Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090643

On Generalized Close-to-Convex Functions

Khalida Inayat Noor and Muhammad Aslam Noor*

Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan.

Received: 8 Mar. 2015, Revised: 6 May 2015, Accepted: 7 May 2015
Published online: 1 Nov. 2015

Abstract: In this paper, we define and study a new class of analytic functions by using the concept of generalized close-to-convexity. Coefficient results, Hankel determinant problem and some other interesting properties of this class are investigated. Results proved in this paper may stimulate further research in this area.

Keywords: Close-to-convex, bounded boundary rotation, Hankel determinant, univalent, functions with positive real part.
2010 AMS Subject Classification: 30C45, 30C50

1 Introduction

Let A be the class of functions analytic in the open unit disc $E = \{z : |z| < 1\}$ and be given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let $S \subset A$ be the class of functions which are univalent and also K, S^*, C be well known subclasses of S which, respectively, contain close-to-convex, starlike and convex functions. For more details, we refer to [2,4,6, 8,9] and the references therein.
Let V_k be the class of functions f with bounded boundary rotation. Paatero [19] showed that a function $f \in A$, $f'(z) \neq 0$ belongs to the class V_k if and only if

$$\int_0^{2\pi} \left|\Re \frac{(zf'(z))'}{f'(z)}\right| d\theta \leq k\pi; \quad z = re^{i\theta}. \tag{2}$$

It is geometrically obvious that $k \geq 2$.
By Paatero representation theorem [19] for $f \in V_k$, we can write

$$\frac{(zf'(z))'}{f'(z)} = h(z),$$

where

$$\begin{aligned} h(z) &= \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z), \Re h_i(z) > 0, i = 1, 2. \end{aligned} \tag{3}$$

The function h, defined by (3) is said to belong to the class P_k, see [20]. Clearly $P_2 = P$, where P is the class of functions with positive real part.
We note that $V_2 = C$ and it is known [19] that V_k, $2 \leq k \leq 4$, consists entirely of univalent functions.
We now define the following.

Definition 1. *Let $f \in A$ and be locally univalent satisfying the condition $f'(z) \neq 0$. Then $f \in M_{m,k}$ if there exists a function $g \in V_k, k \geq 2$, such that, for $z \in E$*

$$\int_0^{2\pi} \left|\Re \frac{f'(z)}{g'(z)}\right| d\theta \leq m\pi, \quad m \geq 2. \tag{4}$$

The condition (4) is equivalent to the following condition that

$$\frac{f'(z)}{g'(z)} \in P_m, m \geq 2, g \in V_k. \tag{5}$$

Clearly $M_{2,2} = K$ and $M_{2,k} = T_k$ is the class introduced and studied in [12].
The following is a necessary condition for the functions f in the class $M_{m,k}$.

Theorem 1. *Let $f \in M_{m,k}$. Then, for all $\theta_1 < \theta_2$ and for all $0 \leq r < 1$, $z = re^{i\theta}$,*

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta > -\left(\frac{m+k}{2} - 1\right)\pi. \tag{6}$$

* Corresponding author e-mail: khalidanoor@hotmail.com

*Proof.*From definition, it follows that

$$|\arg f'(z) - \arg g'(z)| \leq \frac{m\pi}{2}, \quad g \in V_k. \tag{7}$$

Let

$$F(r,\theta) = \arg\{\frac{\partial}{\partial\theta} f(re^{i\theta})\} = \arg f'(re^{i\theta}) + \frac{\pi}{2} + \theta,$$

and

$$G(r,\theta) = \arg\{\frac{\partial}{\partial\theta} g(re^{i\theta})\} = \arg g'(re^{i\theta}) + \frac{\pi}{2} + \theta.$$

Thus

$$|F(re^{i\theta}) - G(r,\theta)| \leq \frac{m\pi}{2}, \tag{8}$$

and so, for $\theta_1 < \theta_2$

$$\begin{aligned} & F(r,\theta_2) - F(r,\theta_1) \\ & = [\{F(r,\theta_2) - G(r,\theta_2)\} + \{G(r,\theta_2) - G(r,\theta_1)\} \\ & + \{G(r,\theta_1) - F(r,\theta_1)\}] \\ & < \frac{m\pi}{4} + (\frac{k}{2} - 1)\pi + \frac{m\pi}{4} = (\frac{m}{2} + \frac{k}{2} - 1)\pi, \end{aligned}$$

where we have used (8) and a necessary condition for $g \in V_k$, see [1]. This proves (6).

Remark 1. From Theorem 1, we can interpret some geometrical meaning for $f \in M_{m,k}$. For simplicity, let us suppose that the image domain is bounded by an analytic curve C_1. At a point on C_1, the outward drawn normal has an angle $\arg\{e^{i\theta} f'(re^{i\theta})\}$. Then it follows that the angle of the outward drawn normal turns back at most $\left(\frac{m}{2} + \frac{k}{2} - 1\right)\pi$.

Remark 2. Goodman [5] defines the class $K(\beta)$ of function f as follows.
Let $f \in A$ and $f'(z) \neq 0$. Then, for $\beta \geq 0$, $f \in K(\beta)$ if and only if, for $z = re^{i\theta}$, $\theta_1 < \theta_2$

$$\int_{\theta_1}^{\theta_2} \Re \frac{(zf'(z))'}{f'(z)} d\theta > -\beta\pi.$$

We note that

$$M_{m,k} \subset K\left(\frac{m+k}{2} - 1\right), \quad m,k \geq 2.$$

The functions in $M_{m,k}$ are univalent for $m+k \leq 4$ and when $(m+k) > 4$, $f \in M_{m,k}$ need not even be finitely valent.

2 Main Results

Theorem 2. *From Remark 2 and the results given in [5] for the class $K(\beta)$, we at once have:*
Let $f \in M_{m,k}$. Then, for $z = re^{i\theta}$, $0 \leq r < 1$,

(i) $|f'(z)| \leq \frac{m(1+r)^{\frac{k}{2}}}{2(1-r)^{\frac{k}{2}+2}}$,

(ii) $|f(z)| \leq \frac{m}{2(k+2)}\left\{\left(\frac{1+r}{1-r}\right)^{\frac{k}{2}+1} - 1\right\}$

The function $F_0 \in M_{m,k}$, defined as

$$\begin{aligned} F_0(z) &= \frac{m}{2(k+2)}\left[\left(\frac{1+z}{1-z}\right)^{\frac{k}{2}-1} - 1\right] \\ &= z + \sum_{n=2}^{\infty} A_n(m,k) z^n, \end{aligned} \tag{9}$$

shows that these upper bounds are sharp.
(iii) *$|a_n| \leq A_n(m,k)$, $n \geq 2$, where $A_n(m,k)$ is defined by (9), a_n is given by (1) and $\frac{m+k}{2}$ is an even integer. This result is sharp for each $n \geq 2$.*

We now deal with the arc length problem for the class $M_{m,k}$ as follows.

Theorem 3. *Let $L(r,f)$ denote the length of the image of the circle $|z| = r$ under f and let $f \in M_{m,k}$.*
Then, for $0 \leq r < 1$,

$$L(r,f) = O(1)\left(\frac{1}{1-r}\right)^{\frac{m+k}{2}}, \quad (r \to 1),$$

where $O(1)$ is a constant.

Proof. Since $M_{m,k} \subset K(\beta_1), \beta_1 = \left(\frac{m+k}{2} - 1\right)$, and it is known [5] that, for $K(\beta_1)$, there exists $\phi \in C$ such that

$$\left|\arg \frac{f'(z)}{\phi'(z)}\right| \leq \frac{\beta_1\pi}{2}, \quad \beta_1 \geq 0.$$

That is, $f \in M_{m,k}$ implies that

$$f'(z) = \phi'(z) h^{\beta_1}(z), \phi \in C, h \in P. \tag{10}$$

From these observations and (10), we have

$$\begin{aligned} & L(r,f) \\ & = \int_0^{2\pi} |zf'(z)| d\theta \\ & = \int_0^{2\pi} |s(z) h^{\beta_1}(z)| d\theta, \quad s = z\phi' \in S^*, \beta_1 = \left(\frac{m+k}{2} - 1\right) \\ & \leq 2\pi \left(\frac{1}{2\pi}\int_0^{2\pi} |s(z)|^2 d\theta\right)^{\frac{1}{2}} \left(\frac{1}{2\pi}\int_0^{2\pi} |h(z)|^{2\beta_1} d\theta\right)^{\frac{1}{2}} \\ & \leq C(m,k)\left(\frac{1}{1-r}\right)^{\beta_1+1} \\ & = O(1)\left(\frac{1}{1-r}\right)^{\frac{m+k}{2}}, \quad m,k \geq 2, \end{aligned}$$

where we have used Schwarz inequality, subordination for starlike functions and a result due to Hayman [8] for the function $h \in P$. □

We can deduce the rate of growth of the coefficients for $f \in M_{m,k}$ from Theorem 2 as:
Let $f \in M_{m,k}$ and be given by (1). Then, for $n \geq 2$

$$a_n = O(1).n^{\left(\frac{m+k}{2}-1\right)},$$

where $O(1)$ is a constant.

Theorem 4. *Let $f \in M_{m,k}$ and be given by (1). Then*

$$a_n = O(1).n^{\frac{1}{2}}, (n \to \infty),$$

and $O(1)$ is a constant depending only on m and k. The function $F_0 \in M_{m,k}$, defined by (8), shows that the exponent $\frac{k}{2}$ is best possible.

Proof. Since $f \in M_{m,k}$, there exists $g \in V_k$ such that

$$f'(z) = g'(z)H(z), \quad H \in P_m, m \geq 2.$$

Set

$$F(z) = (zf'(z))' = g'(z)h(z)H(z)zH'(z),$$

where $(zg'(z))' = g'(z)h(z)$. Now, by Cauchy Theorem, for $z = re^{i\theta}$, we have

$$\begin{aligned} n^2|a_n| &= \frac{1}{2\pi r^{n+2}}\left|\int_0^{2\pi} F(z)e^{-in\theta}d\theta\right| \\ &\leq \frac{1}{2\pi r^{n+2}}\left|\int_0^{2\pi} |g'(z)\{H(z)h(z)+zH'(z)\}|d\theta. \right. \end{aligned} \quad (11)$$

For $g \in V_k$, it is known [1] that

$$g'(z) = \frac{\left(\frac{s_1(z)}{z}\right)^{\frac{k}{4}+\frac{1}{2}}}{\left(\frac{s_2(z)}{z}\right)^{\frac{k}{4}-\frac{1}{2}}}, s_1, s2 \in S^*. \quad (12)$$

Also, see [13,14], for $H \in P_m$, we have

(i) $\frac{1}{2\pi}\int_0^{2\pi} |H(z)|^2 d\theta \leq \frac{1+(m^2-1)r^2}{1-r^2}, \quad z = re^{i\theta},$

and

(i)

$$\frac{1}{2\pi}\int_0^{2\pi} |zH'(z)|d\theta \leq \frac{m}{1-r^2}, \quad z = re^{i\theta}. \quad (13)$$

Thus, on using (12) together with the well known [4] distortion result for $s_1, s_2 \in S^*$ and Schwarz inequality, we have

$$\begin{aligned} &n62|a_n| \\ &\leq \frac{2^{\frac{k}{2}-1}}{r^{n+1}}\left(\frac{1}{1-r}\right)^{\frac{k}{2}-1} \\ &\times\left[\left(\frac{1}{2\pi}\int_0^{2\pi}|H(z)|^2 d\theta\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}|h(z)|^2 d\theta\right)^{\frac{1}{2}}\right. \\ &\left. + \frac{1}{2\pi}\int_0^{2\pi}|zH'(z)|^2 d\theta\right]. \end{aligned} \quad (14)$$

We make use of (13) in (14) and this leads us to the required result. The proof is complete. □

Golusion [3] has shown that we can choose a $z_1 = z_1(r)$ with $|z_1| = r$ such that, for any univalent function $s(z)$

$$\max_{|z|=r} |(z-z_1)s(z)| \leq \frac{2r^2}{1-r^2}. \quad (15)$$

Using similar technique of Theorem 4 with (15), we can easily prove the following.

Theorem 5. *Let $f \in M_{m,k}$ and be given by (1.1). Then, for $k \geq 2$*

$$||a_n| - |a_{n+1}|| \leq c(m,k)n^{\frac{k}{2}-1}, \quad (n \to \infty),$$

where $c(m,k)$ is a constant.

Let $f \in A$ and be given by (1). The qth Hankel determinant of f is defined for $q \geq 1, n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} \ldots & \vdots \\ \vdots & \vdots & \vdots \\ a_{n+q-1} & \ldots & \ldots a_{n+2q-2} \end{vmatrix} \quad (16)$$

Hankel determinants play an important role in the study of singularities and in the theory of power series with integral coefficients (see, for example [2;pp. 320-335].
The problem of determining the rate of growth of $H_q(n)$ as $n \to \infty$ for $f$belonging to certain subclasses of analytic functions is well known, see [6,7.10-13, 15-18, 21,22].
For $f \in S^*$, Pommerenke solved this problem completely. He showed that, if $f \in S^*$,then
$H_q(n) = 0(1).n^{2-q}, \quad n \to \infty$
and the exponent $(2-q)$is best possible, see [22]. Noor [15] generalized this result for close-to-convex functions. We also refer to [16].
Noonan and Thomas [10] have shown that, for a really mean $p-$valent functions f,

$$H_q(n) = O(1)\begin{cases} n^{2p-1}, & q = 1, p > \frac{1}{4}, \\ n^{2pq-q^2}, & q \geq 2, p \geq 2(q-1), \end{cases}$$

where $O(1)$ depends upon p, q and f and the exponent $(2pq - q^2)$ is best possible.
For $p = 1$, Hayman[7] has shown that $H_2(n) = o(1)n^{\frac{1}{2}}$ as $n \to \infty$ and this is best possible.
In [13], it was shown that , if $f \in V_K$, then

$$H_q(n) = O(1)\begin{cases} n^{\frac{k}{2}-1}, & q = 1 \\ n^{\frac{kq}{2}-q^2}, & q \geq 2, k \geq 8q-10, \end{cases}$$

The exponent $\left(\frac{kq}{2} - q^2\right)$ is best possible in some sense.
In this paper, we estimate the rate of growth of Hankel determinant for $f \in M_{n,k}$.

Theorem 6. *Let $f \in M_{m,k}$ and let the Hankel determinant of $f(z)$, for $q \geq 2$ be defined by (16). Then The $O(1)$ is a constant depending upon k, m, q and f.*

To prove this theorem , we need the following known lemmas, see [10]

Lemma 1. *Let $f \in A$ and be given by (1). Let the qth Hankel determinant of f, for $q \geq 1$, $n \geq 1$, be defined by (16). Then writing $\Delta_j(n) = \Delta_j(n, z_1, f)$, we have*

$$H_q(n) = \begin{vmatrix} \Delta_{2q-1}(n) & \Delta_{2q-3}(n+1) \ldots & \Delta_{q-1}(n+q-1) \\ \Delta_{2q-3}(n+1) & \Delta_{2q-4}(n+2) \ldots & \Delta_{q-2}(n+q) \\ \vdots & \vdots & \vdots \\ \Delta_{q-1}(n+q-1) & \ldots & \ldots \Delta_q(n+2q-2) \end{vmatrix},$$

where, with $\Delta_0(n, z_1, f) = a_n$, we define for $J \geq 1$.

$$\Delta_j(n, z_1, f) = \Delta_{j-1}(n, z_1, f) - n\Delta_{j-1}(n+1, z_1, f) \ldots \quad (19)$$

Lemma 2. *With $x = \left(\frac{n}{n+1} y\right), v \geq 0$ and integer*

$$\Delta_j(n+v, x, zf'(z)) = \sum_{k=0}^{j} \binom{j}{k} \frac{y^k (v-(k-1)n)}{(n+1)^k} . \Delta_{j-k}(n+v+k, y, f(z))$$

We now prove Theorem 6.

Proof. We shall prove this result by using the differences (17). Since $f \in M_{m,k}$, there exists $g \in V_u$ such that

$$f'(z) = g'(z)H(z),$$

where $H \in P_m$ and, with $(zg'(z))' = g'(z)h(z), h \in P_k$, we have

$$F(z) = (zf'(z))' = g'(z)\left[H(z)h(z) + zH'(z)\right]$$

Now, for $j \geq 0, z_1$ any non-zero complex, we consider

$$\begin{aligned} &\left|\Delta_j(n, z_1, F(z))\right| \\ &= \frac{1}{2\pi r^{n+j}} \left| \int_0^{2\pi} (z-z_1)^j (zf'(z))' e^{-i(n+j)\theta} d\theta \right| \\ &\leq \frac{1}{2\pi r^{n+j}} \int_0^{2\pi} |(z-z_1)|^j \left|g'(z)\right| \left|H(z)h(z) + zH'(z)\right| d\theta. \end{aligned}$$

We use (12) and (15) and distortion result for S^* to have, with $k \geq 4j-2$,

$$\begin{aligned} &\left|\Delta_j(n, z_1, F(z))\right| \\ &\leq \left(\frac{4}{r}\right)^{\frac{k}{4}-\frac{1}{2}} . \frac{1}{2\pi r^{n+j}} \left(\frac{2r^2}{1-r^2}\right)^j \left(\frac{r}{1-r}\right)^{\frac{k}{2}+1-2j} \\ &\times \int_0^{2\pi} \left|H(z)h(z) + zH'\theta\right| d\theta. \end{aligned} \quad (20)$$

Applying Schwarz inequality and using (13), we obtain

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \left|H(z)h(z) + zH'(z)\right| d\theta \\ &\leq \left(\frac{1}{2\pi} \int_0^{2\pi} |H(z)|^2 d\theta\right)^{\frac{1}{2}} \left(\frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta\right)^{\frac{1}{2}} \\ &\qquad + \frac{1}{2\pi} \int_0^{2\pi} \left|zH'(z)\right| d\theta \\ &\leq c_1(m,k) . \frac{1}{1-r}, \end{aligned} \quad (21)$$

where $c_1(m,k)$ is constant.
From (18), (19) and Lemma 2, it follows that

$$\Delta_j(n, z_1, f(z)) = O(1).n^{\frac{k}{2}-j}, \quad (22)$$

$O(1)$ depends only on m, k and j.

We use similar argument due to Noonan and Thomas [10] together with Lemma 1 to estimate the rate of growth of $H_q(n)$.

For $q = 1$, $H_1(n) = a_n = \Delta_0(n)$ and, from Theorem 4, it follows that

$$H_1(n) = O(1).n^{\frac{k}{2}}$$

For $q \geq 2$, we have, from (20) and Lemma 1,

$$H_q(n) = O(1).n^{q\left\{\frac{h}{2}-(q-1)\right\}}, k \geq 4(q-1) - 2 = 4q - 6.$$

This gives us the required result. □

As a special case, we note that

$$H_2(n) = O(1).n^{k-2}, k \geq 2$$

Also, for $k = 2$, $f \in M_{m,2}$ and in this case

$$H_q(n) = O(1).n^{2q-q^2}.$$

Theorem 7. *Let $f \in M_{m,k}$ then f maps $|z| < R$ onto a convex domain where R is the least positive root of*

$$T(R) = R^3 - (r_2 + 2r_1)R^2 - (2r_1r_2 + r_1^2)R + r_1^2 r_2 = 0, \quad (23)$$

where

$$r_2 = \frac{k - \sqrt{k^2-4}}{2}, \quad r_1 = \frac{m - \sqrt{m^4-4}}{2}$$

As a special case, when $k = m$, then *$r_1 = r_2$ and we have $R = \left(2-\sqrt{3}\right) r_2$.*

Proof. For $f \in M_{m,k}$, we can write

$$f'(z) = g'(z)H(z), g \in V_k \text{ and } H \in P_m$$

. It is known that, for $|z| < r_1$, $\Re H(z) > 0$, see [20].
Let α be any complex number such that $|\alpha| < r_1$.
Then

$$p(z) = H\left(\frac{r_1^2(2+\alpha)}{r_1^2 + \overline{\alpha}z}\right) = H'(\alpha)\left(1 - \frac{|\alpha|^2}{r_1^2}z + \dots\right)$$

is analytic in $|z| < r_1$ and $\Re p(z) > 0$ for all $|z| < r_1$. Hence, by a result due to Nehari [9], we have

$$\left|H'(\alpha)\left(1 - \frac{|\alpha|^2}{r_1^2}\right)\right| \leq \frac{2|H(\alpha)|}{r_1},$$

which implies that

$$\left|\frac{\alpha H'(\alpha)}{H(\alpha)}\right| \leq \frac{2r_1|\alpha|}{r_1^2 - |\alpha|^2}. \tag{24}$$

Since α is any complex number such that $|\alpha| < r_1$, we can write the inequality (22) as

$$\left|\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)}\right| \leq \frac{2r_1|z|}{r_1^2 - |z|^2}.$$

Hence

$$\Re\frac{(zf'(z))'}{f'(z)} \geq \Re\frac{(zg'(z))'}{g'(z)} - \frac{2r_1|z|}{r_1^2 - |z|^2}.$$

Also, for $g \in V_k$, $\Re\frac{(zg'(z))'}{g'(z)} \geq 0$ for $|z| < r_2 = \frac{k-\sqrt{k^2-4}}{2}$.
Using Harnack Inequality, we can write

$$\Re\frac{(zg'(z))'}{g'(z)} \geq \frac{r_2 - |z|}{r_2 + |z|}.$$

Therefore

$$\begin{aligned} \Re\frac{(zf'(z))'}{f'(z)} &\geq \frac{r_2 - |z|}{r_2 + |z|} - \frac{2r_1|z|}{r_1^2 - |z|^2} \\ &= \frac{(r_2 - |z|)(r_1^2 - |z|^2) - 2r_1|z|(r_2 + |z|)}{(r_2 + |z|)(r_1^2 - |z|^2)} \\ &= \frac{T(|z|)}{(r_2 + |z|)(r_1^2 + |z|^2)}, \end{aligned}$$

where, with $|z| = R$, $T(R)$ is given by (21). We note that $T(0) = r_2 r_1^2$ and $T_1 < 0$, so $R \in (0,1)$ exists.
Hence $\Re\frac{(zf'(z))'}{f'(z)} > 0$ for $|z| < R$, where R is the least positive root of $T(R) = 0$. This completes the proof. □

As a special case, let $m = k$. In this case $f \in M_{k,k}$ maps $|z| < (2-\sqrt{3})r_2$ onto a convex domain.
Here

$$\begin{aligned} T(R) &= R^3 - 3r_2R^2 - 3r_2^2R + r_2^3 \\ &= (r_2 + R)(R^2 - 4r_2R + r_2^2). \end{aligned}$$

That is $R = (2-\sqrt{3})r_2$. We note that, by taking

$$\frac{f'(z)}{g'(z)} = \frac{1+z}{1-z}, g \in V_2,$$

it can be shown that $(2-\sqrt{3})$ cannot be replaced by a smaller constant.

Conclusion

We have used the concept of close-t-convexity to introduce and investigate some new classes of analytic functions. The rate of growth for Hankel determinant of coefficients of these functions has been studied. Arclength problem is also a part of our results. Several applications our main results have been pointed out. The ideas and techniques of this paper may motivate further research in this field.

Acknowledgement

The authors are grateful to Dr. S. M. Junaid Zaidi, Rector, COMSATS Institute of Information Technology, Pakistan for providing excellent research and academic environment. This research is supported by HEC Project NRPU No.: 20-1966/R&D/11-2553: titled, Research unit of Academic Excellence in Geometric Function Theory and Applications.

References

[1] D. A. Brannan, On functions of bounded boundry rotation, Proc. Edin. Math. Soc., 2(1968-1969), 339-347.
[2] P. Dienes, The taylor Series, Dover Publicztions, 1957.
[3] G. Golusin, On distortion theorem and coefficients of univalent functions, Mat. Sb. 19 (1946), 183-203.
[4] A. W. Goodman, Univalent Functions, Vol. I,II, Polygonal Publishing House, Washington, New Jersey, 1983.
[5] A. W. Goodman, On close-to-convex functions of higher order, Ann. Univ. Sci. Budapest, Eotous Sect. Math. 25(1972), 17-30.
[6] W. K. Hayman, Research Problems in Function Theory, Athlon Press, 1967.
[7] W. K. Hayman, On the second Hankel determinant of mean univalent functions, Proc. London math. Soc. 18(3) (1968), 77-84.
[8] W. K. Hayman, On functions with positive real part, J. London Math. Soc., 36(1961), 34-48.
[9] Z. Nehari, Conformal Mappings, McGraw Hill, Newyork, 1952.
[10] J. W. Noonan, D. K. Thomas, On the Hankel determinants of arcally p-valent functions, Proc. London Math. Soc. 25(3)(1972), 503-524.
[11] K. I. Noor, Al-Naggar, Hankel determinant problem, J. Natu. Geom., 14(2)(1998), 133-140.

[12] K. I. Noor, On ageneralization of close-to-convexity, Inter. J. Math. Math. Sci., 6(1983), 327-334.

[13] K. I. Noor, Hankel determinant problem for functions of bounded boundry rotations, Rev. Roum. Math. Pures Appl. 28 (1983), 731-739.

[14] K. I. Noor, On certain analytic functions related with functions of bounded boundry rotations, Comment Math. Univ. St. Pauli, 30(1981), 113-118.

[15] K. I. Noor, On the Hankel determinant of close-to-convex univalent functions, Inter. J. Math. Sci., 3 (1980), 447-481.

[16] K. I. Noor, On the Hankel determinant problem for dtrongly close-to-covex functions, J. Natu. Geom. 11(1997), 29-34.

[17] K. I. Noor, On a generalization of close-to-convexity of complex order, Tamkang J. Math., 29(1998), 73-87.

[18] S. Hussain, M. Arif and S. Nawaz Malik, Higher order close-to-convex functions associated with Attiya-Srivastava operator, Bull. Iranian Math. Soc., 40(2014), 911-920.

[19] V. Paatero, Uber die koforme Abbildung von Gebieten, deren von beschrankter Drehung sind, Ann. Acad. Sci. Fenn. Ser. A, 33(9) (1931), 77pp.

[20] B. Pinchuk, Functions with bounded boundary rotation, Isr. J. Math. 10(1971), 7-16.

[21] Ch. Pommerenke, On Hankel determinant of univalent functions, Mathematica, 14(1967), 108-112.

[22] Ch. Pommerenke, On the coefficients and Hankel determinants of univalent functions, J. London. Math. Soc. 41(1966), 111-122.

Khalida Inayat Noor is a leading world-known figure in mathematics. She obtained her PhD from University of Wales (Swansea)(UK). She has a vast experience of teaching and research at university levels in various countries including Iran, Pakistan, Saudi Arabia, Canada and United Arab Emirates. She was awarded HEC best research paper award in 2009 and CIIT Medal for innovation in 2009. She has been awarded by the President of Pakistan: President's Award for pride of performance on August 14, 2010 for her outstanding contributions in mathematical sciences and other fields. Her field of interest and specialization is Complex analysis, Geometric function theory, Functional and Convex analysis. She introduced a new technique, now called as Noor Integral Operator which proved to be an innovation in the field of geometric function theory and has brought new dimensions in the realm of research in this area. She has been personally instrumental in establishing PhD/MS programs at COMSATS Institute of Information Technology, Pakistan. Dr. Khalida Inayat Noor has supervised successfully several Ph.D and MS/M.Phil students. She has been an invited speaker of number of conferences and has published more than 400 (Four hundred) research articles in reputed international journals of mathematical and engineering sciences. She is member of editorial boards of several international journals of mathematical and engineering sciences.

M. Aslam Noor earned his PhD degree from Brunel University, London, UK (1975) in the field of Applied Mathematics (Numerical Analysis and Optimization). He has vast experience of teaching and research at university levels in various countries including Pakistan, Iran, Canada, Saudi Arabia and United Arab Emirates. His field of interest and specialization is versatile in nature. It covers many areas of Mathematical and Engineering sciences such as Variational Inequalities, Operations Research and Numerical Analysis. He has been awarded by the President of Pakistan: President's Award for pride of performance on August 14, 2008, in recognition of his contributions in the field of Mathematical Sciences. He was awarded HEC Best Research paper award in 2009. He has supervised successfully several Ph.D and MS/M.Phil students. He is currently member of the Editorial Board of several reputed international journals of Mathematics and Engineering sciences. He has more than 800 research papers to his credit which were published in leading world class journals.