, and $F(t, u(t))$ is continuous, then $F(t_{\eta}, u_{\eta-1}(t_{\eta})) \rightarrow F(s, u(s))$ as $\eta \rightarrow \infty$.

Proof. Firstly, we will prove that $u_{\eta-1}(t_{\eta}) \rightarrow u(s)$. Since, we can note that

$$\begin{aligned}
 &|u_{\eta-1}(t_{\eta}) - u(s)| \\
 &= |u_{\eta-1}(t_{\eta}) - u_{\eta-1}(s) + u_{\eta-1}(s) - u(s)| \\
 &\leq |u_{\eta-1}(t_{\eta}) - u_{\eta-1}(s)| + |u_{\eta-1}(s) - u(s)|.
 \end{aligned} \tag{42}$$

By reproducing property of the kernel function $R_t(s)$, we have $u_{\eta-1}(t_\eta) = \sum_{j=1}^n \langle u_{\eta-1}(t), (R_{t_\eta}(t))_j e_j \rangle_W e_j$ and $u_{\eta-1}(s) = \sum_{j=1}^n \langle u_{\eta-1}(t), (R_s(t))_j e_j \rangle_W e_j$. Thus,

$$\begin{aligned} & |u_{\eta-1}(t_\eta) - u_{\eta-1}(s)| \\ &= \left| \sum_{j=1}^n \langle u_{\eta-1}(t), (R_{t_\eta}(t))_j e_j - (R_s(t))_j e_j \rangle_W e_j \right| \\ &\leq \sum_{j=1}^n \left| \langle u_{\eta-1}(t), (R_{t_\eta}(t) - R_s(t))_j e_j \rangle_W e_j \right| \\ &\leq \sum_{j=1}^n \|u_{\eta-1}(t)\|_W \|(R_{t_\eta}(t) - R_s(t))_j e_j\|_W |e_j| \\ &\leq \|u_{\eta-1}(t)\|_W \sum_{j=1}^n \|(R_{t_\eta}(t) - R_s(t))_j e_j\|_W |e_j|. \end{aligned} \quad (43)$$

From the symmetry of R , it follows that $\|(R_{t_\eta}(t) - R_s(t))_j e_j\|_W \rightarrow 0$ as $t_\eta \rightarrow s$ and $\eta \rightarrow \infty$. In terms of the boundedness of $\|u_{\eta-1}\|_W$, one obtains that $|u_{\eta-1}(t_\eta) - u_{\eta-1}(s)| \rightarrow 0$ as soon as $t_\eta \rightarrow s$ and $\eta \rightarrow \infty$. Again, by Lemma 4.1, for each $s \in [0, 1]$, it holds that $|u_{\eta-1}(s) - u(s)| \leq M_1 \|u_{\eta-1} - u\|_W \rightarrow 0$. Therefore, $u_{\eta-1}(t_\eta) \rightarrow u(s)$ in the sense of $\|\cdot\|_W$ as $t_\eta \rightarrow s$ and $\eta \rightarrow \infty$. As a result, by the means of the continuation of F , it implies that $F(t_\eta, u_{\eta-1}(t_\eta)) \rightarrow F(s, u(s))$ as $\eta \rightarrow \infty$.

Theorem 4.2 Suppose that $\|u_\eta\|_W$ is bounded in Eqs. (39) and (40), $\{t_i\}_{i=1}^\infty$ is dense on $[0, 1]$, and Eqs. (28) and (29) have a unique solution. Then the η -term numerical solution $u_\eta(t)$ converges to the exact solution $u(t)$ with $u(t) = \sum_{i=1}^\infty \sum_{j=1}^n B_{ij} \bar{\psi}_{ij}(t)$.

Proof. The proof can be divided into two steps. Firstly, we will prove the convergence of $u_\eta(t)$. From Eqs. (39) and (40), we infer that

$$u_{\eta+1}(t) = u_\eta(t) + \sum_{j=1}^n B_{(\eta+1)j} \bar{\psi}_{(\eta+1)j}(t). \text{ From the orthogonality of } \{\bar{\psi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,n)} \text{ it follows that}$$

$$\begin{aligned} \|u_{\eta+1}\|_W^2 &= \|u_\eta\|_W^2 + \sum_{j=1}^n B_{(\eta+1)j}^2 = \|u_{\eta-1}\|_W^2 + \sum_{j=1}^n B_{\eta j}^2 + \sum_{j=1}^n B_{(\eta+1)j}^2 = \dots = \|u_0\|_W^2 + \sum_{i=1}^{\eta+1} \sum_{j=1}^n B_{ij}^2. \end{aligned}$$

In other formulation, it holds that $\|u_{\eta+1}\|_W \geq \|u_\eta\|_W$. Due to the condition that $\|u_\eta\|_W$ is bounded, $\|u_\eta\|_W$ is convergent and there exists a constant c such that $\sum_{i=1}^\infty \sum_{j=1}^n B_{ij}^2 = c$. This implies that $\left\{ \sum_{j=1}^n B_{ij}^2 \right\}_{i=1}^\infty \in l^2$.

On the other hand, since $(u_\zeta(t) - u_{\zeta-1}(t)) \perp (u_{\zeta-1}(t) -$

$u_{\zeta-2}(t)) \perp \dots \perp (u_{\eta+1}(t) - u_\eta(t))$, it follows for $\zeta > \eta$ that

$$\begin{aligned} \|u_\zeta - u_\eta\|_W^2 &= \|u_\zeta - u_{\zeta-1} + u_{\zeta-1} - \dots + u_{\eta+1} - u_\eta\|_W^2 \\ &= \|u_\zeta - u_{\zeta-1}\|^2 + \|u_{\zeta-1} - u_{\zeta-2}\|^2 + \dots + \|u_{\eta+1} - u_\eta\|^2. \end{aligned} \quad (44)$$

Furthermore, $\|u_\zeta - u_{\zeta-1}\|_W^2 = \sum_{j=1}^n B_{\zeta j}^2$. Consequently,

$$\text{as } \eta, \zeta \rightarrow \infty, \text{ we have } \|u_\zeta - u_\eta\|_W^2 = \sum_{l=\eta+1}^\zeta \sum_{j=1}^n B_{lj}^2 \rightarrow 0.$$

Considering the completeness of $W[0, 1]$, there exist $u \in W[0, 1]$ such that $u_\eta(t) \rightarrow u(t)$ as $\eta \rightarrow \infty$ in the sense of $\|\cdot\|_W$.

Secondly, we will prove that $u(t)$ is the solution of Eqs. (28) and (29). Taking the limits in Eqs. (39) and (40), one can get $u(t) = \sum_{i=1}^\infty \sum_{j=1}^n B_{ij} \bar{\psi}_{ij}(t)$. But on the

$$\text{other aspect as well, } (Lu)(t) = \sum_{i=1}^\infty \sum_{j=1}^n B_{ij} L \bar{\psi}_{ij}(t),$$

$$\begin{aligned} \text{thus, } (Lu)_k(t_l) &= \sum_{i=1}^\infty \sum_{j=1}^n B_{ij} \langle L \bar{\psi}_{ij}(t), \varphi_{lk}(t) \rangle_H = \\ &= \sum_{i=1}^\infty \sum_{j=1}^n B_{ij} \langle \bar{\psi}_{ij}(t), L^* \varphi_{lk}(t) \rangle_W = \\ &= \sum_{i=1}^\infty \sum_{j=1}^n B_{ij} \langle \bar{\psi}_{ij}(t), \psi_{lk}(t) \rangle_W. \end{aligned}$$

$$\begin{aligned} \text{Therefore, } \sum_{l'=1}^l \sum_{k'=1}^k B_{l'k'}^{lk} (Lu)_{k'}(t_{l'}) &= \\ \sum_{i=1}^\infty \sum_{j=1}^n B_{ij} \langle \bar{\psi}_{ij}(t), \sum_{l'=1}^l \sum_{k'=1}^k B_{l'k'}^{lk} \psi_{l'k'}(t) \rangle_W &= \\ \sum_{i=1}^\infty \sum_{j=1}^n B_{ij} \langle \bar{\psi}_{ij}(t), \bar{\psi}_{lk}(t) \rangle_W = B_{lk}. \end{aligned}$$

For the conduct of proceedings in the proof, if $l = 1$, then $(Lu)_j(t_1) = f_j(t_1, u_0(t_1))$, $j = 1, 2, \dots, n$, that is, $Lu(t_1) = F(t_1, u_0(t_1))$. Again, if $l = 2$, then $(Lu)_j(t_2) = f_j(t_2, u_1(t_2))$, $j = 1, 2, \dots, n$, that is, $Lu(t_2) = F(t_2, u_1(t_2))$. In the same manner, we can discover the following general pattern form: $Lu(t_\eta) = F(t_\eta, u_{\eta-1}(t_\eta))$. Since $\{t_i\}_{i=1}^\infty$ is dense on $[0, 1]$, then for each $s \in [0, 1]$, there exists subsequence $\{t_{\eta_j}\}_{j=1}^\infty$, such that $t_{\eta_j} \rightarrow s$ as $j \rightarrow \infty$. But since, we

have known that $Lu(t_{\eta_j}) = F(t_{\eta_j}, u_{\eta_j-1}(t_{\eta_j}))$. Hence, let $j \rightarrow \infty$, by the continuity of F , we have $Lu(s) = F(s, u(s))$. Hence, $u(t)$ satisfies Eq. (28). Also, since $\bar{\psi}_{ij}(t) \in W[0, 1]$, then $u(t)$ satisfies the constraint conditions of Eq. (29). In other words, $u(t)$ is the solution of Eqs. (28) and (29). The application of the uniqueness of solution yields that

$$u(t) = \sum_{i=1}^\infty \sum_{j=1}^n B_{ij} \bar{\psi}_{ij}(t).$$

Let $\varepsilon_\eta = \|u - u_\eta\|_W$, where $u(t)$ and $u_\eta(t)$ are given by Eqs. (35) and (37), respectively. Then using the

expansions $u(t) = \sum_{i=1}^{\infty} \sum_{j=1}^n \langle u(t), \bar{\psi}_{ij}(t) \rangle_W \bar{\psi}_{ij}(t)$ and $u_{\eta}(t) = \sum_{i=1}^{\eta} \sum_{j=1}^n \langle u(t), \bar{\psi}_{ij}(t) \rangle_W \bar{\psi}_{ij}(t)$, one can write $\varepsilon_{\eta}^2 = \sum_{i=\eta+1}^{\infty} \sum_{j=1}^n \langle u(t), \bar{\psi}_{ij}(t) \rangle_W^2$ and $\varepsilon_{\eta-1}^2 = \sum_{i=\eta}^{\infty} \sum_{j=1}^n \langle u(t), \bar{\psi}_{ij}(t) \rangle_W^2$. Clearly, $\varepsilon_{\eta-1} \geq \varepsilon_{\eta}$, and consequently $\{\varepsilon_{\eta}\}_{\eta=1}^{\infty}$ are decreasing in the sense of the norm of $W[0, 1]$. By Theorem 3.3, we know that $\sum_{i=1}^{\infty} \sum_{j=1}^n \langle u(t), \bar{\psi}_{ij}(t) \rangle_W \bar{\psi}_{ij}(t)$ is convergent. Thus, $\varepsilon_{\eta}^2 = \sum_{i=\eta+1}^{\infty} \sum_{j=1}^n \langle u(t), \bar{\psi}_{ij}(t) \rangle_W^2 \rightarrow 0$ or $\varepsilon_{\eta} \rightarrow 0$ as $\eta \rightarrow \infty$.

5 Application and numerical simulations

In order to solve the DASs of IICs numerically on a computer, the system is approximated by a discrete one and the continuous functions are approximated by finite arrays of values. Anyhow, to demonstrate the simplicity and effectiveness of the proposed method, numerical solutions for three different DASs of IICs is constructed using the RKHS method. Results obtained by the proposed method are compared systematically with some other well-known methods and are found outperforms in terms of accuracy and generality.

By generating the finite direct sum between the spaces $W_2^1[0, 1]$, $\overline{W}_2^1[0, 1]$, $\overline{W}_2^{2k}[0, 1]$ and merge the kernel functions $R_t^{\{1\}}(s)$, $\underline{R}_t^{\{1\}}(s)$, $\overline{R}_t^{\{1\}}(s)$, $R_t^{\{2,k\}}(s)$ in one vector space that satisfying the corresponding IICs, we can directly obtain the exact and the numerical solutions by applying the following algorithm.

Algorithm 5.1 To approximate the solution $u_{\eta}(t)$ of $u(t)$ for Eqs. (28) and (29), we do the following steps:

Input: The interval $[0, 1]$, the integers η , the kernel functions $R_t(s)$, $r_t(s)$, the differential operator L , and the function F .

Output: Numerical solution $u_{\eta}(t)$ of $u(t)$ at each grid points in the independent interval $[0, 1]$.

Step 1: Fixed t in $[0, 1]$ and set $s \in [0, 1]$;

If $s \leq t$, set

$$R_t(s) = [R_{t,1}^{\{2,k_1\}}(s), \dots, (\underline{R}_{t,1}^{\{1\}})_{(p+1)\text{th}}(s),$$

$$\dots, (\underline{R}_{t,1}^{\{1\}})_{n\text{th}}(s)]^T;$$

$$\text{Else set } R_t = [R_{t,2}^{\{2,k_1\}}(s), \dots, (\underline{R}_{t,2}^{\{1\}})_{(p+1)\text{th}}(s),$$

$$\dots, (\underline{R}_{t,2}^{\{1\}})_{n\text{th}}(s)]^T;$$

For $i = 1, 2, \dots, \eta$ and $j = 1, 2, \dots, n$, do the following:

$$\text{Set } t_i = \frac{i-1}{\eta-1};$$

$$\text{Set } \psi_{i,j}(t_i) = L_s[R_{t_i}(s)]_{s=t_i};$$

Output: the orthogonal function systems $\psi_{i,j}(t_i)$.

Step 2: For $l = 2, 3, \dots, \eta$ and $k = 1, 2, \dots, l$, do Algorithm (3.1) for l and k ;

Output: the orthogonalization coefficients μ_{lk}^{ij} .

Step 3: For $l = 2, 3, \dots, \eta - 1$ and $k = 1, 2, \dots, l - 1$, do the following:

$$\text{Set } \bar{\psi}_{ij}(t_i) = \sum_{l=1}^i \sum_{k=1}^j \mu_{lk}^{ij} \psi_{lk}(t_i);$$

Output: the orthonormal function system $\bar{\psi}_{ij}(t_i)$.

Step 4: Set $B_{ij} = \sum_{l=1}^i \sum_{k=1}^2 \mu_{lk}^{ij} f_k(t_l, u_{l-1}(t_l));$

$$\text{Set } u_i(t_i) = \sum_{i=1}^i \sum_{j=1}^n B_{ij} \bar{\psi}_{ij}(t_i);$$

Output: the numerical solution $u_{\eta}(t_i)$ of $u(t_i)$.

Step 5: Stop.

Using RKHS algorithm, taking $t_i = \frac{i-1}{\eta-1}$, $i = 1, 2, \dots, \eta$ in $u_{\eta}(t)$ of Eq. (37), generating the reproducing kernel functions $r_t(s)$, $R_t(s)$ on $[0, 1]$, and applying Algorithms 3.1 and 5.1 throughout the numerical computations; some graphical results, tabulate data, and numerical comparison are presented and discussed quantitatively at some selected grid points on $[0, 1]$ to illustrate the numerical solutions for the following DASs of IICs. In the process of computation, all the symbolic and numerical computations are performed by using MAPLE 13 software package.

Example 5.1 Consider the following differential equation:

$$u_1'(t) = tu_1(t) - u_2(t) + e^t, \quad (45)$$

subject to the integral condition:

$$u_1(0) = \int_0^1 s u_1(s), \quad (46)$$

simultaneously with the following algebraic equation:

$$u_1(t) + e^{-t} u_2(t) = t + e^t, \quad (47)$$

subject to the initial conditions:

$$u_2(0) = 0. \quad (48)$$

Here, $t \in [0, 1]$, whilst the exact solutions are $u_1(t) = e^t$ and $u_2(t) = te^t$.

Example 5.2 Consider the following set of differential equations:

$$\begin{cases} u_1'(t) = u_1(t) - u_3^2(t) \ln(u_1(t)) + 1, \\ u_2'(t) = u_3^2(t) + u_3^{-1}(t) e^{u_2(t)} - (t+1)^{\frac{3}{2}}, \end{cases} \quad (49)$$

subject to the integral conditions:

$$\begin{cases} u_1(0) = \int_0^1 e^{-s} u_1(s) + 1, \\ u_2(0) = \int_0^1 s u_2(s) + 4^{-1}, \end{cases} \quad (50)$$

Table 1 Numerical values of the dependent variables $u_{\eta 1}(t)$ and $u_{\eta 2}(t)$ in Example 5.1.

t	$u_1(t)$	$u_{\eta 1}(t)$	$u_2(t)$	$u_{\eta 2}(t)$
0	1	0.99999999983888	0	0
0.16	1.173510870991810	1.173510870931385	0.187761739358690	0.187761739302131
0.32	1.377127764335957	1.377127764127835	0.440680884587506	0.440680884556301
0.48	1.616074402192893	1.616074401628453	0.775715713052589	0.775715713003221
0.64	1.896480879304951	1.896480879219793	1.213747762755169	1.213747762024998
0.80	2.225540928492467	2.225540928101454	1.780432742793974	1.780432742279523
0.96	2.611696473423117	2.611696473135761	2.507228614486193	2.507228614240745
1	2.718281828459045	2.718281827751456	2.718281828459045	2.718281828229471

Table 2 Numerical values of the dependent variables $u_{\eta 1}(t)$, $u_{\eta 2}(t)$, and $u_{\eta 3}(t)$ in Example 5.2

t	$u_{\eta 1}(t)$	$u_{\eta 2}(t)$	$u_{\eta 3}(t)$
0	2.718281826327278	-0.000000001895396	1
0.16	3.189933270319385	0.148419981604011	0.928476685564841
0.32	3.743421374024469	0.277631731955104	0.870388271788553
0.48	4.392945675887708	0.392042080909051	0.821994930830145
0.64	5.155169487518942	0.494696207847439	0.78086879917688
0.80	6.049647411643256	0.587786639485744	0.745355983124897
0.96	7.099326989961757	0.672944426438238	0.714285679827479
1	7.389056036977788	0.693147127060917	0.707106780485565

Table 3 Numerical values of the dependent variables $u_{\eta 1}(t)$, $u_{\eta 2}(t)$, $u_{\eta 3}(t)$, and $u_{\eta 4}(t)$ in Example 5.3.

t	$ u_{\eta 1}(t) - u_1(t) $	$ u_{\eta 2}(t) - u_2(t) $	$ u_{\eta 3}(t) - u_3(t) $	$ u_{\eta 4}(t) - u_4(t) $
0	$6.77393774 \times 10^{-8}$	$2.81688636 \times 10^{-8}$	$5.95128789 \times 10^{-7}$	$1.90495767 \times 10^{-7}$
0.16	$3.04644408 \times 10^{-8}$	$4.97586606 \times 10^{-8}$	$7.35618057 \times 10^{-7}$	$2.92327896 \times 10^{-7}$
0.32	$7.02482980 \times 10^{-8}$	$1.00492597 \times 10^{-8}$	$2.34717390 \times 10^{-8}$	$7.87919454 \times 10^{-7}$
0.48	$2.87288290 \times 10^{-7}$	$3.82075576 \times 10^{-8}$	$4.87564335 \times 10^{-7}$	$1.20484809 \times 10^{-8}$
0.64	$6.02059218 \times 10^{-8}$	$1.42838984 \times 10^{-8}$	$1.21798967 \times 10^{-8}$	$3.35573115 \times 10^{-7}$
0.80	$1.36113692 \times 10^{-7}$	$4.67986424 \times 10^{-7}$	$2.60534860 \times 10^{-8}$	$2.96273934 \times 10^{-8}$
0.96	$3.50745738 \times 10^{-7}$	$1.35392753 \times 10^{-7}$	$9.61230455 \times 10^{-9}$	$5.17649545 \times 10^{-9}$
1	$1.06582351 \times 10^{-7}$	$3.86098600 \times 10^{-7}$	0	0

simultaneously with the following algebraic equation:

$$\ln^{-2}(u_1(t)) + 2 \sinh(u_2(t)) u_3^2(t) = 1, \quad (51)$$

subject to the initial conditions:

$$u_3(0) = 1. \quad (52)$$

Here, $t \in [0, 1]$, whilst the exact solutions are $u_1(t) = e^{t+1}$, $u_2(t) = \ln(t+1)$, and $u_3(t) = (t+1)^{-\frac{1}{2}}$.

Example 5.3 Consider the following set of singular differential equations:

$$\begin{cases} (t-1)u_1'(t) = (t-1)u_2(t) - \sqrt{1-u_3^2(t)} + \cos t, \\ t(t-1)u_2'(t) = t(t-1)u_1(t) + u_3^2(t)u_4^{-2}(t) \\ \quad - u_4^{-2}(t) + 1, \end{cases} \quad (53)$$

subject to the integral conditions:

$$\begin{cases} u_1(0) = \int_0^1 (-s) u_1(s) + e^{-1}, \\ u_2(0) = \int_0^1 4e^s u_2(s) - (e^2 + 1), \end{cases} \quad (54)$$

simultaneously with the following set of singular algebraic equations:

$$\begin{cases} \sinh(t)u_1(t) - \cosh(t)u_2(t) + \sin^{-1}(u_4(t)) \\ \quad = -t + 0.5\pi - 1, \\ tu_2(t) - \tan^{-1}(u_3^{-1}(t)u_4(t)) \\ \quad = t \cosh(t)u_2(t) + t - 0.5\pi, \end{cases} \quad (55)$$

subject to the purely conditions:

$$\begin{cases} u_3(1) = \sin(1), \\ u_4(1) = \cos(1). \end{cases} \quad (56)$$

Here, $t \in [0, 1]$, whilst the exact solutions are $u_1(t) = \sinh(t)$, $u_2(t) = \cosh(t)$, $u_3(t) = \sin(t)$, and $u_4(t) = \cos(t)$.

For the conduct of proceedings in the solutions for the last examples, it require the expansion rules for the kernel

functions $R_t^{\{2,k\}}(s)$ when $k(s) = s$, $k(s) = e^{-s}$, $k(s) = -s$, and $k(s) = 4e^s$, respectively. As a special case and without the loss of generality we are finding the rules of $R_t^{\{2,k\}}(s)$ when $k(s) = s$. To do so, we need to find the coefficients $a_i(t)$, $b_i(t)$, and $c(t)$ in

$$R_t^{\{2,s\}}(s) = \begin{cases} a_1(t) + a_2(t)s + a_3(t)s^2 \\ + a_4(t)s^3 + c(t)\frac{s^5}{120}, & s \leq t, \\ b_1(t) + b_2(t)s + b_3(t)s^2 \\ + b_4(t)s^3 + c(t)\frac{s^5}{120}, & s > t. \end{cases} \quad (57)$$

Anyhow, by solving the following sets of algebraic equations by using Maple 13 software package:

$$\begin{aligned} R_t^{\{2,s\}}(0) &= \int_0^1 s R_t^{\{2,s\}}(s) ds, \\ R_t^{\{2,s\}}(0) + \partial_s^3 R_t^{\{2,s\}}(0) + c(t) &= 0, \\ \partial_s^1 R_t^{\{2,s\}}(0) - \partial_s^2 R_t^{\{2,s\}}(0) &= 0, \\ \partial_s^2 R_t^{\{2,s\}}(1) &= 0, \\ \partial_s^3 R_t^{\{2,s\}}(1) &= 0, \\ \partial_s^q R_t^{\{2,k\}}(t+0) - \partial_s^q R_t^{\{2,k\}}(t-0) &= 0, q = 0, 1, 2, \\ \partial_s^3 R_t^{\{2,k\}}(t+0) - \partial_s^3 R_t^{\{2,k\}}(t-0) &= -1. \end{aligned} \quad (58)$$

the mentioned unknown coefficients can be obtained as follows:

$$\begin{aligned} a_1(t) &= -\frac{21}{1952}t^5 + \frac{105}{976}t^3 - \frac{105}{488}t^2 - \frac{105}{244}t - \frac{173}{488}, \\ a_2(t) &= \frac{7}{976}t^5 - \frac{35}{488}t^3 + \frac{35}{244}t^2 - \frac{87}{122}t - \frac{105}{244}, \\ a_3(t) &= \frac{7}{1952}t^5 - \frac{35}{976}t^3 + \frac{35}{488}t^2 - \frac{87}{244}t - \frac{105}{488}, \\ a_4(t) &= -\frac{7}{3904}t^5 + \frac{35}{1952}t^3 - \frac{35}{976}t^2 - \frac{35}{488}t + \frac{803}{2928}, \\ b_1(t) &= -\frac{21}{1952}t^5 + \frac{803}{2928}t^3 - \frac{105}{488}t^2 - \frac{105}{244}t - \frac{173}{488}, \\ b_2(t) &= \frac{7}{976}t^5 - \frac{35}{488}t^3 - \frac{87}{244}t^2 - \frac{87}{122}t - \frac{105}{244}, \\ b_3(t) &= \frac{7}{1952}t^5 - \frac{35}{976}t^3 + \frac{35}{488}t^2 + \frac{35}{244}t - \frac{105}{488}, \\ b_4(t) &= -\frac{7}{3904}t^5 + \frac{35}{1952}t^3 - \frac{35}{976}t^2 - \frac{35}{488}t + \frac{105}{976}, \\ c(t) &= \frac{21}{976}t^5 + \frac{105}{488}t^3 + \frac{105}{244}t^2 + \frac{105}{122}t + \frac{315}{244}. \end{aligned} \quad (59)$$

Our next goal is to illustrate some numerical results of the RKHS solutions of the aforementioned DASs of IICs in numeric values. In fact, results from numerical analysis are an approximation, in general, which can be made as accurate as desired. Because a computer has a finite word length, only a fixed number of digits are stored and used during computations. Next, the agreement between the exact and the numerical solutions is investigated for Examples 5.1 and 5.2 at various t in $[0, 1]$ by computing the numerical approximating of their exact solutions for the corresponding equivalent equations as shown in Tables 1 and 2, respectively. Whilst, in Table 3 the absolute difference between the exact values and the

values obtained using the RKHS are tabulated for Example 5.3.

6 Concluding remarks

The reproducing kernel algorithm is a powerful method for solving various linear and nonlinear differential systems of different types and orders. In this study, we introduce the reproducing kernel approach to enlarge its application range. It is analyzed that the proposed method is well suited for use in DASs of IICs for ordinary differential equations of volatile orders and resides in its simplicity in dealing with those conditions. However, the RKHS method does not require discretization of the variables, it provides the best solution in a less number of iterations and reduces the computational work. Numerical experiments are carried out to illustrate that the present method is an accurate and reliable analytical technique for treating DASs of IICs. It is worth to be pointed out that the RKHS method is still suitable and can be employed for solving other strongly linear and nonlinear systems of differential equations.

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