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# On Optimal Coefficient in Augmented Lagrangian Method for Saddle Point Problem

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**Abstract:** The optimal coefficient in the augmented Lagrangian method for the Saddle Point Problem is found. As the criterion the minimum of the condition number of the diagonal block is taken. The application of the commonly used preconditioners requires the proper approximations of the inverse of this block and of the Schur's complement. The condition number plays the important role in the calculation of such approximations. The result confirms the experimental value of the coefficient commonly used in the augmented Lagrangian technique.

**Keywords:** saddle point problem, augmented Lagrangian algorithm, linear algebra, condition number, preconditioning

## 1 Introduction

The Saddle Point Problem (SPP) appears in many areas. Let $H_1$ and $H_2$ be finite-dimensional Hilbert spaces with inner product denoted by (.,.). The abstract generalized nonsymmetric saddle point problem is formulated as the set of equations with the block operator $G$:

$$G \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & D \\ B & -C \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{1}$$

The symmetric SPP is presented in many papers in the following form:

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{2}$$

where $f_1 \in H_1$ , $f_2 \in H_2$ are given, $x \in H_1$, $y \in H_2$ are unknowns. $A : H_1 \mapsto H_1$ is linear, symmetric and positive definite, $C : H_2 \mapsto H_2$ is linear, symmetric and semipositive definite, $B : H_1 \mapsto H_2$ and $D : H_2 \mapsto H_1$ are linear maps, $B^T : H_2 \mapsto H_1$ is $B$'s adjoint. In the paper the spaces are: $H_1 = R^{n\times n}$ and $H_2 = R^{m\times m}$ with $m \leq n$.
The problems in which such a structure appears can be the following [2]: computational fluid dynamics, elasticity problems, mixed (FE) formulations of II and IV order elliptic PDEs, linearly constrained programs, weighted least squares (image restoration), FE formulations of consolidation problem.
The block factorization

$$\begin{pmatrix} A & D \\ B & -C \end{pmatrix} = \begin{pmatrix} I & 0 \\ BA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -C - BA^{-1}D \end{pmatrix} \begin{pmatrix} I & A^{-1}D \\ 0 & I \end{pmatrix}$$

is commonly used to solve block equations. A very important part of the method is the calculation of the Schur's complement $S = -C - BA^{-1}D$. Computing the inverse is costly and often ill-conditioned. Additionally many preconditioning techniques are based then on approximating $S^{-1}$ and $A^{-1}$. Two classes of iterative methods are commonly used for solving large SPP: Uzawa algorithm and Krylov methods. For Krylov solvers the condition number and the distribution of the eigenvalues play a role in the speed of convergence. The preconditioning is applied to change the spectrum of the system for more convenient for the Krylov solvers. Several block preconditioners were introduced [2,6,16,18,19,22,25,27] specially for the saddle point problem. The most popular of them are built using the Schur's complement. An augmented Lagrangian technique can be used to improve the numerical properties of $(1,1)$ block $A$ which can be possibly singular or ill-conditioned. Let us recall the idea of this method. Let $W$ be a $n \times n$ matrix. Multiplying the second block-row of system (1) by $B^T W$ and adding the resulting equation to the first block

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equation, we obtain another saddle point problem:

$$\begin{pmatrix} A+B^TWB & D-B^TWC \\ B & -C \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1+B^TWf_2 \\ f_2 \end{pmatrix}. \tag{3}$$

where $G_\gamma = \begin{pmatrix} A+B^TWB & D-B^TWC \\ B & -C \end{pmatrix}$. The new linear system has the same solution and may be easier to solve using existing methods. For methods that rely on the Schur's complement there may be one very important benefit: even if the original $(1,1)$ block $A$ was singular or ill-conditioned, the $(1,1)$ block $A+B^TWB$ of the modified linear system (3), may be nonsingular, positive definite and has a small condition number. The augmented Lagrangian technique has been studied by several authors [9,10,12,15]. In [12] the specific choice $C=0$ and $W=\gamma I$ (where $\gamma$ is a scalar) was considered which leads to the following system:

$$\begin{pmatrix} A+\gamma B^TB & D \\ B & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1+\gamma B^Tf_2 \\ f_2 \end{pmatrix}. \tag{4}$$

The case $W=\gamma I$ with $\gamma = ||A||_2/||B^TB||_2$ may often force the norm of the matrix $A$ to be of the same magnitude as the norm of the added term $B^TB$. This in turn may cause a significant difference in the spectrum and the condition number of the matrix $A_\gamma = A+\gamma B^TB$ in comparison to $A$. The convergence rate of the augmented Lagrangian algorithm is calculated for symmetric case in [14]. The Uzawa algorithm for non-symmetric case is analyzed in [1]. The condition number of the matrix $A_\gamma$ has been analyzed in [10,15]. There is no analytical way of determining the optimal value of $\gamma$ but $\gamma = ||A||/||B||^2$ worked well in the examples presented in [3,12]. For an augmented Lagrangian approach the preconditioners are applied and analyzed by several authors [4,5,8,23,24]. No condition which guaranties the good numerical properties of $A_\gamma$ was given.
In the paper the optimal value of $\gamma$ is calculated theoretically. We will estimate the upper bound of the condition number of the block $A_\gamma$. The real condition number can be much smaller.
In the first section the analysis of the condition number of the block $A_\gamma$ is presented and the main theorem is proved. Then the numerical examples are described. In the examples the condition numbers of the $A_\gamma$ and the appropriate Schur's complements are compared for different $\gamma$. The condition numbers of the Schur's complements for equations (2) and (3) depend on the condition number of $A$ or $A_\gamma$ appropriately. The last part contains the similar problem of updating the singular values of the matrix $A$ with the appended row $a$ scaled by the parameter $\begin{pmatrix} A \\ \beta a \end{pmatrix}$. The problem is recalled after [11, 7]. The condition number of the appended scaled matrix $\begin{pmatrix} A \\ \beta a \end{pmatrix}$ is analyzed. It is shown that the result similar to the one from the Theorem 2 can be obtained for one row matrix $B$ using the results presented in [7,11].

## 2 Theoretical analysis

In the paper we consider the case with matrix symmetric $A \in \mathbb{R}^{n\times n}$: $A = A^T$ and positive definite, $B \in \mathbb{R}^{m\times n}$, $n \geq m$ with full rank: $rank(B) = m$. Let

$$A_\gamma = A+\gamma B^TB \tag{5}$$

with $\gamma > 0$. The appropriate Schur complent is calculated by the following formula:

$$S_\gamma = -C - BA_\gamma^{-1}D. \tag{6}$$

The aim is to find the upper bound of the condition number

$$\kappa_2(A_\gamma) = ||A_\gamma||_2 \cdot ||A_\gamma^{-1}||_2.$$

As the block $A$ is positive definite we have the following inequality for $||.||_2$ norm:

$$||A_\gamma||_2 = ||A+\gamma B^TB||_2 \leq ||A||_2 + \gamma||B^TB||_2. \tag{7}$$

To analyze the dependency of $||A_\gamma^{-1}||_2$ on $\gamma$ the GSVD [13,20,21] is used. As $A$ is the symmetric and positive definite matrix, it has a Cholesky decomposition i.e. there exists an upper triangular matrix $L$, with strictly positive diagonal elements, such that $A = A^T = L^TL$. Let us make the singular value decomposition (GSVD) of the matrix $\tilde{A} = \begin{pmatrix} L \\ B \end{pmatrix}$. Let $U^TLX = D_A$ and $V^TBX = D_B$ where the matrices $U$ and $V$ are orthogonal and the matrix $X$ is invertible. Thus

$$L = UD_AX^{-1} \qquad \text{and} \qquad B = VD_BX^{-1}.$$

In [26] the forms of $D_A$ and $D_B$ were given. Let us recall the main theorem from [26].

**Theorem 1.**( *[26], Theorem 1.1)*
*Let $Q \in \mathbb{R}^{k\times n}$ have orthonormal columns. Partition $Q$ in the form*

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \begin{matrix} l \\ m \end{matrix} \qquad (l+m=k)$$

*Then there are orthonormal matrices $U \in \mathbb{R}^{l\times l}$, $V \in \mathbb{R}^{(k-l)\times(k-l)}$ and $X \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \cdot \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \cdot X = \begin{bmatrix} UQ_1X \\ VQ_2X \end{bmatrix}$$

*assumes one of the following forms:*

*1. $l \geq n$, $m \geq n$*

$$\begin{array}{cc} \left[\begin{array}{c} C \\ 0 \\ \hline S \\ 0 \end{array}\right] & \begin{array}{c} n \\ l-n \\ \hline n \\ m-n \end{array} \end{array}$$

*2. $l \geq n, m \leq n$*

$$\begin{array}{cc} \left[\begin{array}{cc} C & 0 \\ 0 & I \\ 0 & 0 \\ \hline S & 0 \end{array}\right] & \begin{array}{c} m \\ n-m \\ l-n \\ \hline m \end{array} \end{array}$$

*3. $l \leq n, m \geq n$*

$$\begin{array}{cc} \left[\begin{array}{cc} C & 0 \\ \hline S & 0 \\ 0 & I \\ 0 & 0 \end{array}\right] & \begin{array}{c} l \\ \hline l \\ n-l \\ m-n \end{array} \end{array}$$

*4. $l \leq n, m \leq n$*

$$\begin{array}{cc} \left[\begin{array}{ccc} C & 0 & 0 \\ 0 & I & 0 \\ \hline S & 0 & 0 \\ 0 & 0 & I \end{array}\right] & \begin{array}{c} k-n \\ n-m \\ \hline k-n \\ n-l \end{array} \end{array}$$

*Here $C$ and $S$ are nonnegative diagonal matrices satisfying*

$$C^2 + S^2 = I.$$

If the rank of $A$ is less than $n$ and $\tilde{A}$ has full rank we have:

$$D_A = \begin{pmatrix} C_A & 0 \\ 0 & I_{(n-m)\times(n-m)} \end{pmatrix} = \begin{pmatrix} diag(c_i)_{m\times m} & 0 \\ 0 & I_{(n-m)\times(n-m)} \end{pmatrix}$$

and

$$D_B = \begin{pmatrix} S_B & 0 \end{pmatrix} = \begin{pmatrix} diag(s_i)_{m\times m} & 0 \end{pmatrix}.$$

$C_A$ and $S_B$ are diagonal matrices satisfying $C_A^T C_A + S_B^T S_B = I$. Then the matrix $A_\gamma = L^T L + \gamma B^T B$ can be calculated in the following way: $A_\gamma = X^{-T} D_A U^T U D_A X^{-1} + \gamma X^{-T} D_B V^T V D_B X^{-1} = X^{-T}(D_A^2 + \gamma D_B^2) X^{-1}$ and for the inverse of $A_\gamma$ we obtain the following formula:

$$A_\gamma^{-1} = X(D_A^2 + \gamma D_B^2)^{-1} X^T. \tag{8}$$

The inverse of the diagonal matrix $D_A^2 + \gamma D_B^2$ can be calculated very easily as follows:

$$(D_A^2 + \gamma D_B^2)^{-1} = \begin{pmatrix} (diag(\frac{1}{c_i^2+\gamma s_i^2}))_{m\times m} & 0 \\ 0 & I_{(n-m)\times(n-m)} \end{pmatrix}.$$

Let us notice that

$$c_i^2 + \gamma s_i^2 \geq 2\sqrt{\gamma} c_i s_i$$

and

$$\max(\tfrac{1}{c_i^2+\gamma s_i^2}) \leq \tfrac{1}{2\sqrt{\gamma}} \max(\tfrac{1}{|c_i|\cdot|s_i|}) \leq$$
$$\tfrac{1}{2\sqrt{\gamma}} \max(\tfrac{1}{|c_i|},1)\cdot\max(\tfrac{1}{|s_i|},1) \leq \tfrac{1}{2}\tfrac{1}{\sqrt{\gamma}} \cdot ||D_A^{-1}||_2 \cdot ||D_B^\dagger||_2,$$

where $D_B^\dagger$ in the pseudoinverse of $B$. Thus:

$$||(D_A^2 + \gamma D_B^2)^{-1}|| \leq \max(\frac{1}{c_i^2 + \gamma s_i^2})$$

$$\leq \frac{1}{2}\frac{1}{\sqrt{\gamma}} \cdot ||D_A^{-1}||_2 \cdot ||D_B^\dagger||_2 \tag{9}$$

From (8) and (9) the condition number $\kappa_2(A_\gamma) = ||A_\gamma||_2 \cdot ||A_\gamma^{-1}||_2$ calculated in the norm $||.||_2$ has the upper bound depending on $\gamma$:

$$\kappa_2(A_\gamma) \leq \frac{1}{2\sqrt{\gamma}}(||A||_2 + \gamma||B^T B||_2)||X||_2||X^T||_2||D_A^{-1}||_2||D_B^\dagger||_2.$$

Finally the upper bound of the condition number $\kappa_2(A_\gamma)$ is the function:

$$\kappa_2(A_\gamma) \leq Const \cdot \frac{1}{\sqrt{\gamma}}(||A||_2 + \gamma||B^T B||_2) = \Phi(\gamma) \tag{10}$$

where $Const = \frac{1}{2}\cdot||X||_2\cdot||X^T||_2\cdot||D_A^{-1}||_2\cdot||D_B^\dagger||_2$ does not depend on $\gamma$.

**Theorem 2.***The minimum value of the upper bound of the condition number of the matrix $\kappa_2(A_\gamma) = \kappa_2(A + \gamma B^T B)$ in the augmented Lagrangian method is obtained for the coefficient $\gamma_0 = \frac{||A||_2}{||B^T B||_2}$.*

*Proof.*The bound of the condition number is the function of $\gamma$ written by (10). The derivative of it is the following:

$$\Phi'(\gamma) = Const \cdot (-\gamma^{-\frac{3}{2}} \cdot ||A||_2 + \gamma^{-\frac{1}{2}} \cdot ||B^T B||_2).$$

Thus $\Phi'(\gamma_0) = 0 \Rightarrow \gamma_0 = \frac{||A||_2}{||B^T B||_2}$.

## *2.1 Case study for full rank $B^T B$*

In this case $B^T B$ is invertible because it has full rank. As $A$ is the Hermitian matrix and positive definite, it has a Cholesky decomposition i.e. there exists an lower triangular matrix $L$, with strictly positive diagonal elements, such that $A = A^T = L^T L$. Applying the Woodbury formula:

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$$

to $A_\gamma^{-1} = (L^T L + \gamma B^T B)^{-1}$ we have two equations:

$$A_\gamma^{-1} + \gamma A^{-1} B^T (I + \gamma B A^{-1} B^T)^{-1} B A^{-1} = A^{-1}, \tag{11}$$

$$A_\gamma^{-1} + \frac{1}{\gamma}(B^T B)^{-1} L^T (I + L\frac{1}{\gamma}(B^T B)^{-1} L^T)^{-1} L \frac{1}{\gamma}(B^T B)^{-1} =$$

$$= \frac{1}{\gamma}(B^T B)^{-1}. \quad (12)$$

Notice, that the matrices $\gamma A^{-1}B^T(I+\gamma BA^{-1}B^T)^{-1}BA^{-1})$ and $\frac{1}{\gamma}(B^TB)^{-1}L^T(I+L\frac{1}{\gamma}(B^TB)^{-1}L^T)^{-1}L\frac{1}{\gamma}(B^TB)^{-1}$ in (11)) and (12) are positive definite for $\gamma > 0$. Thus from (11) and (12) we have two inequalities (see [17]):

$$||A_\gamma^{-1}||_2 \le ||A^{-1}||_2, \qquad ||A_\gamma^{-1}||_2 \le \frac{1}{\gamma}\cdot||(B^TB)^{-1}||_2. \quad (13)$$

Next, from (13) the bounds of the norm of $A_\gamma^{-1}$ are obtained:

$$||A_\gamma^{-1}||_2 \le \frac{1}{\sqrt{\gamma}}\sqrt{||A^{-1}||_2||(B^TB)^{-1}||_2} \le$$

$$\le \frac{1}{2}(||A^{-1}||_2 + \frac{1}{\gamma}||(B^TB)^{-1}||_2). \quad (14)$$

Basing on (7) and (13) the upper bound of the condition number of $A_\gamma$ is represented by the following function of $\gamma$:

$$\kappa_2(A_\gamma) \le \frac{1}{\sqrt{\gamma}}\cdot\sqrt{||A^{-1}||_2\cdot||(B^TB)^{-1}||_2}\cdot(||A||+\gamma||B^TB||_2). \quad (15)$$

Let denote:

$$\Phi(\gamma) = \frac{1}{\sqrt{\gamma}}\cdot Cons_1\cdot(||A||+\gamma||B^TB||_2). \quad (16)$$

where $Cons_1 = \sqrt{||A^{-1}||_2\cdot||(B^TB)^{-1}||_2}$

**Theorem 3.** *Let us assume that $B^TB$ has full rank. The minimum value of the upper bound of the condition number of the diagonal block $\kappa_2(A_\gamma)$ in the augmented Lagrangian method is obtained for the coefficient $\gamma_0 = \frac{||A||_2}{||B^TB||_2}$.*

*Proof.* The proof is similar as in the Theorem 2. The bound of the condition number is the function of $\gamma$ written by (15). The derivative of it is the following:

$$\Phi'(\gamma) = \frac{1}{2}\cdot Cons_1\cdot(-\gamma^{-\frac{3}{2}}\cdot||A||_2+\gamma^{-\frac{1}{2}}\cdot||B^TB||_2).$$

Thus

$$\Phi'(\gamma_0) = 0 \quad \Rightarrow \quad \gamma_0 = \frac{||A||_2}{||B^TB||_2}.$$

*Remark.* The following facts can be noticed:

1. $||\gamma_0\cdot B^TB||_2 = ||A||_2$,
2. the upper bound for the condition number for $\gamma_0$ has the following value:
$\kappa_2(A_{\gamma_0}) \le$
$\frac{1}{\sqrt{\gamma_0}}\cdot\sqrt{||A^{-1}||_2\cdot||(B^TB)^{-1}||_2}\cdot(||A||_2+\gamma_0||B^TB||_2) =$
$= 2\cdot\sqrt{||A^{-1}||_2\cdot||(B^TB)^{-1}||_2\cdot||A||_2\cdot||(B^TB)||_2} =$
$= 2\cdot\sqrt{\kappa_2(A)\cdot\kappa_2(B^TB)}.$

**Table 1:** Results for the condition numbers of diagonal block $A_\gamma$ for $m = 10$, $n = 3$. $A$ is the Hilbert matrix. Here $\kappa_2(A) = 1.6025\cdot10^{13}$, $\kappa_2(S) = 2.8755\cdot10^4$, $\kappa_2(B) = 2.9281$, $\gamma_0 = 0.2677$, $\kappa_2(A_{\gamma_0}) = 2.5306\cdot10^8$, $\kappa_2(S_{\gamma_0}) = 1.00$.

| $\gamma$ | 0 | 0.1 | $\gamma_0 = 0.2677$ |
|---|---|---|---|
| $\kappa_2(A_\gamma)$ | $1.6025e+13$ | $1.7826e+08$ | $2.5306e+08$ |
| $\kappa_2(S_\gamma)$ | $2.8755e+04$ | 1.0000 | 1.0000 |
| $\gamma$ | 0.5 | 0.7 | 1.0 |
| $\kappa_2(A_\gamma)$ | $3.6934e+08$ | $4.7318e+08$ | $6.3117e+08$ |
| $\kappa_2(S_\gamma)$ | 1.0000 | 1.0000 | $6.8413e+08$ |
| $\gamma$ | 1.1 | 1.3 | 1.5 |
| $\kappa_2(A_\gamma)$ | $7.9030e+08$ | $8.9669e+08$ | $1.1099e+09$ |
| $\kappa_2(S_\gamma)$ | 1.0000 | 1.0000 | 1.0000 |

## 3 Numerical experiments

We present the comparison of the condition number of the diagonal block $\kappa_2(A_\gamma) = \kappa_2(A+\gamma B^TB)$ in the augmented Lagrangian method for the saddle point problem for different coefficient $\gamma$. Numerical tests were done in *MATLAB* with machine precision round off $\varepsilon \approx 10^{-16}$. Examples 1, 2 show that we can improve the condition numbers of the block $A_\gamma$ and of the Schur's complement. Both examples are constructed in such way that the block $A$ is ill-conditioned and matrix $B^TB$ can improve the condition number of the block $A_\gamma$. In Example 3 we have the opposite situation - the condition number of $A_\gamma$ is bigger than that of $A$ as $A$ is the Hilbert matrix and $B$ is the Pascal matrix.

*Example 1.* For generating the matrices $A(m\times m)$, $B(m\times n)$ and we used the following *MATLAB* code:

```
H=hilb(m); A=H;
B=rand(m,n);
```

Here $H(m\times m)$ is the Hilbert matrix:

$$H = (h_{ij}),\ h_{ij} = \frac{1}{i+j-1},\ i,j = 1,\dots,m.$$

The results are presented in Table 1.

*Example 2.* For generating the matrices $A(m\times m)$, $B(m\times n)$ we used the following *MATLAB* code:

```
a=200*ones(m,1);b=-100*ones(m-1,1);
A=diag(a)+diag(b,1)+diag(b,-1);
B=rand(m,n);
```

$A$ is tridiagonal:

$$A(m\times m) = 100\cdot\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}_{m\times m},$$

**Table 2:** Results for the condition numbers of diagonal block $A_\gamma$ for $m = 2000$, $n = 2000$. $A$ is tridiagonal. Here $\kappa_2(A) = 1.6228 \cdot 10^6$, $\kappa_2(S) = 5.5904 \cdot 10^{12}$, $\kappa_2(B) = 2.0593 \cdot 10^5$, $\gamma_0 = 3,9991 \cdot 10^{-4}$, $\kappa_2(A_{\gamma_0}) = 1.2981 \cdot 10^4$, $\kappa_2(S_{\gamma_0}) = 2.0803 \cdot 10^6$, $\kappa_2(X) = 158,489$

| $\gamma$ | 0 | $\gamma_0 = 3,9991e-04$ | 0.5 |
|---|---|---|---|
| $\kappa_2(A_\gamma)$ | $1.6228e+06$ | $1.2981e+04$ | $1.8697e+04$ |
| $\kappa_2(S_\gamma)$ | $5.5904e+12$ | $2.0803e+06$ | $1.6700e+07$ |
| $\gamma$ | 0.8 | 1.0 | 1.1 |
| $\kappa_2(A_\gamma)$ | $2.2723e+04$ | $2.5119e+04$ | $2.6258e+04$ |
| $\kappa_2(S_\gamma)$ | $1.0438e+07$ | $8.3502e+06$ | $7.5911e+06$ |
| $\gamma$ | 1.4 | 1.7 | 2.6 |
| $\kappa_2(A_\gamma)$ | $2.9504e+04$ | $3.2570e+04$ | $4.1501e+04$ |
| $\kappa_2(S_\gamma)$ | $5.9645e+06$ | $4.9119e+06$ | $3.2116e+06$ |

**Table 3:** Results for the condition numbers of diagonal block $A_\gamma$ for $m = 10$, $n = 10$. $A$ is a Pacal matrix. Here $\kappa_2(A) = 4.1552 \cdot 10^9$, $\kappa_2(S) = 1.7894 \cdot 10^{18}$, $\kappa_2(B) = 1.6025 \cdot 10^{13}$, and $\gamma_0 = 21001$

| $\gamma$ | 0 | 1.5 | 10000 |
|---|---|---|---|
| $\kappa_2(A_\gamma)$ | $4.1552e+09$ | $4.0737e+09$ | $2.8916e+09$ |
| $\kappa_2(S_\gamma)$ | $1.7894e+18$ | $6.0847e+16$ | $6.4367e+15$ |
| $\gamma$ | $\gamma_0 = 21001$ | 30000 | 40000 |
| $\kappa_2(A_\gamma)$ | $3.2580e+09$ | $4.0411e+09$ | $5.0724e+09$ |
| $\kappa_2(S_\gamma)$ | $1.4267e+15$ | $2.1323e+14$ | $9.3018e+13$ |
| $\gamma$ | 50000 | 60000 | 70000 |
| $\kappa_2(A_\gamma)$ | $6.1205e+09$ | $7.1546e+09$ | $8.1688e+09$ |
| $\kappa_2(S_\gamma)$ | $1.7385e+14$ | $1.0884e+14$ | $8.0417e+13$ |

The results are presented in Table 2. In this case the experimental $\gamma_0$ is very small.

*Example 3.*For generating the matrices $A(m \times m)$, $B(m \times n)$ we used the following *MATLAB* code:

```
A=pascal(n);
B=hilb(m);
```

The results are shown in Table 3. The function pascal(n) returns the Pascal matrix of order $n$: a symmetric positive define matrix with integer entries taken from the Pascal's triangle. As $B^TB$ is ill-conditioned, the augmented Lagrangian method does not improve the condition number of the diagonal block $A$.

## 4 Eigenvalues of a matrix modified by a rank one matrix

In [7,11] the similar problem is presented - the singular value decomposition is updated when the row is appended. Let $A \in R^{m\times n}$, $m \geq n$, $\tilde{A} = \begin{pmatrix} A \\ a^T \end{pmatrix}$ and $A_1$ the matrix: $A_1 = A^TA$. We are interested in the solution of the following eigenvalue problem: given a symmetric matrix $A_1$ with known eigensystem $A_1 = QDQ^T$, $Q^TQ = QQ^T = I$, calculate the eigensystem of $\tilde{A}_1 = A_1 + \rho b^T b$, $b^Tb = 1$, $\rho \in R$. This problem may be simplified making an observation that $A_1 + \rho b^T b = Q(D + \rho zz^T)Q^T$, where $b = Qz$. Thus, if $C = D + \rho zz^T = X\tilde{D}X^T$ is the orthogonal decomposition of $D + \beta zz^T$ then the orthogonal decomposition of $\tilde{A}_1$ is $\tilde{A}_1 = \tilde{Q}\tilde{D}\tilde{Q}^T$ where $\tilde{Q} = QX$. Let us assume that we are working with an $n \times n$ problem for which no deflation is possible. We consider the problem where $D = diag(d_i)$, $d_i$ are distinct for all $i$, $z = \left(z_1\ z_2\ \ldots\ z_n\right)^T$ and $z_i \neq 0$ for all $i$. In [11] Golub has shown that in the above situation the eigenvalues of $C$ are the zeros of $w(\lambda)$, where $w(\lambda) = 1 + \rho \Sigma_{j=1}^{n} \frac{z_j^2}{(d_j - \lambda)}$. Let us denote the eigenvalues of $C$ by $\tilde{d}_1 < \tilde{d}_2 < \ldots < \tilde{d}_n$. In [7] it is proved that $\tilde{d}_i = d_i + \rho\mu_i$ where $\Sigma_{j=1}^{n}\mu_i = 1$, and $0 \leq \mu_i \leq 1$. Moreover $d_1 \leq \tilde{d}_1 \leq d_2 \leq \tilde{d}_2 \leq \ldots \leq d_n \leq \tilde{d}_n$ if $\rho > 0$ and $\tilde{d}_1 \leq d_1 \leq \tilde{d}_2 \leq \ldots \leq \tilde{d}_n \leq d_n$ if $\rho < 0$. Let us scale the vector $a$ by $\beta$. Let us notice that $|d_n| = ||D||_2$, $\rho = ||a||_2^2 \geq 0$, $\frac{1}{d_1} = ||D||^{-1}$. Then the condition number of $\tilde{A_{1\beta}} = A^TA + \beta^2 aa^T$ has the following bound for $\rho \geq 0$:

$$\kappa_2(\tilde{A_{1\beta}}) = \kappa_2(C_\beta = D + \beta^2 \rho z^T z) = \kappa_2(\tilde{D_\beta}) =$$

$$= \frac{|\tilde{d}_n|}{|\tilde{d}_1|} = \frac{|d_n + \beta^2 \rho \mu_n|}{|d_1 + \beta^2 \rho \mu_1|}$$

$$\leq \frac{||D||_2 + \beta^2 ||a||_2^2}{2\sqrt{\beta^2 \rho \mu_1 d_1}} \leq \frac{1}{2\sqrt{\rho \mu_1 d_1}} \frac{1}{\beta}(||D||_2 + \beta^2 ||a||_2^2)$$

We have the similar function as in the Theorem 2 with $\gamma = \beta^2$. The minimum of the bound of the condition number of $\kappa_2(\tilde{A_{1\beta}}) = \kappa_2(C_\beta)$ is reached for $\beta_0 = \frac{\sqrt{||D||_2}}{||a||_2} = \frac{||A||_2}{||a||_2}$.

## 5 Conclusion

The analysis presented in the paper is the argumentation for the practice. The results are obtained for the $||.||_2$ norm. The choice of the norm is the open question.

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