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Efficient estimators of population coefficient of variation under simple random sampling using single auxiliary variable

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Abstract: In this article, we have proposed two new estimators to estimate the coefficient of variation (CV) incorporating transform ratio type and log type estimators of the study variable using the known information on an auxiliary variable. These estimators utilize information on logarithm transformation on both the population and sample mean of auxiliary character.The proposed estimators utilize logarithmic transformations of certain data points to improve the accuracy of estimating the coefficient of variation for a population.We have derived expressions for the bias and mean squared errors (MSEs) of the proposed estimators up to the first order of approximation using Taylor series techniques . The efficiencies of proposed estimators are evaluated by comparing their MSE and percent relative efficiency (PRE). A real data set is used to verify the efficiency conditions. The results showed that the proposed estimators are more efficient than the existing estimators considered in this study.

Keywords: Study variable, auxiliary variable, bias, mean square error, and coefficient of variation

1 Introduction

Sampling is a method which allows researchers to estimate population parameters by applying data from a subset of the population. This is generally more practical than gathering data from the entire population. The CV is a statistical measure that expresses the relative variability of a set of data points, accounting for differences in their magnitudes. It is calculated as the ratio of the standard deviation (SD) to the mean of the data set, expressed as a percentage.

Whenever it's not possible to calculate the CV directly from the entire population due to its large size, a researcher use sampling technique to select a subset of the population. From this sample, they calculate the sample CV using the sample standard deviation and mean. This sample CV serves as an estimate of the population CV, providing valuable insights into the variability of the population.

The auxiliary information refers to additional information or characteristics related to the study variable that are available for the population but may not be directly measured in the sample. It gives better precision of any estimator and it improves the efficiency of that estimator. It is true when the study variable Y is highly correlated with auxiliary variables X. McKay (1931)[1] worked on the consideration of the approximate distribution of the estimated CV of the study variable. Das and Tripathi (1981a) [2] employed specific methods to estimate the CV of the primary variable within finite sampling theory under the simple random sampling without replacement (SRSWOR) framework. Shafer and Sullivan (1986) [3] performed a simulation study to investigate whether the CV of the main variable being analyzed was consistent or equal. Miller (1991)[4] introduced an asymptotic test statistic for the CV of the primary variable under investigation. Das and Tripathi (1981b)[2] suggested a set of estimators for the CV utilizing both ratio and regression estimation techniques. Tripathi et al. (2002)[5] introduced a novel category of CV estimators, demonstrating that the CV estimators proposed by Das and Tripathi (1981a)[2] are subsets within this newly proposed class of estimators. Sharma and Singh (2014)[6] suggested three improved dual to variance ratio type estimators for estimating the unknown population variance using auxiliary information. Adichwal et al. (2015)[7] suggested some improved class of estimators of population variance using auxiliary information in form of attribute. Patel and Rina (2009)[8]compared number of estimators for CV

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using simulated data under the SRSWOR scheme. Rajyaguru and Gupta (2006)[9] introduced a new set of CV estimators tailored for both simple and stratified random sampling methods, aiming to enhance the accuracy of population CV estimation. Singh and Kumari (2022) [10] proposed four estimators for coefficient of variation based on information on a single auxiliary variable.

Several researchers have played a role in enhancing the accuracy of CV estimation. Some to mention are Archana and Rao (2011)[11], Shabbir and Gupta (2017) [12], Singh et al. (2018a)[13], Muneer et al. (2018), Singh and Mishra (2019) [14], Sanaullah et al. (2022) [15], Zaman and Kadilar (2021) [16], Ijaz et al. (2020)[17], Yadav and Zaman (2021)[18], Ahmed and Shabbir (2021)[19], Yunusa et al. (2021)[20], Audu et al. (2021)[21]. To achieve precise estimation of any given parameter, it is crucial to utilize increasingly efficient estimators whose sampling distributions closely approximate the true population parameter.

In the pursuit of more effective estimators, in this study, a ratio and logarithmic ratio-type estimator for population coefficient of variation is suggested. The proposed estimator utilize information on logarithm transformation on both the population and sample mean of the auxiliary character. This paper is structured into sections. Section 1 outlines the problem being investigated, while section 2 reviews existing estimators for the coefficient of variation. In section 3 i.e. methodology section the proposed estimators are detailed , along with an analysis of their sampling properties under the first-order approximation and theoretical efficiency comparisons between the proposed estimators and existing estimators are provided. Section 4 presents empirical examples. Section 5 presents the discussion.In section 6 we have concluded about the proposed estimators.In the last the various references given for more about CV.

2 Existing estimators

Let's consider a finite population P = $\{P_1,P_2,\ldots.P_N\}$ of size N and each unit are uniquely defined. Let Y and X defined as study and auxiliary variable and Y_i and X_i are the values corresponding their unit i(i=1,2,3.......N).
Let us consider a simple random sample of size n drawn from the given population of N units.
Let,

$\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ and $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$ are the population means of the study and auxiliary variables Y and X.

$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i-\overline{Y})^2$ are the population variance of the study variable Y.

$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i-\overline{X})^2$ are the population variance of the auxiliary variable X.

$S_{xy} = \frac{1}{N-1}\sum_{i=1}^{N}(X_i-\overline{X}(Y_i-\overline{Y})$ is the population covariance of the auxiliary and study variable Y and X.

$\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ are the sample mean of the study and auxiliary variables y and x.
$s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i-\overline{y})^2$ is the sample variance of the study variable y.

$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x})^2$ is the sample variance of the study variable x.

$s_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(y_i-\overline{y}(x_i-\overline{x})$ is the sample covariance of the auxiliary and study variable y and x.
Let,
$e_0 = \frac{\overline{y}-\overline{Y}}{\overline{Y}}$, $e_1 = \frac{\overline{x}-\overline{X}}{\overline{X}}$, $e_2 = \frac{(s_y^2-S_y^2)}{S_y^2}$, $e_3 = \frac{(s_x^2-S_X^2)}{S_X^2}$,such that
$\overline{y} = \overline{Y}(1+e_0), \overline{x} = \overline{X}(1+e_1), \overline{y} = \overline{Y}(1-e_0), s_y^2 = S_y^2(1+e_2), s_x^2 = S_x^2(1+e_3)$
$E(e_0) = E(e_1) = E(e_2) = E(e_3) = 0,$
$E(e_0^2) = \gamma C_y^2,$
$E(e_1^2) = \gamma C_x^2,$
$E(e_2^2) = \gamma(\lambda_{40}-1),$
$E(e_3^2) = \gamma(\lambda_{04}-1),$
$E(e_0e_1) = \gamma\rho C_yC_x\ , E(e_0e_2) = \gamma C_y\lambda_{30},$
$E(e_0e_3) = \gamma C_y\lambda_{12}, E(e_1e_2) = \gamma C_x\lambda_{21},$
$E(e_1e_3) = \gamma C_x\lambda_{03}, E(e_2e_3) = \gamma(\lambda_{22}-1).$
Here, $\gamma = \frac{1}{n}(1-f)$, $f = \frac{n}{N}$, f is known as sampling fraction. C_y and C_x are the population coefficient of variation of study variable Y and auxiliary variable X and defined as $C_y = \frac{S_y}{\overline{Y}}$ and $C_x = \frac{S_x}{\overline{X}}$. ρ is the correlation coefficient between X and Y.
In general form,

$\mu_{rs} = \frac{\sum_{i=1}^{N}(y_i-\overline{y})^r(x_i-\overline{x})^s}{(N-1)}$ and $\lambda_{rs} = \frac{\mu_{rs}}{(\mu_{20}^{r/2}\mu_{02}^{s/2})}$ respectively

The usual estimator for the population coefficient of variation is given by

$$t_0 = \hat{C}_y \tag{1}$$

The MSE expression for usual estimator t_0 given by

$$MSE(t_0) = C_y^2\gamma\left(C_y^2 + \frac{1}{4}(\lambda_{40}-1) - C_y\lambda_{30}\right) \tag{2}$$

Archana and Rao (2014)[22] proposed following estimators for estimating finite population CV given as-

$$t_{AR} = \hat{C}_y\left(\frac{\overline{X}}{\overline{x}}\right) \tag{3}$$

$$t_{AR_1} = \hat{C}_y\left(\frac{\overline{x}}{\overline{X}}\right) \tag{4}$$

$$t_{AR_2} = \hat{C}_y\left(\frac{S_x^2}{s_x^2}\right) \tag{5}$$

$$t_{AR_3} = \hat{C}_y\left(\frac{s_x^2}{S_x^2}\right) \tag{6}$$

The MSE expressions for the estimators t_{AR}, t_{AR1}, t_{AR2} and t_{AR3} are respectively given by :

$$MSE(t_{AR}) = C_y^2\gamma\left(C_y^2 + \frac{1}{4}(\lambda_{40}-1) + C_x^2 - C_x\lambda_{21} - C_y\lambda_{30} + 2\rho C_yC_x\right) \tag{7}$$

$$MSE(t_{AR1}) = C_y^2\gamma\left(C_y^2 + \frac{1}{4}(\lambda_{40}-1) + C_x^2 + C_x\lambda_{21} - C_y\lambda_{30} - 2\rho C_yC_x\right) \tag{8}$$

$$MSE(t_{AR2}) = C_y^2\gamma\left(C_y^2 + \frac{1}{4}(\lambda_{40}-1) + (\lambda_{04}-1) - (\lambda_{22}-1) - C_y\lambda_{21} + 2C_y\lambda_{30}\right) \tag{9}$$

$$MSE(t_{AR3}) = C_y^2\gamma\left(C_y^2 + \frac{1}{4}(\lambda_{40}-1) + (\lambda_{04}-1) + (\lambda_{22}-1) - C_y\lambda_{21} - 2C_y\lambda_{30}\right) \tag{10}$$

Singh et al. (2018b)[13] proposed ratio-type, exponential ratio-type and difference–type estimators for coefficient of variation of the study variable Y using mean of auxiliary variable as-

$$t_1 = \hat{C}_y\left[\frac{\overline{X}}{\overline{x}}\right]^{\alpha} \tag{11}$$

$$t_2 = \hat{C}_y\exp\left[\beta\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right)\right] \tag{12}$$

$$t_3 = \hat{C}_y + d_1\left(\overline{X}-\overline{x}\right) \tag{13}$$

The MSE expressions for the estimators t_1, t_2 and t_3 are respectively given as-

$$MSE(t_1) = C_y^2\gamma\left(C_y^2 + \frac{1}{4}(\lambda_{40}-1) + \alpha^2C_x^2 - \alpha C_x\lambda_{03} - C_y\lambda_{30} + 2\alpha\rho C_yC_x\right) \tag{14}$$

$$MSE(t_2) = C_y^2\gamma\left(C_y^2 + \frac{1}{4}(\lambda_{40}-1) + \frac{1}{4}\beta^2C_x^2 - \frac{1}{2}\beta C_x\lambda_{21} - C_y\lambda_{30} + \beta\rho C_yC_x\right) \tag{15}$$

$$MSE(t_3) = \gamma\left[C_y^2\left(C_y^2 - C_y\lambda_{30} + \frac{1}{4}(\lambda_{40}-1)\right) + d_1^2\overline{X}^2C_x^2 + 2d_1\overline{X}\rho C_yC_x - d_1\overline{X}C_yC_x\lambda_{21}\right] \tag{16}$$

Where,

$$\alpha=\left(\frac{(\lambda_{03}-2\rho_{yx}C_y)}{2C_x}\right),\ \beta=\frac{(\lambda_{21}-2\rho_{yx}C_y)}{C_x},\ d_1=\frac{(\lambda_{21}-2\rho_{yx}C_y)}{2\overline{X}C_x}.$$

Singh et al. (2018b)[13] proposed the arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on t_0 and t_1 estimators for estimating the CV of study variable Y with their MSEs as:

$$t_4^{AM}=\frac{1}{2}\hat{C}_y\left[1+\left(\frac{\overline{X}}{\overline{x}}\right)^{\alpha}\right] \tag{17}$$

$$t_4^{GM}=\hat{C}_y\left[\frac{\overline{X}}{\overline{x}}\right]^{\alpha/2} \tag{18}$$

$$t_4^{HM}=2\hat{C}_y\left[1+\left(\frac{\overline{X}}{\overline{x}}\right)^{\alpha}\right]^{-1} \tag{19}$$

$$MSE(t_4^k)=C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+\alpha^2\frac{1}{4}C_x^2-C_y\lambda_30+\alpha\rho_{yx}C_yC_x-\frac{\alpha}{2}C_x\lambda_{21}\right) \tag{20}$$

Where,
$\alpha=\left(\frac{(\lambda_{21}-2\rho_{yx}C_y)}{2C_x}\right)$, k= AM, GM and HM.
The arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on the estimators (t_0) and t_2) and their MSE are respectively given as-

$$t_5^{AM}=\frac{1}{2}\hat{C}_y\left[1+\exp\left\{\beta\left(\frac{\overline{X}-\overline{x}}{\overline{X}-\overline{x}}\right)\right\}\right] \tag{21}$$

$$t_5^{GM}=\hat{C}_y\exp\left[\frac{\beta}{2}\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right)\right] \tag{22}$$

$$t_5^{HM}=2\hat{C}_y\left[1+\exp\left\{-\beta\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right)\right\}\right]^{-1} \tag{23}$$

$$MSE(t_5^k)=C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+\frac{1}{16}\beta^2C_x^2-C_y\lambda_30+\frac{1}{2}\beta\rho_{yx}C_yC_x-\frac{1}{4}\beta C_x\lambda_{21}\right) \tag{24}$$

Where,
$\beta=2\left(\frac{(\lambda_{21}-2\rho_{yx}C_y)}{2C_x}\right)$, k= AM, GM and HM.

The arithmetic, geometric and harmonic mean estimators (AM, GM, HM) based on (t_1 and t_2) estimators of the study variable Y and their MSE are respectively given as-

$$t_6^{AM}=\frac{1}{2}\hat{C}_y\left[\left(\frac{\overline{X}}{\overline{x}}\right)^{\alpha}+\exp\left\{\beta\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right)\right\}\right] \tag{25}$$

$$t_6^{GM}=\hat{C}_y\left[\left(\frac{\overline{X}}{\overline{x}}\right)^{\alpha}\exp\left\{\beta\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right)\right\}\right]^{1/2} \tag{26}$$

$$t_6^{HM}=2\hat{C}_y\left[\left(\frac{\overline{x}}{\overline{X}}\right)^{\alpha}+\exp\left\{-\beta\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right)\right\}\right]^{-1} \tag{27}$$

$$MSE(t_6^k)=C_y^2\gamma\left[C_y^2+\frac{1}{4}(\lambda_{40}-1)+\frac{1}{4}\left(\alpha+\frac{\beta}{2}\right)^2C_x^2-C_y\lambda_{30}+\left(\alpha+\frac{\beta}{2}\right)\rho_{yx}C_yC_x-\frac{1}{2}\left(\alpha+\frac{\beta}{2}\right)C_x\lambda_{21}\right] \tag{28}$$

Where,
$\beta=2\left(\frac{\lambda_{21}-2\rho_{yx}C_y}{C_x}-\alpha\right)$, k= AM, GM and HM.

3 Methodology

3.1 Proposed estimators :

Here we have proposed two estimators t_{rs} and t_{rs_1} incorporating transform ratio type and log type estimators for estimating the unknown population mean $\overline{Y}$.

$$t_{rs} = \left[\frac{\hat{C}_y}{2}\left(\frac{\overline{X}}{\overline{x}} + \frac{\overline{x}}{\overline{X}}\right) + r_1\left(\overline{X} - \overline{x}\right) + r_2\hat{C}_y\right]\left[1 + \log\left(\frac{\overline{x}}{\overline{X}}\right)\right] \tag{29}$$

$$t_{rs_1} = \left[\frac{\hat{C}_y}{2}\left\{\exp\left(\frac{\overline{X} - \overline{x}}{\overline{X} + \overline{x}}\right) + \exp\left(\frac{\overline{x} - \overline{X}}{\overline{x} + \overline{X}}\right)\right\} + r_3\left(\overline{X} - \overline{x}\right) + r_4\hat{C}_y\right]\left[1 + \log\left(\frac{\overline{x}}{\overline{X}}\right)\right] \tag{30}$$

Expressing estimators t_{rs} and t_{rs_1} in terms of e_i,(i = 0, 1, 2, 3) and simplifying respectively, we have

$$t_{rs} = \left[\frac{S_y(1+e_2)^{1/2}}{2\overline{Y}(1+e_0)}\left\{\frac{\overline{X}}{\overline{X}(1+e_1)} + \frac{\overline{X}(1+e_1)}{\overline{X}}\right\} + r_1\left(\overline{X} - \overline{X}(1+e_1)\right) + r_2\left(\frac{S_y(1+e_2)^{1/2}}{\overline{Y}(1+e_0)}\right)\right]\left[1 + \log\left(\frac{\overline{X}(1+e_1)}{\overline{X}}\right)\right] \tag{31}$$

$$t_{rs_1} = \left[\frac{S_y(1+e_2)^{1/2}}{2\overline{Y}(1+e_0)}\left\{\exp\left(\frac{\overline{X}(1+e_1) - \overline{X}}{\overline{X}(1+e_1) + \overline{X}}\right) + \exp\left(\frac{\overline{X} - \overline{X}(1+e_1)}{\overline{X} - \overline{X}(1+e_1)}\right)\right\} + r_3\left(\overline{X} - \overline{X}(1+e_1)\right) + r_4\left(\frac{S_y(1+e_2)^{1/2}}{\overline{Y}(1+e_0)}\right)\right] \times\left[1 + \log\left(\frac{\overline{X}(1+e_1)}{\overline{X}}\right)\right] \tag{32}$$

$$t_{rs} = \hat{C}_y\left[\left(1 - e_0 + e_1 - e_0e_1 + e_0^2 + \frac{1}{2}e_2 + \frac{1}{2}e_1e_2 - \frac{1}{2}e_0e_1 - \frac{1}{8}e_2^2\right) - \frac{r_1\overline{X}}{C_y}(e_1 + e_1^2) + r_2\left(1 - e_0 + e_1 + e_0^2 - \frac{1}{2}e_1^2 - e_0e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_1e_2 - \frac{1}{2}e_0e_2 - \frac{1}{8}e_2^2\right)\right] \tag{33}$$

$$t_{rs_1} = \hat{C}_y\left[\left(1 - e_0 + e_1 - e_0e_1 + e_0^2 + \frac{1}{2}e_2 + \frac{1}{2}e_1e_2 + \frac{3}{8}e_1^2 - \frac{1}{2}e_0e_1 - \frac{1}{8}e_2^2\right) - r_3\frac{\overline{X}}{C_y}(e_1 + e_1^2) + r_2\left(1 - e_0 + e_1 + e_0^2 - \frac{1}{2}e_1^2 - e_0e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_1e_2 - \frac{1}{2}e_0e_2 - \frac{1}{8}e_2^2\right)\right] \tag{34}$$

Substracting Cy from equations (33) and (34) and taking expectations on both sides, we get the bias expression respectively up to the first order of approximation.

$$Bias(t_{rs}) = \hat{C}_y\left[\gamma\left(C_y^2 - \rho C_yC_x + \frac{1}{2}C_x\lambda_{21} - \frac{1}{2}C_y\lambda_{30} - \frac{1}{8}(\lambda_{40} - 1)\right) - r_1\frac{\overline{X}}{C_y}\gamma C_x^2 + r_2\left(1 + \gamma\left(C_y^2 - C_x^2 - \rho C_yC_x + \frac{1}{2}C_x\lambda_{21} - \frac{1}{8}(\lambda_{40} - 1) - \frac{1}{2}C_x\lambda_{21}\right)\right)\right] \tag{35}$$

$$Bias(t_{rs_1}) = \hat{C}_y\left[\gamma\left(C_y^2 - \frac{3}{8}C_x^2 - \rho C_yC_x + \frac{1}{2}C_x\lambda_{21} - \frac{1}{2}C_y\lambda_{30} - \frac{1}{8}(\lambda_{40} - 1)\right) - r_3\frac{\overline{X}}{C_y}\gamma C_x^2 + r_4\left(1 + \gamma\left(C_y^2 - \frac{1}{2}C_x^2 - \rho C_yC_x + \frac{1}{2}C_x\lambda_{21} - \frac{1}{8}(\lambda_{40} - 1) - \frac{1}{2}C_x\lambda_{21}\right)\right)\right] \tag{36}$$

Using equations (33) and (34), we get the MSE expressions of the estimators t_{rs} and t_{rs1} respectively as follows-

$$MSE(t_{rs}) = C_y^2(A_1 + r_1^2 B_1 + r_2^2 C_1 + 2r_1 D_1 - 2r_2 E_1 - 2r_1 r_2 F_1) \quad (37)$$

$$MSE(t_{rs_1}) = C_y^2(A_2 + r_3^2 B_2 + r_4^2 C_2 + 2r_3 D_2 - 2r_4 E_2 - 2r_3 r_4 F_2) \quad (38)$$

Where,
$A_1 = \gamma\left(C_y^2 + C_x^2 - 2\rho C_y C_x + C_x\lambda_{21} - C_y\lambda_{30} + \frac{1}{4}(\lambda_{40} - 1)\right)$
,[5pt]
$B_1 = \gamma h^2 C_x^2$, $h = \frac{\overline{X}}{C_y}$,

$C_2 = 1 + \gamma(3C_y^2 - 4\rho C_y C_x + 2C_x\lambda_{21} - 2C_y\lambda_{30})$,

$D_1 = \gamma h\left(\rho C_y C_x - C_x^2 - \frac{1}{2}C_x\lambda_{21}\right)$,

$E_1 = \gamma\left[\frac{3}{2}C_y\lambda_{30} + 3\rho C_y C_x - C_x^2 - 2C_y^2 - \frac{3}{2}C_x\lambda_{21} - \frac{3}{8}(\lambda_{40} - 1)\right]$,

$F_1 = \gamma h\left[2C_x^2 - \rho C_y C_x + \frac{1}{2}C_x\lambda_{21}\right]$,

$A_2 = \gamma\left(C_y^2 + C_x^2 - 2\rho C_y C_x + C_x\lambda_{21} - C_y\lambda_{30} + \frac{1}{4}(\lambda_{40} - 1)\right)$,

$B_2 = \gamma h^2 C_x^2$, $h = \frac{\overline{X}}{C_y}$,

$C_2 = 1 + \gamma(3C_y^2 - 4\rho C_y C_x + 2C_x\lambda_{21} - 2C_y\lambda_{30})$,

$D_2 = \gamma h\left(\rho C_y C_x - C_x^2 - \frac{1}{2}C_x\lambda_{21}\right)$,

$E_2 = \gamma\left[\frac{1}{2}C_y\lambda_{30} + \rho C_y C_x - \frac{5}{8}C_x^2 - 2C_y^2 - \frac{3}{2}C_x\lambda_{21} - \frac{3}{8}(\lambda_{40} - 1)\right]$,

$F_2 = \gamma h\left[2C_x^2 - \rho C_y C_x + \frac{1}{2}C_x\lambda_{21}\right]$.

Partially differentiating equations (37) and (38) with respect to r_1 and r_2 respectively we get optimum values of r_1 and r_2 as -
$r_{1opt} = \left(\frac{C_1 D_1 - E_1 F_1}{F_1^2 - B_1 C_1}\right)$, $r_{2opt} = \left(\frac{D_1 F_1 - B_1 E_1}{F_1^2 - B_1 C_1}\right)$

Substituting optimum values of r_{1opt} and r_{2opt} in equation (37), we get the minimum MSE for the estimator t_{rs} as-

$$MSE(t_{rs})_{min} = C_y^2\left[A_1 + \frac{(C_1 D_1^2 + B_1 E_1^2 - 2D_1 E_1 F_1)}{F_1^2 - B_1 C_1}\right] \quad (39)$$

Partially differentiating equations (37) and (38) with respect to r_3 and r_4 respectively we get optimum values of r_3 and r_4 as -
$r_{3opt} = \left(\frac{C_2 D_2 - E_2 F_2}{F_2^2 - B_2 C_2}\right)$, $r_{4opt} = \left(\frac{D_2 F_2 - B_2 E_2}{F_2^2 - B_2 C_2}\right)$

Substituting optimum values of r_{3opt} and r_{4opt} in equation (38), we get the minimum MSE for the estimator t_{rs1} as-

$$MSE(t_{rs1})_{min} = C_y^2\left[A_2 + \frac{(C_2 D_2^2 + B_2 E_2^2 - 2D_2 E_2 F_2)}{F_2^2 - B_2 C_2}\right] \quad (40)$$

3.2 Efficiency comparisons

In this section, efficiency conditions of t_{rs} and t_{rs1} over sample coefficient of variation t_0, t_{AR}, t_{AR1}, t_{AR2}, t_{AR3}, t_1, t_2, t_3, t_4^k, t_5^k and t_6^k are established.

A. Efficiency comparison for the estimator t_{rs} .

i. t_{rs} is more efficient than t_0 , if

$MSE(t_{rs})_{min} < MSE(t_0)$

$$C_y^2\left[A_1+\frac{(C_1D_1^2+B_1E_1^2-2D_1E_1F_1)}{F_1^2-B_1C_1}\right]<C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)-C_y\lambda_{30}\right) \tag{41}$$

ii. t_{rs} is more efficient than t_{AR} , if

$MSE(t_{rs})_{min} < MSE(t_{AR})$

$$C_y^2\left[A_1+\frac{(C_1D_1^2+B_1E_1^2-2D_1E_1F_1)}{F_1^2-B_1C_1}\right]<C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+C_x^2-C_x\lambda_{21}-C_y\lambda_{30}+2\rho C_yC_x\right) \tag{42}$$

iii. t_{rs} is more efficient than t_{AR1} , if

$MSE(t_{rs})_{min} < MSE(t_{AR1})$

$$C_y^2\left[A_1+\frac{(C_1D_1^2+B_1E_1^2-2D_1E_1F_1)}{F_1^2-B_1C_1}\right]<C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+C_x^2+C_x\lambda_{21}-C_y\lambda_{30}-2\rho C_yC_x\right) \tag{43}$$

iv. t_{rs} is more efficient than t_{AR2} , if

$MSE(t_{rs})_{min} < MSE(t_{AR2})$

$$C_y^2\left[A_1+\frac{(C_1D_1^2+B_1E_1^2-2D_1E_1F_1)}{F_1^2-B_1C_1}\right]<C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+(\lambda_{04}-1)-(\lambda_{22}-1)-C_y\lambda_{21}+2C_y\lambda_{30}\right) \tag{44}$$

v. t_{rs} is more efficient than t_{AR3} , if

$MSE(t_{rs})_{min} < MSE(t_{AR3})$

$$C_y^2\left[A_1+\frac{(C_1D_1^2+B_1E_1^2-2D_1E_1F_1)}{F_1^2-B_1C_1}\right]<C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+(\lambda_{04}-1)+(\lambda_{22}-1)-C_y\lambda_{21}-2C_y\lambda_{30}\right) \tag{45}$$

vi. t_{rs} is more efficient than t_1 , if

$MSE(t_{rs})_{min} < MSE(t_1)$

$$C_y^2\left[A_1+\frac{(C_1D_1^2+B_1E_1^2-2D_1E_1F_1)}{F_1^2-B_1C_1}\right]<C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+\alpha^2C_x^2-\alpha C_x\lambda_{03}-C_y\lambda_{30}+2\alpha\rho C_yC_x\right) \tag{46}$$

vii. t_{rs} is more efficient than t_2 , if

$MSE(t_{rs})_{min} < MSE(t_2)$

$$C_y^2\left[A_1+\frac{(C_1D_1^2+B_1E_1^2-2D_1E_1F_1)}{F_1^2-B_1C_1}\right]<C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+\frac{1}{4}\beta^2C_x^2-\frac{1}{2}\beta C_x\lambda_{21}-C_y\lambda_{30}+\beta\rho C_yC_x\right) \tag{47}$$

viii. t_{rs} is more efficient than t_3 , if

$MSE(t_{rs})_{min} < MSE(t_3)$

$$C_y^2\left[A_1+\frac{(C_1D_1^2+B_1E_1^2-2D_1E_1F_1)}{F_1^2-B_1C_1}\right]<\gamma\left[C_y^2\left(C_y^2-C_y\lambda_{30}+\frac{1}{4}(\lambda_{40}-1)\right)+d_1^2\overline{X}^2C_x^2+2d_1\overline{X}\rho C_yC_x-d_1\overline{X}C_yC_x\lambda_{21}\right] \tag{48}$$

ix. t_{rs} is more efficient than t_4^k , if

$MSE(t_{rs})_{min} < MSE(t_4^k)$

$$C_y^2\left[A_1+\frac{(C_1D_1^2+B_1E_1^2-2D_1E_1F_1)}{F_1^2-B_1C_1}\right]<C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+\alpha^2\frac{1}{4}C_x^2-C_y\lambda_30+\alpha\rho_{yx}C_yC_x-\frac{\alpha}{2}C_x\lambda_{21}\right) \tag{49}$$

x. t_{rs} is more efficient than t_5^k , if

$MSE(t_{rs})_{min} < MSE(t_5^k)$

$$C_y^2\left[A_1+\frac{(C_1D_1^2+B_1E_1^2-2D_1E_1F_1)}{F_1^2-B_1C_1}\right]<C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+\frac{1}{16}\beta^2C_x^2-C_y\lambda_30+\frac{1}{2}\beta\rho_{yx}C_yC_x-\frac{1}{4}\beta C_x\lambda_{21}\right) \tag{50}$$

xi. t_{rs} is more efficient than t_6^k , if

$MSE(t_{rs})_{min} < MSE(t_6^k)$

$$C_y^2\left[A_1+\frac{(C_1D_1^2+B_1E_1^2-2D_1E_1F_1)}{F_1^2-B_1C_1}\right]<C_y^2\gamma\Big(C_y^2+\frac{1}{4}(\lambda_{40}-1)+\frac{1}{4}(\alpha+\frac{\beta}{2})^2C_x^2-C_y\lambda_30+ (\alpha+\frac{\beta}{2})\rho_{yx}C_yC_x-\frac{1}{4}(\alpha+\frac{\beta}{2})C_x\lambda_{21}\Big) \tag{51}$$

B. Efficiency comparison for the estimator t_{rs1} .

i. t_{rs1} is more efficient than t_0 , if
$MSE(t_{rs1})_{min} < MSE(t_0)$

$$C_y^2\left[A_2+\frac{(C_2D_2^2+B_2E_2^2-2D_2E_2F_2)}{F_2^2-B_2C_2}\right]<C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)-C_y\lambda_{30}\right) \tag{52}$$

ii. t_{rs1} is more efficient than t_{AR} , if
$MSE(t_{rs1})_{min} < MSE(t_{AR})$

$$C_y^2\left[A_2+\frac{(C_2D_2^2+B_2E_2^2-2D_2E_2F_2)}{F_2^2-B_2C_2}\right]<C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+C_x^2-C_x\lambda_{21}-C_y\lambda_{30}+2\rho C_yC_x\right) \tag{53}$$

iii. t_{rs1} is more efficient than t_{AR1} , if
$MSE(t_{rs1})_{min} < MSE(t_{AR1})$

$$C_y^2\left[A_2+\frac{(C_2D_2^2+B_2E_2^2-2D_2E_2F_2)}{F_2^2-B_2C_2}\right]<C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+C_x^2+C_x\lambda_{21}-C_y\lambda_{30}-2\rho C_yC_x\right) \tag{54}$$

iv. t_{rs1} is more efficient than t_{AR2} , if

$MSE(t_{rs1})_{min} < MSE(t_{AR2})$

$$C_y^2\left[A_2+\frac{(C_2D_2^2+B_2E_2^2-2D_2E_2F_2)}{F_2^2-B_2C_2}\right] < C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+(\lambda_{04}-1)-(\lambda_{22}-1)-C_y\lambda_{21}+2C_y\lambda_{30}\right) \tag{55}$$

v. t_{rs1} is more efficient than t_{AR3} , if

$MSE(t_{rs1})_{min} < MSE(t_{AR3})$

$$C_y^2\left[A_2+\frac{(C_2D_2^2+B_2E_2^2-2D_2E_2F_2)}{F_2^2-B_2C_2}\right] < C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+(\lambda_{04}-1)+(\lambda_{22}-1)-C_y\lambda_{21}-2C_y\lambda_{30}\right) \tag{56}$$

vi. t_{rs1} is more efficient than t_1 , if

$MSE(t_{rs1})_{min} < MSE(t_1)$

$$C_y^2\left[A_2+\frac{(C_2D_2^2+B_2E_2^2-2D_2E_2F_2)}{F_2^2-B_2C_2}\right] < C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+\alpha^2C_x^2-\alpha C_x\lambda_{03}-C_y\lambda_{30}+2\alpha\rho C_yC_x\right) \tag{57}$$

vii. t_{rs1} is more efficient than t_2 , if

$MSE(t_{rs1})_{min} < MSE(t_2)$

$$C_y^2\left[A_2+\frac{(C_2D_2^2+B_2E_2^2-2D_2E_2F_2)}{F_2^2-B_2C_2}\right] < C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+\frac{1}{4}\beta^2C_x^2-\frac{1}{2}\beta C_x\lambda_{21}-C_y\lambda_{30}+\beta\rho C_yC_x\right) \tag{58}$$

viii. t_{rs1} is more efficient than t_3 , if

$MSE(t_{rs1})_{min} < MSE(t_3)$

$$C_y^2\left[A_2+\frac{(C_2D_2^2+B_2E_2^2-2D_2E_2F_2)}{F_2^2-B_2C_2}\right] < \gamma\left[C_y^2\left(C_y^2-C_y\lambda_{30}+\frac{1}{4}(\lambda_{40}-1)\right)+d_1^2\overline{X}^2C_x^2+2d_1\overline{X}\rho C_yC_x-d_1\overline{X}C_yC_x\lambda_{21}\right] \tag{59}$$

ix. t_{rs1} is more efficient than t_4^k , if

$MSE(t_{rs1})_{min} < MSE(t_4^k)$

$$C_y^2\left[A_2+\frac{(C_2D_2^2+B_2E_2^2-2D_2E_2F_2)}{F_2^2-B_2C_2}\right] < C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+\alpha^2\frac{1}{4}C_x^2-C_y\lambda_30+\alpha\rho_{yx}C_yC_x-\frac{\alpha}{2}C_x\lambda_{21}\right) \tag{60}$$

x. t_{rs1} is more efficient than t_5^k , if

$MSE(t_{rs1})_{min} < MSE(t_5^k)$

$$C_y^2\left[A_2+\frac{(C_2D_2^2+B_2E_2^2-2D_2E_2F_2)}{F_2^2-B_2C_2}\right] < C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+\frac{1}{16}\beta^2C_x^2-C_y\lambda_30+\frac{1}{2}\beta\rho_{yx}C_yC_x-\frac{1}{4}\beta C_x\lambda_{21}\right) \tag{61}$$

xi. t_{rs1} is more efficient than t_6^k , if

$MSE(t_{rs1})_{min} < MSE(t_6^k)$

$$C_y^2\left[A_2+\frac{(C_2D_2^2+B_2E_2^2-2D_2E_2F_2)}{F_2^2-B_2C_2}\right] < \left[C_y^2\gamma\left(C_y^2+\frac{1}{4}(\lambda_{40}-1)+\frac{1}{4}\left(\alpha+\frac{\beta}{2}\right)^2C_x^2-C_y\lambda_30+\right.\right.$$
$$\left.\left.\left(\alpha+\frac{\beta}{2}\right)\rho_{yx}C_yC_x-\frac{1}{2}\left(\alpha+\frac{\beta}{2}\right)C_x\lambda_{21}\right)\right] \tag{62}$$

4 Empirical Study

In this section, empirical study is carried out to demonstrate the performance of the proposed estimators over existing ones. Data are taken from the Murthy (1967)[23] and Singh (2003)[24].

Population 1
$[Source : Murthy(1967), p.399]$
X: Area under wheat in 1963, Y: Area under wheat in 1964 N=34, n=15 , $C_x = 0.72, C_y = 0.75, \rho = 0.98,$
$\lambda_{21} = 1.0045, \lambda_{12} = 0.9406, \lambda_{40} = 3.6161, \lambda_{04} = 2.8266, \lambda_{30} = 1.1128, \lambda_{03} = 0.9206, \lambda_{22} = 3.01133, \overline{Y} = 199.44, \overline{X} = 208.88$

Table1. The MSE and PRE of the existing and the proposed estimators

Estimators	MSE	PRE
t_0	0.00800	100.00
t_{AR}	0.02715	29.47
t_{AR1}	0.01184	67.5812
t_{AR2}	0.03365	23.78054
t_{AR3}	0.05890	13.58789
t_1	0.00686	116.53
t_2	0.00686	116.53
t_3	0.00686	116.53
t_4^k	0.00686	116.53
t_5^k	0.00686	116.53
t_6^k	0.00686	116.53
t_{rs}	0.00679	117.8445
t_{rs1}	0.00575	139.0549

Population 2:
$[Source : Singh(2003), p.1116]$
X: Number of fish caught in year 1993, Y: Number of fish caught in year 1995 N=69, n=40 , $C_x = 1.38, C_y = 1.35, \rho = 0.96, \lambda_{21} = 2.19, \lambda_{12} = 2.3, \lambda_{40} = 7.66, \lambda_{04} = 9.84, \lambda_{30} = 1.11, \lambda_{03} = 2.52, \lambda_{22} = 8.19, \overline{Y} = 4514.89, \overline{X} = 4591.07$

Table2. The MSE and PRE of the existing and the proposed estimators

Estimators	MSE	PRE
t_0	0.03808	100.00
t_{AR}	0.08517	44.71481
t_{AR1}	0.06393	59.57474
t_{AR2}	0.18860	20.1948
t_{AR3}	0.22613	16.84297
t_1	0.03806	100.0657
t_2	0.03731	102.0726
t_3	0.03654	104.2332
t_4^k	0.03731	102.0726
t_5^k	0.03750	101.5463
t_6^k	0.03750	101.5463
t_{rs}	0.03638	104.6949
t_{rs1}	0.02883	132.0712

5 Discussion

The formula for Percentage Relative Efficiency (PRE) is given as:

$PRE(estimators) = \frac{MSE_{t_0}}{MSE(estimatots)} \times 100$

In Table 1. and Table 2. the mean square error of the proposed estimators t_{rs}, t_{rs1} and existing estimators $t_0, t_{AR}, t_{AR1}, t_{AR2}, t_{AR3}, t_1, t_2, t_3, t_4^k, t_5^k, t_6^k$ with their percentage relative efficiency are presented. From Table 1. and Table 2. we observe that the proposed estimators have a lower mean square error and a higher percentage relative efficiency. This indicates that the proposed estimators are more efficient than the existing ones. It shows that estimators are more likely to provide estimates that are closer to the true cofficeint of variation.

6 Conclusion

Using the information from auxiliary variables, in this study, we have proposed two new estimators to estimate the coefficient of variation incorporating transform ratio type and log type estimators. In real life problems if certain conditions discussed in efficiency comparison section are satisfied, our proposed estimators can be used by researchers.Therefore, we suggest that the proposed estimators be utilized in practical applications.

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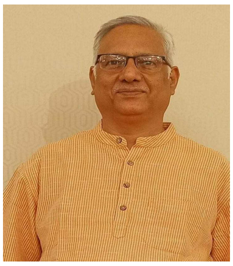

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