Quantum Physics Letters
An International Journal

http://dx.doi.org/10.18576/qpl/090202

Quantum Anharmonic Oscillator with Velocity- and Position-Dependent Anharmonicities: an Exactly Solvable Model under Rotating Wave Approximation

Swapan Mandal, Monojit Kora, Dolan Krishna Bayen, Arjun Mukhopadhyay and Kartick Chandra Saha*

Department of Physics, Visva-Bharati, Santiniketan-731235, India

Received: 2 May. 2020, Revised: 2 Jun. 2020, Accepted: 2 Jul. 2020
Published online: 1 Aug. 2020

Abstract: The electromagnetic field coupled to a nonlinear medium of having nonvanishing polarizations and magnetizations could be modeled as a classical anharmonic oscillator with velocity- and position-dependent anharmonicities. The Hamiltonian corresponding to the quantum anharmonic oscillator with velocity- and position-dependent anharmonicities is obtained from the knowledge of its classical counterpart. Under rotating wave approximation, the solution of the oscillator with $q-dependent$ and $p-dependent$ anharmonicities exhibit the shifts of the resonance peak frequency. Interestingly, the shifts of the resonance peak of the oscillator due to the $q-dependent$ anharmonicity is opposite to those of the corresponding shifts due to the $p-dependent$ anharmonicity. Therefore, the shifts of the resonance peak frequency asserts the presence of particular anharmonicity as well (i.e $p-$ or $q-type$)

Keywords: Quantum anharmonic oscillator ; Normal ordered form ; Rotating wave approximation.

1 Introduction

The model of a simple harmonic oscillator (SHO) arises when a particle moves under the action of a restoring force. The SHO model is an ideal one and is extremely useful for the explanation of basic physics. However, for real physical systems, the inclusion of damping and or anharmonicities are inevitable. In the present investigation, we neglect the damping altogether. By anharmonic oscillator, we normally mean the presence of $q-dependent$ (q is the position coordinate of the oscillator) anharmonicity. Because of the wide range of applications and of the fundamental nature of the problem, the problems of anharmonic oscillator have attracted people from various branches of physics [1-8]. In addition to the $q-dependent$ anharmonicity, we often encounter the $p-dependent$ (p is the velocity of the oscillator with rest mass unity) anharmonic contribution due to the relativistic correction of the kinetic energy term [9-11]. Now, the Hamiltonian of a classical oscillator with unit mass and unit frequency with $q-dependent$ and $p-dependent$ anharmonicities is given by

$$H=\frac{p^2}{2}+\frac{q^2}{2}-k_1\,p^{2l}+\lambda_1\,q^{2m} \tag{1}$$

where k_1 and λ_1 are small positive constants. Of course, the Hamiltonian (1) is extremely simple in structure since we neglect the coupling between k_1 and λ_1 if any. Note that $l\geq 2$ and $m\geq 2$ are integers. For $\lambda_1=0\,(k_1=0)$, the equation (1) corresponds the Hamiltonian of an $l\,(m)-th$ anharmonic oscillator with $p\,(q)-dependent$ anharmonicity. Now, the quantum mechanical counterpart of the Hamiltonian (1) is obtained by the replacement of the classically conjugate position $q(t)$ and momentum $p(t)$ by their corresponding operators. During the passage from classical anharmonic oscillator governed by the Hamiltonian (1) to the corresponding quantum mechanical oscillator, the fundamental equal time commutation relation between the position and momentum operators should be respected. Depending upon the problems of interests, we solve the quantum anharmonic oscillator in two different formalism. In Schrödinger formalism (SF), the time development of the eigenfunction and hence the energy eigenvalues are obtained. On the other hand, the Heisenberg formalism (HF) gives rise to the time development of the operators. Most of the problems involving quantum anharmonic oscillators are solved under SF. However, the quantum anharmonic oscillator under HF still unexplored to its full

* Corresponding author e-mail: swapanvb@rediffmail.com

potential. To our knowledge, there are no investigations till to date involving the quantum anharmonic oscillator with two different types of anharmonicities. From the fundamental and from the practical point of views, we would like to investigate the analytical solutions to the quantum oscillator with $q-$ and $p-dependent$ anharmonicities. These are quite relevant since a huge number of problem leads to the model of anharmonic oscillators with $p-dependent$ and/or $q-dependent$ anharmonicities. Therefore, before we go for the solution, we would like to give few physical situations where these types of anharmonicities will come into picture.

2 The physical model

The electromagnetic field **E** interacting with a nonlinear medium produces polarization **P**

$$\mathbf{P}=\chi^{(1)}\mathbf{E}+\chi^{(2)}\mathbf{EE}+\chi^{(3)}\mathbf{EEE}+\ldots+\ldots\ldots \tag{2}$$

where $\chi^{(i)}$ is the $i-th$ order susceptibility tensor and is positive. The first term in the right hand side of the equation (2) corresponds the linear polarization. The remaining terms are responsible for the nonlinear part of the polarization. For inversion symmetric medium, the susceptibilities of even order would vanish. Hence, the leading contribution to the nonlinear polarization for an inversion symmetric medium comes from the third order nonlinear susceptibility $\chi^{(3)}$. The rough estimates show that the ratio of the magnitudes of two successive orders of nonlinear susceptibilities are $\chi^{(i)}/\chi^{(i-1)}\approx 10^{-6}$ where $i\geq 2$. Therefore, the values of the susceptibility tensors decrease rapidly with the increase of the orders. Consequently, the terms corresponding to the nonlinear polarizations do not contribute significantly unless the field strength is quite high. Now, the energy density follows as [12]

$$H_d=\frac{1}{8\pi}\mathbf{E}.\mathbf{D} \tag{3}$$

where the displacement vector $\mathbf{D}=\mathbf{E}+4\pi\mathbf{P}$. In a similar way, the magnetization (**M**) in a material medium due to the magnetic induction **B** is given by [13]

$$\mathbf{M}=\chi_m^{(1)}\mathbf{H}+\chi_m^{(2)}\mathbf{HH}+\chi_m^{(3)}\mathbf{HHH}+\ldots+\ldots\ldots \tag{4}$$

where $\chi_m^{(i)}$ stands for the magnetic susceptibility of $i-th$ order. $\chi_m^{(1)}$ correspond the linear magnetic susceptibility term. The rest of the contributions come from the nonlinear magnetic susceptibility. Again, for inversion symmetric medium the even order of the nonlinear magnetic susceptibilities would vanish. The first nonlinear contribution to the magnetic susceptibilities is a third order one for paramagnetic substances. As a matter of fact it is envisaged that the third order magnetic susceptibility is an ideal test of the quadrupolar Kondo effect in UBe_3 [14]. However, for higher order of the magnetic susceptibilities are possible in a spin glass. In other words, it is possible to select a particular type of nonlinearity by the proper choice of the medium. Now, the energy density due to the magnetic induction in the magnetic material medium follows as [12]

$$H_m=\frac{1}{8\pi}\mathbf{B}.\mathbf{H} \tag{5}$$

where, $\mathbf{H}=\mathbf{B}-4\pi\mathbf{M}$. Therefore, the total electromagnetic field energy is given by

$$H_{tot}=\frac{1}{8\pi}\int_v(\mathbf{E}.\mathbf{E}.+\mathbf{B}.\mathbf{B})\,d^3x+\frac{1}{2}\int_v(\mathbf{E}.\mathbf{P}.-\mathbf{B}.\mathbf{M})\,d^3x \tag{6}$$

In the entire investigation of the present problem, we assume the *field as monochromatic* in nature. Two integrals in the right hand side correspond free field and interaction parts of the Hamiltonian respectively. Now, to calculate the Hamiltonian (6), we require the microscopic Maxwell equations without source [12]

$$\begin{aligned}\nabla.\mathbf{E}&=0\\ \nabla.\mathbf{B}&=0\\ \nabla\times\mathbf{E}&=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\\ \nabla\times\mathbf{B}&=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\end{aligned} \tag{7}$$

where $\mathbf{B}=\nabla\times\mathbf{A}$ and $\mathbf{E}=-\nabla\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$. The vector potential **A** obeys the wave equation

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=0 \tag{8}$$

and the Coulomb gauze condition $\nabla.\mathbf{A}=0$. Again, the scalar potential $\phi=0$. Now, the solution of the wave equation (8) follows as

$$\mathbf{A}(\mathbf{r},t)=\mathbf{A_0}\exp i(\mathbf{k}.\mathbf{r}-\omega t) \tag{9}$$

where $\mathbf{A}_0$ and **k** are in general complex constant vectors with $|\mathbf{k}|c=\omega$. Therefore, the fields **E** and **B** are

$$\begin{aligned}\mathbf{E}(\mathbf{r},t)&=i|\mathbf{k}|\mathbf{A_0}\exp i(\mathbf{k}.\mathbf{r}-\omega t)\\ \mathbf{B}(\mathbf{r},t)&=i(\mathbf{A_0}\times\mathbf{k})\exp i(\mathbf{k}.\mathbf{r}-\omega t)\end{aligned} \tag{10}$$

where $\mathbf{n}=\frac{\mathbf{k}}{|\mathbf{k}|}$ is the unit propagation vector. Now, we rewrite the equations (10) in the following forms

$$\begin{aligned}\mathbf{E}(\mathbf{r},t)&=i\mathbf{A_0}|\mathbf{k}|q(t)\,\exp i\mathbf{k}.\mathbf{r}\\ \mathbf{B}(\mathbf{r},t)&=\frac{i}{c}(\mathbf{A_0}\times\mathbf{n})p(t)\exp i\mathbf{k}.\mathbf{r}\end{aligned} \tag{11}$$

where $\dot{q}=p$ and $\dot{p}=-\omega^2q$. By using the normalization condition $\int|\mathbf{A_0}\exp(i\mathbf{k}.\mathbf{r})|^2|\mathbf{k}|^2d^3x=4\pi$, we obtain the free field part of the Hamiltonian (6)

$$H_{free}=\frac{1}{2}(\omega^2q^2+p^2) \tag{12}$$

It is clear that H_{free} coincides exactly with the Hamiltonian (1) when $k_1=0=\lambda_1$ and $\omega=1$. It is evidently clear that the variables q and p are proportional to electric and magnetic fields respectively and are canonically conjugate. Now, it is possible to calculate the interaction part of the Hamiltonian (6) by using equations (11). After performing the algebraic calculations, we obtain the exact coincidence between the Hamiltonian's (1) and (6) where k_1 and λ_1 are proportional to $\chi_m^{(2l-1)}$ and $\chi^{(2m-1)}$ respectively. Of course, in these calculations we assume that the particular order of nonlinear magnetic and electric susceptibilities are invoked through the nonlinear interaction. Therefore, it is clear that the anharmonic oscillators with $q-$ and $p-dependent$ anharmonicities are obtained through the nonlinear interactions between the electromagnetic field and material medium. If the medium is nonmagnetic in nature, the corresponding nonlinear oscillator contains the $q-dependent$ anharmonicity alone. Again, the $p-dependent$ anharmonicity arises when the medium is highly magnetic in nature where the magnetic interaction dominates over the electric interaction. In addition to these, the particle executing anharmonic vibration with very high velocity gives rise to the model of an anharmonic oscillator with $q-$ and $p-dependent$ anharmonicity.

3 Quantum Oscillator with velocity and position dependent anharmonicity

In earlier section, we have given the examples of physical problems where the model Hamiltonian (1) could be realized. Of course, we have talked so far about the classical anharmonic oscillator with two different types of anharmonicities. To obtain the quantum mechanical counterpart of the Hamiltonian (1), we replace the classical canonical position (q) and momentum (p) by their equivalent operators and we impose the following equal time commutation relation [15-16]

$$[\hat{q}(t), \hat{p}(t)] = [\hat{q}(0), \hat{p}(0)] = i \tag{13}$$

where $\hbar = 1$. In order to differentiate between the $c-numbers$ and the operators, we use the caret for the operators. Assuming the relation (13) is valid, we define the usual relations connecting the position and momentum operators with those of the dimensionless annihilation $(\hat{a})$ and creation $(\hat{a}^\dagger)$ operators

$$\begin{aligned} \hat{a} &= \tfrac{1}{\sqrt{2}}(\hat{q}(t) + i\hat{p}(t)) \\ \hat{a}^\dagger &= \tfrac{1}{\sqrt{2}}(\hat{q}(t) - i\hat{p}(t)) \end{aligned} \tag{14}$$

Therefore, we have

$$\begin{aligned} \hat{q}(t) &= \tfrac{1}{\sqrt{2}}\left(\hat{a}(t) + \hat{a}^\dagger(t)\right) \\ \hat{p}(t) &= -\tfrac{i}{\sqrt{2}}\left(\hat{a}(t) - \hat{a}^\dagger(t)\right) \end{aligned} \tag{15}$$

It is clear that the annihilation operator $\hat{a}$ and creation operator $\hat{a}^\dagger$ obey the following commutation relation

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1 \tag{16}$$

Now, the Hamiltonian corresponding to the quantum mechanical counterpart of the Hamiltonian (1) follows immediately

$$\hat{H} = \hat{a}^\dagger\hat{a} - \frac{(-1)^l k_1}{2^l}\left(\hat{a} - \hat{a}^\dagger\right)^{2l} + \frac{\lambda_1}{2^m}\left(\hat{a} + \hat{a}^\dagger\right)^{2m} \tag{17}$$

where the equations (14) and (15) are used. In equation (17), we neglect the zero point energy term since it is irrelevant for the further development of the present investigation. We observe that we are unable to solve the Hamiltonian (17) as it stands and hence the closed form analytical solutions for the field operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ are unreachable. Before we go further, we put annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators involving $\hat{q}^{2m}(t)$ and $\hat{p}^{2l}(t)$ in the normal ordered forms. The meaning of the normal order is to put all the creation operators $\hat{a}^\dagger$ left to those of the annihilation operators $\hat{a}$. Therefore, the normal ordered form of $\hat{q}^{2m}(t)$ and $\hat{p}^{2l}(t)$ are

$$\begin{aligned} \hat{q}^{2m}(t) &= :\hat{q}^{2m}(t): + \sum_{r=1}^{m} \tfrac{(2r-1)!!}{2^r}\,{}^{2m}C_{2r} :\hat{q}^{2m-2r}(t): \\ \hat{p}^{2l}(t) &= :\hat{p}^{2l}(t): + \sum_{r=1}^{l} \tfrac{(2r-1)!!}{2^r}\,{}^{2l}C_{2r} :\hat{p}^{2l-2r}(t): \end{aligned} \tag{18}$$

where the notation :: stands for the normal ordered form, the binomial coefficient ${}^xC_y = \frac{x!}{(x-y)!y!}$ and $(2r-1)!! = (2r-1)(2r-3)....3.1$. For example, $\hat{q}^4(t)$ assumes the following normal ordered form

$$\hat{q}^4(t) =: \hat{q}^4(t): +3: \hat{q}^2(t): +\frac{3}{4} \tag{19}$$

where first one of the equation (18) is used. Therefore, the normal ordered forms (18) involving annihilation and creation operators are useful for the purpose of the calculation of the expectation values of those operators. For example, in terms of the number state basis, the expectation value of $\hat{q}^4(t)$ is given by

$$\begin{aligned} \langle n|\hat{q}^4(t)|n\rangle &= \langle n|:\hat{q}^4(t): +3:\hat{q}^2(t): +\tfrac{3}{4}|n\rangle \\ &= \tfrac{3}{4}(2n^2+2n+1) \end{aligned} \tag{20}$$

where $\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$ and $\langle m|n\rangle = \delta_{m,n}$ are used. On expansion of $:\hat{q}^4(t):$, we have five terms involving $\hat{a}$ and $\hat{a}^\dagger$. However, it is the middle term which contributes in (20) and the remaining four terms do not contribute. Similarly, the middle term of the expansion of $:\hat{q}^2(t):$ contribute in equation (20). Interestingly, these middle terms conserve the photon number and hence the energy. Therefore, the middle term of the operator $:\hat{q}^{2m}(t):$ $\frac{{}^{2m}C_m\hat{a}^{\dagger m}(t)\hat{a}^m(t)}{2^m}$ is the energy conserving term. After neglecting the energy nonconserving terms, the explicit normal ordered form of $\hat{q}^{2m}(t)$ is given by

$$\begin{aligned} \hat{q}^{2m}(t) = \tfrac{1}{2^m}\Big\{ &{}^{2m}C_m\hat{a}^{\dagger m}\hat{a}^m + \sum_{r=1}^{m}(2r-1)!!\,{}^{2m}C_{2r} \\ &\times {}^{2m}C_{m-r}\hat{a}^{\dagger m-r}\hat{a}^{m-r}\Big\} \end{aligned} \tag{21}$$

In spite of the complicated nature of the analytical expressions (17) and (18), it is possible to adopt an approximate method to obtain the analytical solution to the Hamiltonian (17). However, it is our purpose to explore the analytical solution of the oscillator by neglecting the non-conserving energy terms. In order to obtain an exact analytical solution to the above Hamiltonian (17), we use the *rotating wave approximation* (RWA). The idea behind the RWA is to remove the fast rotating terms from the Hamiltonian (17). Removal of the fast rotating terms also ensure the conservation of the total energy. Under the RWA, the Hamiltonian (17) reduces to

$$\begin{aligned}\hat{H} &= \hat{a}^{\dagger}\hat{a} - \tfrac{k}{l}\hat{a}^{\dagger l}\hat{a}^{l} + \tfrac{\lambda}{m}\hat{a}^{\dagger m}\hat{a}^{m} \\ &- \tfrac{k_1}{2^l}\sum_{r=1}^{l}(2r-1)!!^{2l}C_{2r}{}^{2l}C_{l-r}\times\hat{a}^{\dagger l-r}\hat{a}^{l-r} \\ &+ \tfrac{\lambda_1}{2^m}\sum_{r=1}^{m}(2r-1)!!^{2m}C_{2r}\times{}^{2m}C_{m-r}\hat{a}^{\dagger m-r}\hat{a}^{m-r}\end{aligned} \quad (22)$$

where $k = \frac{k_1\times\{1\cdot3\cdot5\cdot7\cdot9\cdots\cdots(2l-1)\}}{(l-1)!}$ and $\lambda = \frac{\lambda_1\times\{1\cdot3\cdot5\cdot7\cdot9\cdots\cdots(2m-1)\}}{(m-1)!}$ are proportional to k_1 and λ_1 and are called the anharmonic constants for $p-dependent$ and $q-dependent$ anharmonic oscillators respectively. The last two terms under summation signs appear due to the ordering of the field operators in normal form. These terms are proportional to k_1 and λ_1 respectively. Upon dropping the terms under summation signs, the Hamiltonian (22) reduces to the following form

$$\hat{H}_c = \hat{a}^{\dagger}\hat{a} - \frac{k}{l}\hat{a}^{\dagger l}\hat{a}^{l} + \frac{\lambda}{m}\hat{a}^{\dagger m}\hat{a}^{m} \quad (23)$$

For $l=0$, the equation (23) corresponds the Hamiltonian for a $m-photon$ anharmonic oscillator [17-19]. Interestingly, this Hamiltonian is widely used to investigate the squeezing, phase properties and other nonclassical properties of the coherent light coupled to the $m-photon$ anharmonic oscillator [17-19]. By analogy, for $\lambda=0$, the equation (23) corresponds the Hamiltonian for a $l-photon$ anharmonic oscillator. Of course, the nature of anharmonicities in these two cases are completely different. Admittedly, under Schroedinger formalism, Maduemezia [9-10] obtained the solution of an oscillator with $p-dependent$ anharmonicity. However, under Heisenberg formalism, the solution of an oscillator with $p-dependent$ anharmonicity is yet to be explored. In this way, the present investigation is a first one which takes care both these $p-dependent$ and $q-dependent$ anharmonicities. Now, the equation of motion for the annihilation operator $\hat{a}$ corresponding to the Hamiltonian (22) is given by

$$\dot{\hat{a}} = -i\hat{O}\hat{a} \quad (24)$$

where the operator

$$\begin{aligned}\hat{O}(t) &= 1 - k\hat{a}^{\dagger l-1}\hat{a}^{l-1} + \lambda\hat{a}^{\dagger m-1}\hat{a}^{m-1} \\ &- \tfrac{k_1}{2^l}\sum_{r=1}^{l}(l-r)(2r-1)!!^{2l}C_{2r}\times{}^{2l}C_{l-r}\hat{a}^{\dagger l-r-1}\hat{a}^{l-r-1} \\ &+ \tfrac{\lambda_1}{2^m}\sum_{r=1}^{m}(m-r)(2r-1)!!^{2m}C_{2r} \\ &\times{}^{2m}C_{m-r}\hat{a}^{\dagger m-r-1}\hat{a}^{m-r-1}\end{aligned} \quad (25)$$

is constant of motion $(i.e[\hat{H},\hat{O}]=0)$. The time independent nature of the operator $\hat{O}(t)=\hat{O}(0)$ helps us to find the exact solution to the differential equation involving the annihilation operator $\hat{a}$ (24). The corresponding solution is given by

$$\hat{a}(t) = \exp[-it\hat{O}(0)]\hat{a}(0) \quad (26)$$

Obviously, the solution for the creation operator $\hat{a}^{\dagger}$ follows immediately by taking the Hermitian conjugate of the equation (26)

$$\hat{a}^{\dagger}(t) = \hat{a}^{\dagger}(0)\exp[it\hat{O}(0)] \quad (27)$$

The equations (26) and (27) could be used to establish that the relation (16) is valid indeed. By using the equations (14), the position and momentum operators are easily calculated to obtain the solution of the quantum oscillator governed by the Hamiltonian (23). Clearly, the solutions (26), (27) along with the operators $\hat{q}(t)$ and $\hat{p}(t)$ could be used to investigate the quantum statistical properties of the radiation field coupled to a nonlinear medium of having $q-$ and $p-dependent$ anharmonicities. Of course, these studies are altogether different issues and we do not have any intention to discuss here. Now, we rearrange the term $\hat{a}^{\dagger m}\hat{a}^{m}$ in the following convenient form

$$\hat{a}^{\dagger m}\hat{a}^{m} = \hat{a}^{\dagger}\hat{a}(\hat{a}^{\dagger}\hat{a}-1)(\hat{a}^{\dagger}\hat{a}-2)\cdots\cdots(\hat{a}^{\dagger}\hat{a}-\overline{m-1}) \quad (28)$$

where $m\geq1$ is an integer. In terms of the number state basis, we calculate the expectation values of the operator $\hat{a}^{\dagger m}\hat{a}^{m}$

$$\begin{aligned}\langle n|\hat{a}^{\dagger m}\hat{a}^{m}|n\rangle &= n(n-1)(n-2)\cdots\cdots(n-\overline{m-1}) \\ &= n!/(n-m)!\end{aligned} \quad (29)$$

where n is the eigenvalue of the number operator $\hat{a}^{\dagger}\hat{a}$ corresponding to the eigenstate (number state) $|n\rangle$. By using the equations (14) and (26), the time evolution of the position and momentum operators of the oscillator under investigation may also be found. It is possible to calculate the dipole moment matrix elements from the knowledge of the position operator. The shifts of the frequency of the oscillator may also be evaluated from the knowledge of the dipole moment matrix elements. As a matter of fact, the shift of the frequency of the oscillator may also be obtained from the knowledge of the matrix elements $\langle n|\hat{a}(t)|n+1\rangle$. Hence, we have

$$\langle n|\hat{a}(t)|n+1\rangle = \sqrt{n+1}\exp(-itf) \quad (30)$$

where f may be identified as the shifted frequency and is given by

$$\begin{aligned} f &= \langle n|\hat{O}(t)|n\rangle \\ &= 1-k\left\{n!/(n-\overline{l-1})!\right\}+\lambda\left\{n!/(n-\overline{m-1})!\right\} \\ &- \frac{k_1}{2^l}\sum_{r=1}^{l} {}^{2l}C_{2r}\times {}^{2l}C_{l-r}\frac{(l-r)(2r-1)!!n!}{(n-\overline{l-r-1})!} \\ &+ \frac{\lambda_1}{2^m}\sum_{r=1}^{m}\frac{(m-r)(2r-1)!!n!}{(n-\overline{m-r-1})!}{}^{2m}C_{2r}\times {}^{2m}C_{m-r} \end{aligned} \tag{31}$$

Clearly, we obtain the frequency of the unperturbed oscillator $(i.e\, f=1)$ if we put $k_1=0=\lambda_1$. It is obvious that the shifts in the frequency of the oscillator occur due to the presence of the $p-dependent$ and $q-dependent$ anharmonicities. Interestingly, the shifts due to the $q-dependent$ anharmonicity are opposite in nature to those of the shifts due to the $p-dependent$ anharmonicities. The reduction (increase) of the frequency from the unperturbed oscillator corresponds the predominant presence of the $q-dependent$ $(p-dependent)$ anharmonicity. These characteristics in the frequency shifts could be utilized to determine the presence of a particular type of nonlinearity.

4 Conclusion

We investigate the problem of an quantum anharmonic oscillator with $q-$ and $p-dependent$ anharmonicities. The problem is of fundamental in nature and occurs in many branches of physics. By exploiting the matter-field interaction, we establish the model of a quantum anharmonic oscillator with two types of anharmonicities. By using RWA, we solve the problem of the said quantum anharmonic oscillator. It is true that we failed to solve the problem in an exact way. Still, the present solution will be of great help for investigating these unsolvable (at least analytically) problems. In addition to the fundamental academic interests, the present solution will find potential applications in the investigation of the quantum statistical properties of the radiation field coupled to a nonlinear medium of having $q-$ and $p-dependent$ anharmonicities. We claim that the solutions of the quantum oscillator with two different types of anharmonicities are definitely fresh and interesting. However, the present investigation involving the exact solution of the quantum anharmonic oscillator is obtained at the cost of the sacrifice of the nonconserving energy terms in the original Hamiltonian (17).

Acknowledgements:

MK and DKB are thankful to UGC (F1-17.1/2016-17/NFST-2015-17-ST-WES-878/(SA-III/Website) *and CSIR* (09/202(0062)/2017-EMR-I) *for the awrd of Senior Research Fellowship respectively.*

References

[1] A. H. Nayfeh, *Introduction to perturbation techniques* (Wiley, New york, 1981).
[2] A. H. Nayfeh and D. T. Mook, *Non-linear oscillations* (John Wiley, New york, 1979).
[3] R. Bellman, *Methods of nonlinear analysis Vol.1* (Academic press, New york, 1970) p.198.
[4] S. L. Ross, *Differential equation 3rd eds.* (John Wiley, New york, 1984).
[5] S. Mandal, *Physics Letters A,* **299,** 531-542 (2002).
[6] S. Mandal, *J.Phys.A* **31,** L501-L505 (1998).
[7] A. Pathak and S. Mandal, *Physics Letters A* **286**, 261-276 (2001).
[8] C. M Bender, and L. M. A. Bettencourt, *Phys.Rev.Lett* **77,** 4114-4117 (1996).
[9] A. Maduemezia, *J.Phys.A,* **6,** 778-782 (1973).
[10] A. Maduemezia, *J.Phys.A,* **7,** 1520-1526 (1974).
[11] S. Mandal, *Modern Physics Letters B,* **18,** 1453-1466 (2004).
[12] J. D. Jackson, *Classical Electrodynamics,* (John Wiley, New york, 1975).
[13] S. C. Lim, J. Osman, and D R Tilley, *J.Phys.D,* **32,** 755-763 (1999).
[14] A. P. Ramirez, P. Chandra, P. Coleman, Z. Fisk, J. L. Smith, and H. R. Ott, *Phys.Rev.Lett* **73,** 3018-3021 (1994).
[15] S. Gasiorowicz, *Quantum Physics,* (John-Wiley, New york, 1974) p.271.
[16] L. I. Schiff, *Quantum Mechanics, 3rd Vol.* (Mc.Graw Hill Book Company, New york, 1987) p.176.
[17] C. C. Gerry, *Physics Letters A,* **124,** 237-239 (1987).
[18] V. Buzek, *Physics Letters A,* **136,** 188-192 (1989).
[19] R. Tanas, *Physics Letters A,* **141,** 217-220 (1989).

Swapan Mandal, a Professor of Physics, Department of Physics, Siksha Bhavana (Institute of Science), Visva-Bharati, Santiniketan-731235, West-Bengal, India. He received his Ph.D degree in theoretical Physics from the University of Calcutta, Kolkata, West-Bengal, India. His research interests are Quantum Optics, Quantum Information Theory, Mathematical Physics, Laser Spectroscopy, Atomic and Molecular Physics.

Monojit Kora is a Ph.D student at the Physics Department, Siksha Bhavana (Institute of Science), Visva-Bharati, Santiniketan-731235, West-Bengal, India. He received the Master degree in Physics in 2012. His main research interests are Quantum Optics, Laser Spectroscopy, Atomic and Molecular Physics.

Dolan Krishna Bayen is a Ph.D student at the Physics Department, Siksha Bhavana (Institute of Science), Visva-Bharati, Santiniketan-731235, West-Bengal, India. He received the Master degree in Physics in 2014 and his main research interests are Quantum Optics, Quantum Information Theory, Mathematical Physics, Laser Spectroscopy, Atomic and Molecular Physics.

Arjun Mukhopadhyay is an Assistant Professor of Physics at Raja Narendralal Khan Women's College, Midnapore, West-Bengal, India. He is currently pursuing his Ph.D in the Department of Physics, Siksha Bhavana (Institute of Science), Visva-Bharati University, West Bengal, India. The area of his research interest is Quantum Optics and Quantum Information Theory.

Kartick Chandra Saha is an Assistant Professor in Physics at the Department of Physics, Gushkara Mahavidyalaya, Burdwan University, West-Bengal, India. and he is also a Ph.D student at the Physics Department, Siksha Bhavana (Institute of Science), Visva-Bharati, Santiniketan-731235, India. His main research interests are Quantum Optics and Quantum Information Theory.