Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090452

Weibull Rayleigh Distribution: Theory and Applications

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Received: 21 Nov. 2014, Revised: 21 Feb. 2015, Accepted: 23 Feb. 2015
Published online: 1 Jul. 2015

Abstract: For the first time, a three-parameter lifetime model, called the Weibull Rayleigh distribution, is defined and studied. We obtain some of its mathematical properties. Some structural properties of the new distribution are studied. The method of maximum likelihood and least squares methods is used for estimating the model parameters and the observed Fisher's information matrix is derived. We illustrate the usefulness of the proposed model by applications to real data.

Keywords: Weibull-Rayleigh distribution, Hazard function, Moments, Maximum likelihood estimation.

1 Introduction

In many applied sciences such as medicine, engineering and finance, amongst others, modeling and analyzing lifetime data are crucial. Several lifetime distributions have been used to model such kinds of data. The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models.

In this article we present a new generalization of the Rayleigh distribution called the Weibull-Rayleigh distribution. Rayleigh [16] derived it from the amplitude of sound resulting from many important sources. The Rayleigh distribution has a wide range of applications including life testing experiments, reliability analysis, applied statistics and clinical studies. This distribution is a special case of the two parameter Weibull distribution with the shape parameter equal to 2. The origin and other aspects of this distribution can be found in Siddiqui [17], Hirano [7] and Howlader and Hossian [8]. Dyer and Whisenand [4] demonstrated the importance of this distribution in communication engineering and Polovko [15] noted its importance in electrovacuum devices. Several authors have contributed to this model, namely, Sinha and Howlader [18], Ariyawansa and Templeton [2], Howlader [8], Lalitha and Mishra [11] and Abd Elfattah et al. [1].

A random variable X is said to have the Rayleigh distribution (RD) with parameter θ if its probability density function is given by

$$g(x) = \theta x e^{-\frac{\theta}{2}x^2}, x > 0, \theta > 0 \tag{1}$$

while the cumulative distribution function is given by

$$G(x,\theta) = 1 - e^{-\frac{\theta}{2}x^2}, x > 0, \theta > 0. \tag{2}$$

where θ denote the scale parameter.

Weibull distribution introduced by Weibull [21] is a popular distribution for modeling phenomenon with monotonic failure rates. But this distribution does not provide a good fit to data sets with bathtub shaped or upside-down bathtub shaped (unimodal) failure rates, often encountered in reliability, engineering and biological studies. Hence a number of new distributions modeling the data in a better way have been constructed in literature as ramifications of Weibull distribution.

Marcelo et al. [13] introduced and studied in generality a family of univariate distributions with two additional parameters, similarly as the extended Weibull (Gurvich et al. [6]) and gamma (Zografos and

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Balakrishnan, [20]) families, using the Weibull generator applied to the odds ratio $\frac{G(x)}{1-G(x)}$. The term "generator" means that for each baseline distribution G we have a different distribution F.

If $G(x)$ is the baseline cumulative distribution function(cdf) of a random variable, with probability density function(pdf) $g(x)$ and the Weibull cdf $F(x,\alpha,\beta) = 1 - e^{\alpha x^{\beta}}$ (for $x > 0$) with positive parameters α and β. Based on this density, by replacing x with $\frac{G(x)}{\overline{G}(x)}(\overline{G}(x) = 1 - G(x))$. The cdf of Weibull- generalized distribution, say Weibull- G (Wei- G for short) distribution with two extra parameters α and β, is defined by (Marcelo et al. [13])

$$F(x,\alpha,\beta,\zeta) = \int_0^{\frac{G(x;\zeta)}{1-G(x;\zeta)}} \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}} dt$$
$$= 1 - e^{\left\{-\alpha\left[\frac{G(x;\zeta)}{\overline{G}(x)}\right]^{\beta}\right\}}, \quad x \in R; \alpha,\beta > 0, \tag{3}$$

where $G(x;\zeta)$ is a basline cdf, which depends on a parameter vector ζ. The corresponding family pdf becomes

$$f(x,\alpha,\beta,\zeta) = \alpha\beta g(x;\zeta)\frac{(G(x;\zeta))^{\beta-1}}{\left(\overline{G}(x)\right)^{\beta+1}} \times e^{\left\{-\alpha\left[\frac{G(x;\zeta)}{\overline{G}(x)}\right]^{\beta}\right\}}, \quad x \in R; \alpha,\beta > 0. \tag{4}$$

A random variable X with pdf (4) is denoted by $X \backsim Wei - G(\alpha,\beta,\zeta)$. The additional parameters induced by the Weibull generator are sought as a manner to furnish a more flexible distribution. If $\beta = 1$, it corresponds to the exponential-generator. An interpretation of the $Wei - G$ family of distributions can be given as follows (Cooray, [3] in a similar context. Let Y be a lifetime random variable having a certain continuous G distribution. The odds ratio that an individual (or component) following the lifetime Y will die (failure) at time x is $\frac{G(x)}{\overline{G}(x)}$. Consider that the variability of this odds of death is represented by the random variable X and assume that it follows the Weibull model with scale α and shape β. We can write

$$\Pr(Y \le x) = \Pr(X \le \frac{G(x)}{\overline{G}(x)}) = F(x,\alpha,\beta,\zeta),$$

which is given by (3). The survival and hazard rate functions of the $Wei - G$ family are given by

$$R(x,\alpha,\beta,\zeta) = 1 - F(x,\alpha,\beta,\zeta) = e^{\left\{-\alpha\left[\frac{G(x;\zeta)}{\overline{G}(x)}\right]^{\beta}\right\}}, \tag{5}$$

and

$$h(x,\alpha,\beta,\zeta) = \frac{f(x,\alpha,\beta,\zeta)}{R(x,\alpha,\beta,\zeta)} = \frac{\alpha\beta g(x;\zeta)\,(G(x;\zeta))^{\beta-1}}{\left(\overline{G}(x)\right)^{\beta+1}} = \frac{\alpha\beta\,(G(x;\zeta))^{\beta-1}}{\left(\overline{G}(x)\right)^{\beta}} h(x,\zeta), \tag{6}$$

respectively, where $h(x,\zeta) = \frac{g(x;\zeta)}{\left(\overline{G}(x)\right)}$.The multiplying quantity $\frac{\alpha\beta(G(x;\zeta))^{\beta-1}}{\left(\overline{G}(x)\right)^{\beta}}$ works as a corrected factor for the hazard rate function of the baseline model. (1.3) can deal with general situations in modeling survival data with various shapes of the hazard rate function.

1.1 Mathematical Properties.

By using the power series for the exponential function, we obtain

$$e^{-\alpha\left(\frac{G(x;\zeta)}{\overline{G}(x)}\right)^{\beta}} = \sum_{i=0}^{\infty} \frac{(-1)^i\alpha^i}{i!}\left(\frac{G(x;\zeta)}{\overline{G}(x)}\right)^{i\beta}, \tag{7}$$

Substituting from (7) into (4), we get

$$f(x,\alpha,\beta,\zeta) = \alpha\beta g(x;\zeta) \times \sum_{i=0}^{\infty} \frac{(-1)^i\alpha^i}{i!}\frac{(G(x;\zeta))^{\beta(i+1)-1}}{\left(\overline{G}(x)\right)^{\beta(i+1)+1}}. \tag{8}$$

Using the generalized binomial theorem we have

$$[1 - G(x;\zeta)]^{-(\beta(i+1)+1)} = \sum_{j=0}^{\infty} \frac{\Gamma(\beta(i+1)+j+1)}{j!\Gamma(\beta(i+1)+1)} G(x;\zeta)^j. \tag{9}$$

Inserting (9) in (8) , the $Wei - G$ density function is

$$f(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega_{i,j}\, g(x;\zeta)\,(G(x;\zeta))^{\beta(i+1)+j-1}, \tag{10}$$

where

$$\omega_{i,j} = \frac{(-1)^i\alpha^{i+1}\beta\Gamma(\beta(i+1)+j+1)}{i!j!\Gamma(\beta(i+1)+1)}.$$

The paper is outlined as follows. In Section 2, we define the cumulative, density and hazard functions of the Weibull Rayleigh (WR) distribution. In Section 3, we introduced the statistical properties include, quantile function, skewness and kurtosis, r_{th} moment and moment generating function. The distribution of order statistics is expressed in Section 4. The least squares and weighted least squares estimators are introduced in Section 5. Finally, Maximum likelihood estimation of the parameters is determined in Section 6.

2 The Weibull-Rayleigh Distribution

In this section we studied the three parameter Weibull Rayleigh (WR) distribution. Using $G(x)$ and $g(x)$ in (10) to be the cdf and pdf of (1) and (2). The cdf of the Weibull Rayleigh distribution is given by

$$F(x,\alpha,\beta,\theta) = 1 - e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta}}, x > 0. \quad (11)$$

The corresponding pdf of the WR distribution is given by

$$f(x,\alpha,\beta,\theta) = \alpha\beta\theta x e^{\frac{\theta}{2}x^2}\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta-1} e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta}}. \quad (12)$$

Figures 1 and 2 illustrates some of the possible shapes of the pdf and cdf of Weibull-Rayleigh distribution for selected values of the parameters α, β and θ, respectively.

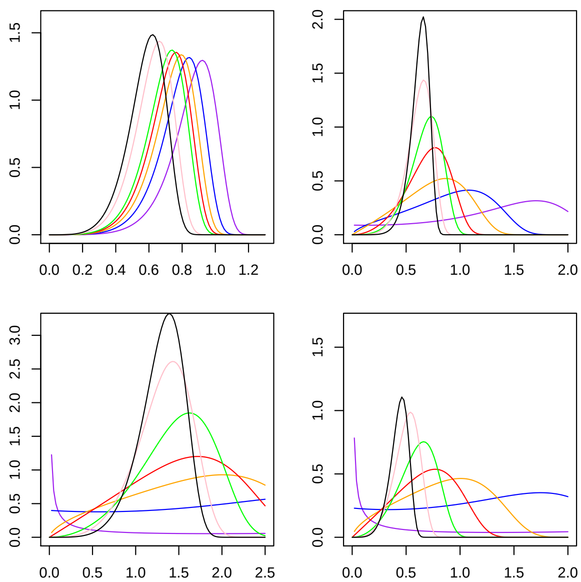

Fig. 1: The pdf function's of various Weibull-Rayleigh distributions for values of parameters: $\alpha = 0.2; 0.4; 0.6; 0.8; 1; 2; 3;$ $a)\beta = 2.5,$ $\theta = 2.5;$ $b)\beta = 0.5; 0.8; 1; 1.5; 2; 2.5; 3.5,$ $\theta = 2.5;$ $c)\beta = 0.1; 0.5; 0.8; 1; 1.5; 2; 2.5,$ $\theta = 0.5;$ $d)\beta = 0.1; 0.5; 0.8; 1; 1.5; 2; 2.5,$ $\theta = 1; 1.5; 2; 2.53; 3.5; 4.5$ with color shapes purple, blue, orange, red, green, pink and black, respectively.

[] and the hazard rate function is

$$h(x,\alpha,\beta,\theta) = \alpha\beta\theta x e^{\frac{\theta}{2}x^2}\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta-1}.$$

Figure 3 illustrates some of the possible shapes of the hazard function of Weibull-Rayleigh distribution for selected values of the parameters α, β and θ, respectively.

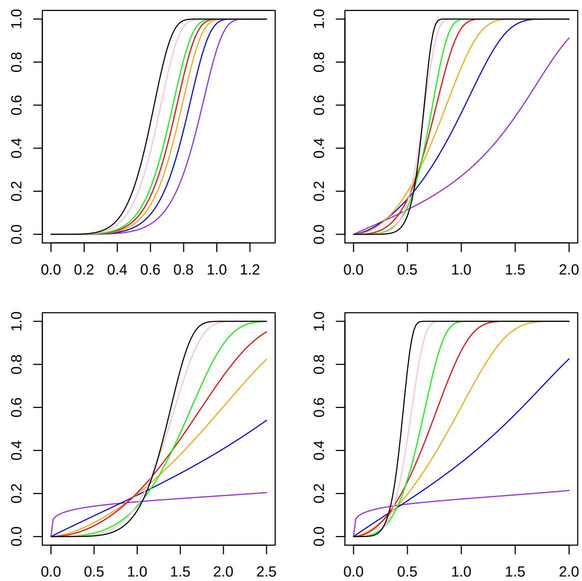

Fig. 2: The cdf function's of various Weibull-Rayleigh distributions for values of parameters: $\alpha = 0.2; 0.4; 0.6; 0.8; 1; 2; 3; a)\beta = 2.5, \theta = 2.5; b)\beta = 0.5; 0.8; 1; 1.5; 2; 2.5; 3.5, \theta = 2.5; c)\beta = 0.1; 0.5; 0.8; 1; 1.5; 2; 2.5, \theta = 0.5; d)\beta = 0.1; 0.5; 0.8; 1; 1.5; 2; 2.5, \theta = 1; 1.5; 2; 2.53; 3.5; 4.5$ with color shapes purple, blue, orange, red, green, pink and black, respectively.

3 Statistical Properties

In this section we study the statistical properties of the WR distribution, specifically quantile function, skewness and kurtosis, moments and moment generating function.

3.1 Quantile Function and Simulation

We present a method for simulating from the WR distribution. The quantile function corresponding to (11) is

$$Q(u) = F^{-1}(u) = \sqrt{\frac{2}{\theta}\ln\left[1+\left(\frac{-\ln(1-u)}{\alpha}\right)^{\frac{1}{\beta}}\right]} \quad (13)$$

Simulating the TEF random variable is straight forward. Let U be a uniform variate on the unit interval $(0,1)$. Thus, by means of the inverse transformation method, we consider the random variable X given by

$$X = \sqrt{\frac{2}{\theta}\ln\left[1+\left(\frac{-\ln(1-u)}{\alpha}\right)^{\frac{1}{\beta}}\right]}, \quad (14)$$

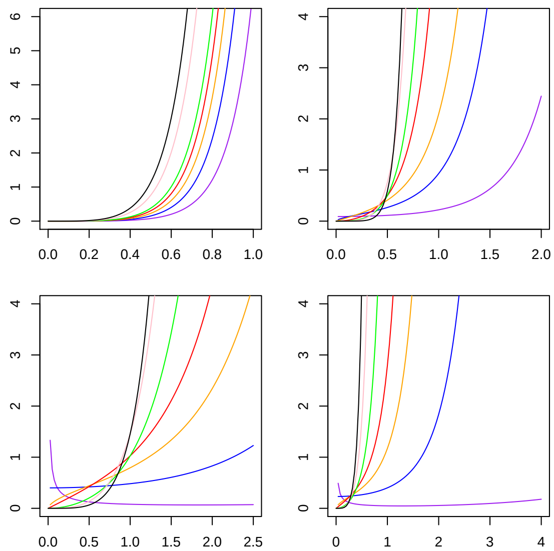

Fig. 3: The hazard function's of various Weibull-Rayleigh distributions for values of parameters: $\alpha = 0.2; 0.4; 0.6; 0.8; 1; 2; 3; a)\beta = 2.5, \theta = 2.5; b)\beta = 0.5; 0.8; 1; 1.5; 2; 2.5; 3.5, \theta = 2.5; c)\beta = 0.1; 0.5; 0.8; 1; 1.5; 2; 2.5, \theta = 0.5; d)\beta = 0.1; 0.5; 0.8; 1; 1.5; 2; 2.5, \theta = 1; 1.5; 2; 2.53; 3.5; 4.5$ with color shapes purple, blue, orange, red, green, pink and black, respectively.

Setting $u = 1/2$ in (14), it follows the median M of X

$$M = \sqrt{\frac{2}{\theta} \ln \left[1 + \left(\frac{\ln 2}{\alpha} \right)^{\frac{1}{\beta}} \right]}.$$

3.2 Skewness and Kurtosis

In this subsection we present the shortcomings of the classical kurtosis measure are well-known. There are many heavy tailed distributions for which this measure is infinite. So, it becomes uninformative precisely when it needs to be. The Bowley's skewness (1962) is based on quartiles

$$S_K = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}},$$

And the Moors' kurtosis (1998) is based on octiles

$$K_u = \frac{Q_{0.875} - Q_{0.625} - Q_{0.375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}}.$$

Where $Q(.)$ represents the quantile function.

3.3 Moments

In this subsection we discuss the r_{th} moment for WR distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem 3.1. If X has $WR(\theta, \beta, \alpha)$, then the r_{th} moment of X is given by the following

$$\mu'_r = \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} \omega_{i,j} (-1)^k \binom{\beta(i+1)+j-1}{k} \frac{\theta \Gamma(1+\frac{r}{2})}{2\left[\frac{2}{\theta(k+1)}\right]^{1+\frac{r}{2}}}. \quad (15)$$

Proof.

We start with the well known definition of the rth moment of the random variable X with probability density function $f(x)$ given by

$$\mu'_r = \int_0^{\infty} x^r f(x) dx.$$

Substituting from (1) and (2) into (10) we get

$$\mu'_r = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_{i,j} \theta \int_0^{\infty} x^{r+1} e^{-\frac{\theta}{2}x^2} \left[1 - e^{-\frac{\theta}{2}x^2}\right]^{\beta(i+1)+j-1} dx, \quad (16)$$

since $0 < 1 - e^{-\frac{\theta}{2}x^2} < 1$ for $x > 0$, the binomial series expansion of $\left[1 - e^{-\frac{\theta}{2}x^2}\right]^{\beta(i+1)+j-1}$ yields

$$\left[1 - e^{-\frac{\theta}{2}x^2}\right]^{\beta(i+1)+j-1} = \sum_{k=0}^{\infty} (-1)^k \binom{\beta(i+1)+j-1}{k} e^{-k\frac{\theta}{2}x^2}, \quad (17)$$

thus we get

$$\mu'_r = \sum_{i,j,k=0}^{\infty} (-1)^k \binom{\beta(i+1)+j-1}{k} \omega_{i,j} \theta \int_0^{\infty} x^{r+1} e^{-(k+1)\frac{\theta}{2}x^2} dx, \quad (18)$$

let $(k+1)\frac{\theta}{2}x^2 = t$,we get

$$\mu'_r = \delta_{i,j,k} \frac{\theta \Gamma(1+\frac{r}{2})}{2\left[\frac{2}{\theta(k+1)}\right]^{1+\frac{r}{2}}}, \quad (19)$$

where

$$\delta_{i,j,k} = \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} \omega_{i,j} (-1)^k \binom{\beta(i+1)+j-1}{k},$$

which completes the proof.

Based on Theorem (3.1) the measures of variation, skewness and kurtosis of the $WR(x; \alpha, \theta, \beta)$ distribution can be obtained according to the following relation

$$CV = \sqrt{\frac{\mu_2}{\mu_1} - 1},$$
$$CS = \frac{\mu_3(\phi) - 3\mu_1(\phi)\mu_2(\phi) + 2\mu_1^3(\phi)}{\left[\mu_2(\phi) - \mu_1^2(\phi)\right]^{\frac{3}{2}}} \text{ and}$$
$$CK = \frac{\mu_4(\phi) - 4\mu_1(\phi)\mu_3(\phi) + 6\mu_1^2(\phi)\mu_2(\phi) - 3\mu_1^4(\phi)}{\left[\mu_2(\phi) - \mu_1^2(\phi)\right]^2}.$$

Theorem 3.2.
The moment generating function of WR distribution is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \delta_{i,j,k} \frac{\theta \Gamma(1+\frac{r}{2})}{2\left[\frac{2}{\theta(k+1)}\right]^{1+\frac{r}{2}}}. \tag{20}$$

Proof.
We start with the well known definition of the $M(t)$ of the random variable X with probability density function $f(x)$ given by

$$\begin{aligned} M_X(t) &= \int_0^{\infty} e^{tx} f_{WR}(x)dx \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f_{WR}(x)dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} \delta_{i,j,k} \frac{\theta \Gamma(1+\frac{r}{2})}{2\left[\frac{2}{\theta(k+1)}\right]^{1+\frac{r}{2}}}. \end{aligned} \tag{21}$$

which completes the proof.

4 Distribution of the order statistics

In this section, we derive closed form expressions for the pdfs of the r_{th} order statistic of the WR distribution, also, the measures of skewness and kurtosis of the distribution of the r_{th} order statistic in a sample of size n for different choices of $n;r$ are presented in this section. Let $X_1, X_2, ..., X_n$ be a simple random sample from WR distribution with pdf and cdf given by (11) and (12), respectively. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denote the order statistics obtained from this sample. We now give the probability density function of $X_{r:n}$, say $f_{r:n}(x,\Phi)$ and the moments of $X_{r:n}$,$r = 1,2,...,n$. Therefore, the measures of skewness and kurtosis of the distribution of the $X_{r:n}$ are presented. The probability density function of $X_{r:n}$ is given by

$$\begin{aligned} f_{r:n}(x,\varphi) &= \frac{1}{B(r,n-r+1)} [F(x,\varphi)]^{r-1} \\ &\quad \times [1 - F(x,\varphi)]^{n-r} f(x,\varphi) \end{aligned} \tag{22}$$

where $F(x,\varphi)$ and $f(x,\varphi)$ are the cdf and pdf of the WR distribution given by (11), (12), respectively, and $B(.,.)$ is the beta function, since $0 < F(x,\varphi) < 1$, for $x > 0$, by using the binomial series expansion of $[1 - F(x,\Phi)]^{n-r}$, given by

$$[1 - F(x,\Phi)]^{n-r} = \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} [F(x,\varphi)]^j, \tag{23}$$

we have

$$f_{r:n}(x,\varphi) = \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} [F(x,\varphi)]^{r+j-1} f(x,\Phi), \tag{24}$$

substituting from (11) and (12) into (24), we can express the k_{th} ordinary moment of the r_{th} order statistics $X_{r:n}$ say $E(X_{r:n}^k)$ as a liner combination of the k_{th} moments of the WR distribution with different shape parameters. Therefore, the measures of skewness and kurtosis of the distribution of $X_{r:n}$ can be calculated.

The r_{th} order statistic for Weibull Rayleigh distribution is given by

$$\begin{aligned} f_{r:n}(x,\varphi) &= \frac{\alpha\beta\theta x e^{\frac{\theta}{2}x^2}}{B(r,n-r+1)} \left[1 - e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta}}\right]^{r-1} \\ &\quad \times \left[e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta}}\right]^{n+\beta-r} \left(e^{\frac{\theta}{2}x^2} - 1\right)^{\beta-1}. \end{aligned}$$

The pdf of the smallest order statistic $X(1)$ is

$$\begin{aligned} f_{1:n}(x,\varphi) =& n\alpha\beta\theta x e^{\frac{\theta}{2}x^2} \left(e^{\frac{\theta}{2}x^2} - 1\right)^{\beta-1} \\ &\times \left[e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta}}\right]^{n+\beta-1}, \end{aligned}$$

and the pdf of the largest order statistic $X(n)$ is

$$\begin{aligned} f_{n:n}(x,\varphi) &= n\alpha\beta\theta x e^{\frac{\theta}{2}x^2} \left(e^{\frac{\theta}{2}x^2} - 1\right)^{\beta-1} \\ &\quad \times \left[e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta}}\right]^{\beta} \left[1 - e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta}}\right]^{n-1}. \end{aligned}$$

5 Least Squares Estimators

In this section we provide the regression based method estimators of the unknown parameters of the Weibull Rayleigh distribution, which was originally suggested by Swain, Venkatraman and Wilson [19] to estimate the parameters of beta distributions. It can be used some other cases also. Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from a distribution function $G(.)$ and suppose $X_{(i)}$; $i=1,2,\dots,n$ denotes the ordered sample. The proposed method uses the distribution of $G(Y_{(i)})$. For a sample of size n, we have

$$E\left(G(X_{(i)})\right)=\frac{i}{n+1},$$
$$V\left(G(Y_{(i)})\right)=\frac{i(n-i+1)}{(n+1)^2(n+2)}$$
$$Cov\left(G(X_{(i)}),G(X_{(j)})\right)=\frac{i(n-j+1)}{(n+1)^2(n+2)};\text{for } i<j,$$

see Johnson, Kotz and Balakrishnan [10]. Using the expectations and the variances, two variants of the least squares methods can be used.

Method 1 (Least Squares Estimators)

The least square estimators (LSES) are obtained by minimizing

$$\sum_{i=1}^{n}\left(G(X_{(i)}-\frac{i}{n+1}\right)^2, \quad (25)$$

with respect to the unknown parameters. Therefore in case of WR distribution the least squares estimators of α,β and θ, say $\widehat{\alpha}_{LSE}$, $\widehat{\beta}_{LSE}$,and $\widehat{\theta}_{LSE}$. respectively, can be obtained by using (11) and (25), we have the following equation

$$Q(\alpha,\beta,\theta)=\sum_{i=1}^{n}\left[1-e^{-\alpha\left(e^{\frac{\theta}{2}x_{(i)}^2}-1\right)^{\beta}}-\frac{i}{n+1}\right]^2, \quad (26)$$

To minimize equation (26) with respect to α,β,and θ, we differentiate with respect to these parameters, which leads to the following equations

$$\frac{\partial Q}{\partial\alpha}=\sum_{i=1}^{n}\left[1-e^{-\alpha\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}}-\frac{i}{n+1}\right]\times\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}e^{-\alpha\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}}, \quad (27)$$

$$\frac{\partial Q}{\partial\beta}=\sum_{i=1}^{n}\left(1-e^{-\alpha\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}}-\frac{i}{n+1}\right)\times\alpha\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}e^{-\alpha\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}}\times\log\left(e^{\frac{\theta}{2}x_i^2}-1\right), \quad (28)$$

$$\frac{\partial Q}{\partial\theta}=\sum_{i=1}^{n}\left(1-e^{-\alpha\left(e^{\frac{\theta}{2}x_{(i)}^2}-1\right)^{\beta}}-\frac{i}{n+1}\right)\times xe^{\frac{\theta}{2}x_{(i)}^2}\left(e^{\frac{\theta}{2}x_{(i)}^2}-1\right)^{\beta-1}e^{-\alpha\left(e^{\frac{\theta}{2}x_{(i)}^2}-1\right)^{\beta}}. \quad (29)$$

The estimates of the parameters are obtained by equating the above equations to zero. Although the proposed estimators cannot be expressed in closed form, they can be obtained through the use of an appropriate numerical solution algorithm.

6 Maximum likelihood and Fisher's information matrix

In this section we determine the maximum likelihood estimates $(MLEs)$ of the parameters of the $WR(x,\varphi)$ distribution from complete samples only. Let $X_1,X_2,\dots,X_n$ be a random sample from $X\sim WR((x,\varphi)$ with observed values $x_1,x_2,\dots,x_n$ and let $\Psi=(\alpha,\beta,\theta)^T$ be the vector of the model parameters. The log likelihood function of (4) when is defined as

$$\ell(\varphi)=n\log\alpha+n\log\beta+n\log\theta+\sum_{i=1}^{n}\log x_i+\frac{\theta}{2}\sum_{i=1}^{n}x_i^2+(\beta-1)\sum_{i=1}^{n}\log\left(e^{\frac{\theta}{2}x_i^2}-1\right)-\alpha\sum_{i=1}^{n}\left[e^{\frac{\theta}{2}x_i^2}-1\right]^{\beta}. \quad (30)$$

Differentiating $L(\varphi)$ with respect to each parameter α,β, and θ and setting the result equals to zero, we obtain maximum likelihood estimates. The partial derivatives of $L(\varphi)$ with respect to each parameter or the score function is given by:

$$U_n(\varphi)=\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\theta}\right)$$

where

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\left[e^{\frac{\theta}{2}x_i^2}-1\right]^{\beta}=0, \quad (31)$$

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log\left(e^{\frac{\theta}{2}x_i^2}-1\right)-\alpha\sum_{i=1}^{n}\left[e^{\frac{\theta}{2}x_i^2}-1\right]^{\beta}\log\left[e^{\frac{\theta}{2}x_i^2}-1\right]=0, \quad (32)$$

and

$$\frac{\partial\ell}{\partial\theta}=\frac{n}{\theta}+\frac{1}{2}\sum_{i=1}^{n}x_i^2+(\beta-1)\sum_{i=1}^{n}\frac{x_i^2e^{\frac{\theta}{2}x_i^2}}{2\left(e^{\frac{\theta}{2}x_i^2}-1\right)}-\frac{\alpha\beta}{2}\sum_{i=1}^{n}x_i^2e^{\frac{\theta}{2}x_i^2}\left[e^{\frac{\theta}{2}x_i^2}-1\right]^{\beta-1}=0. \quad (33)$$

The MLE of the parameters α, β, and θ, say $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{\theta}$ are obtained by solving the following equations, $\frac{\partial \ell}{\partial \alpha} = \frac{\partial \ell}{\partial \beta} = \frac{\partial \ell}{\partial \theta} = 0$.There is no closed form solution to these equations, so numerical technique must be applied. For the three parameters Weibull Rayleigh distribution $WR(x;\alpha,\beta,\theta)$. If all the second order derivatives exist. Thus we have the inverse dispersion matrix is given by

$$\begin{pmatrix} \widehat{\alpha} \\ \widehat{\beta} \\ \widehat{\theta} \end{pmatrix} \sim N\left[\begin{pmatrix} \alpha \\ \beta \\ \theta \end{pmatrix}, \begin{pmatrix} \widehat{V\alpha\alpha} & \widehat{V_{\alpha\beta}} & \widehat{V_{\alpha\theta}} \\ \widehat{V_{\beta\alpha}} & \widehat{V_{\beta\beta}} & \widehat{V_{\beta\theta}} \\ \widehat{V_{\theta\alpha}} & \widehat{V_{\theta\beta}} & \widehat{V_{\theta\theta}} \end{pmatrix}\right]. \tag{34}$$

$$V^{-1} = -E\begin{bmatrix} V\alpha\alpha & V_{\alpha\beta} & V_{\alpha\theta} \\ V_{\beta\alpha} & V_{\beta\beta} & V_{\beta\theta} \\ V_{\theta\alpha} & V_{\theta\beta} & V_{\theta\theta} \end{bmatrix} \tag{35}$$

where

$$V\alpha\alpha = \frac{\partial^2 L}{\partial \alpha^2} = -\frac{n}{\alpha^2}$$

$$V_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2} = -\frac{n}{\beta^2} - \alpha\sum_{i=1}^{n}\left(e^{1/2\,\theta\,{x_i}^2}-1\right)^{\beta}\left(\ln\left(e^{1/2\,\theta\,{x_i}^2}-1\right)\right)^2$$

$$V_{\theta\theta} = \frac{\partial^2 L}{\partial \theta^2} = -\frac{n}{\theta^2} - (\beta-1)\sum_{i=1}^{n} 1/4\,\frac{{x_i}^4 e^{1/2\,\theta\,{x_i}^2}}{\left(e^{1/2\,\theta\,{x_i}^2}-1\right)^2} - \frac{\alpha}{4}\sum_{i=1}^{n}\frac{\left(e^{1/2\,\theta\,{x_i}^2}-1\right)^{\beta}\beta\,{x_i}^4 e^{1/2\,\theta\,{x_i}^2}\left(\beta\,e^{1/2\,\theta\,{x_i}^2}-1\right)}{\left(e^{1/2\,\theta\,{x_i}^2}-1\right)^2}$$

$$V_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta} = -\sum_{i=1}^{n}\left(e^{1/2\,\theta\,{x_i}^2}-1\right)^{\beta}\ln\left(e^{1/2\,\theta\,{x_i}^2}-1\right)$$

$$V_{\alpha\theta} = \frac{\partial^2 L}{\partial\alpha\partial\theta} = -\frac{1}{2}\sum_{i=1}^{n}\frac{\left(e^{1/2\,\theta\,{x_i}^2}-1\right)^{\beta}\beta\,{x_i}^2 e^{1/2\,\theta\,{x_i}^2}}{e^{1/2\,\theta\,{x_i}^2}-1}$$

$$V_{\beta\theta} = \frac{\partial^2 L}{\partial b\partial\theta} = \sum_{i=1}^{n} 1/2\,\frac{{x_i}^2 e^{1/2\,\theta\,{x_i}^2}}{e^{1/2\,\theta\,{x_i}^2}-1} - \frac{\alpha}{2}\sum_{i=1}^{n}\frac{\left(e^{1/2\,\theta\,{x_i}^2}-1\right)^{\beta}{x_i}^2 e^{1/2\,\theta\,{x_i}^2}\cdot K_i}{e^{1/2\,\theta\,{x_i}^2}-1}$$

$$K_i = \left(\beta\ln\left(e^{1/2\,\theta\,{x_i}^2}-1\right)+1\right)$$

By solving this inverse dispersion matrix these solutions will yield asymptotic variance and covariances of these ML estimators for $\widehat{\alpha}, \widehat{\beta}$ and $\widehat{\theta}$. Using (35), we approximate $100(1-\gamma)\%$ confidence intervals for α, β and θ are determined respectively as

$$\widehat{\alpha} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{\alpha\alpha}}}, \widehat{\beta} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{\beta\beta}}}, \text{ and } \widehat{\theta} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{\theta\theta}}},$$

where z_{γ} is the upper $100\gamma_{the}$ percentile of the standard normal distribution.

Using R we can easily compute the Hessian matrix and its inverse and hence the standard errors and asymptotic confidence intervals.

We can compute the maximized unrestricted and restricted log-likelihood functions to construct the likelihood ratio (LR) test statistic for testing on some the Weibull-Rayleigh sub-models. For example, we can use the LR test statistic to check whether the Weibull-Rayleigh distribution for a given data set is statistically *superior* to the Weibull distribution. In any case, hypothesis tests of the type $H_0 : \lambda = \lambda_0$ versus $H_0 : \lambda \neq \theta_0$ can be performed using a LR test. In this case, the LR test statistic for testing H_0 versus H_1 is $\omega = 2(\ell(\hat{\theta};x) - \ell(\hat{\lambda}_0;x))$, where $\hat{\lambda}$ and $\hat{\lambda}_0$ are the MLEs under H_1 and H_0, respectively. The statistic ω is asymptotically (as $n \to \infty$) distributed as χ^2_k, where k is the length of the parameter vector λ of interest. The LR test rejects H_0 if $\omega > \chi^2_{k;\gamma}$, where $\chi^2_{k;\gamma}$ denotes the upper $100\gamma\%$ quantile of the χ^2_k distribution.

7 Application

In this section we compare the results of fitting the Weibull-Rayleigh, Exponentiated Weibull, Beta-Weibull([12]) and Weibull distribution to the data set studied by Meeker and Escobar (1998, p. 383, [14]), which gives the times of failure and running times for a sample of devices from a eld-tracking study of a larger system. At a certain point in time, 30 units were installed in normal service conditions. Two causes of failure were observed for each unit that failed: the failure caused by an accumulation of randomly occurring damage from power-line voltage spikes during electric storms and failure caused by normal product wear. The times are:

2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00, 3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66.

The variance covariance matrix $I(\hat{\lambda})^{-1}$ of the MLEs under the Weibull-Rayleigh distribution for data set is computed as

$$I(\hat{\lambda})^{-1} = \begin{pmatrix} 0.012 & 0.001 & -0.040 \\ 0.001 & 0.007 & -0.044 \\ -0.040 & -0.044 & 0.364 \end{pmatrix}.$$

Thus, the variances of the MLE of α, β and θ is $var(\hat{\alpha}) = 0.012, var(\hat{\beta}) = 0.007$ and $var(\hat{\theta}) = 0.364$. Therefore, 95% confidence intervals for α, β and θ are $[0.060, 0.490], [0.122, 0.462]$ and $[0.379, 2.745]$ respectively. We plot the profile likelihood of α in Figure 4.

Table 1: The ML estimates, standard error and Log-likelihood and LSE estimates for data set

Model	ML Est.	St. Err.	LL	LSES
Weibull	$\hat{\alpha} = 0.275$	0.109	35.409	0.628
Rayleigh	$\hat{\beta} = 0.292$	0.086		0.039
	$\hat{\theta} = 1.562$	0.603		1.561
Beta	$\hat{\alpha} = 6.104$	0.003	39.562	1.595
Weibull	$\hat{\beta} = 0.205$	3.179e-06		0.045
	$\hat{a} = 0.149$	0.029		6.793
	$\hat{b} = 8.114$	3.932		3.933
Exponentiated	$\hat{\alpha} = 0.314$	0.024	39.463	0.032
Weibull	$\hat{\beta} = 5.877$	0.199		5.885
	$\hat{\theta} = 0.156$	0.029		0.038
Weibull	$\hat{\alpha} = 0.449$	0.115	46.158	0.133
	$\hat{\beta} = 1.265$	0.204		0.055

Table 2: The AIC and AICC of the models based on data set

Model	-2LL	AIC	AICC
Weibull-Rayleigh	70.818	76.818	77.741
Beta-Weibull	79.124	87.124	88.266
Exponentiated Weibull	78.926	84.926	85.849
Weibull	92.316	96.316	96.760

Table 1 shows parameter MLEs to each one of the two fitted distributions for data set , Tables 2 shows the values of $-2\log(L)$, AIC and AICC values. The values in Table 2, indicate that the Weibull-Rayleigh is a strong competitor to other distribution used here for fitting data set. A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Fig. 5). The fitted density for the Weibull-Rayleigh model is closer to the empirical histogram than the fits of the Weibull models. Figures (Fig.6) shows fitted P-P plots for WR and W distribution for data set.

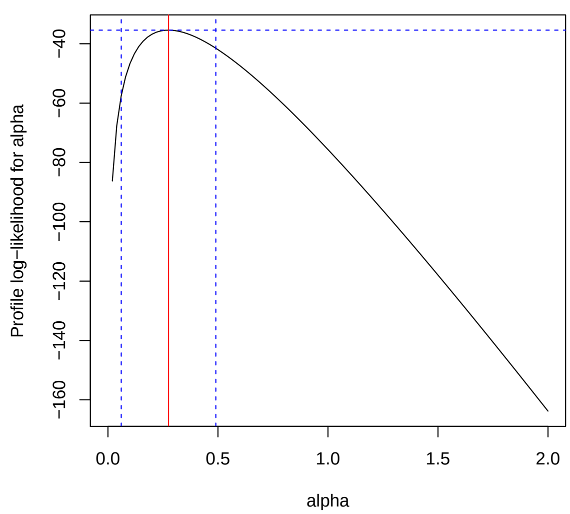

Fig. 4: The profile log-likelihood function of α for the data set.

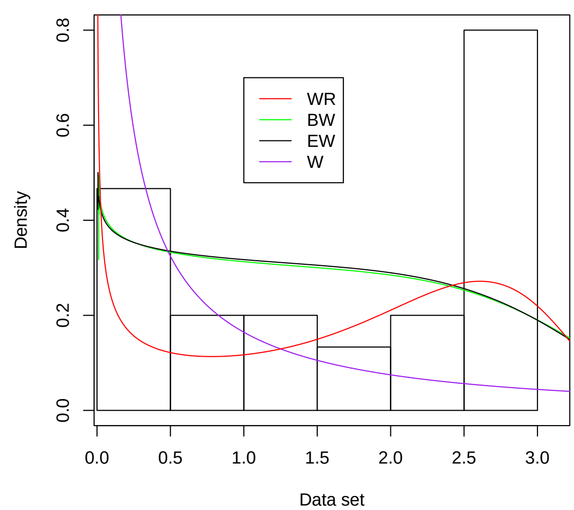

Fig. 5: Estimated densities of the models for data set.

In order to compare the two distribution models, we consider criteria like -2ℓ, AIC (Akaike information criterion)and AICC (corrected Akaike information criterion) for the data set. The better distribution corresponds to smaller -2ℓ, AIC and AICC values:‘

$$\text{AIC} = 2k - 2\ell, \quad \text{and} \quad \text{AICC} = \text{AIC} + \frac{2k(k+1)}{n-k-1},$$

where k is the number of parameters in the statistical model, n the sample size and ℓ is the maximized value of the log-likelihood function under the considered model.

The LR test statistic to test the hypotheses $H_0 : \theta = 2$ versus $H_1 : \theta \neq 2$ for data set is $\omega = 21.498 > 3.841 = \chi^2_{2;0.05}$, so we reject the null hypothesis.

8 Simulated data

In this subsection, we provided an algorithm to generated a random sample from the WR distribution for the given values of its parameters and sample size n. The simulation process consists the following steps:

Step 1.Set n, and $\Theta = (\alpha, \beta, \theta)$.
Step 2.Set initial value x^0 for the random starting.

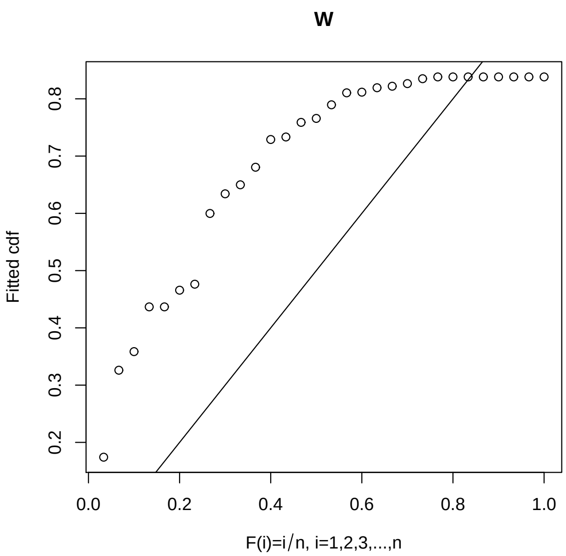

Fig. 6: P-P plots for fitted Weibullfor data set.

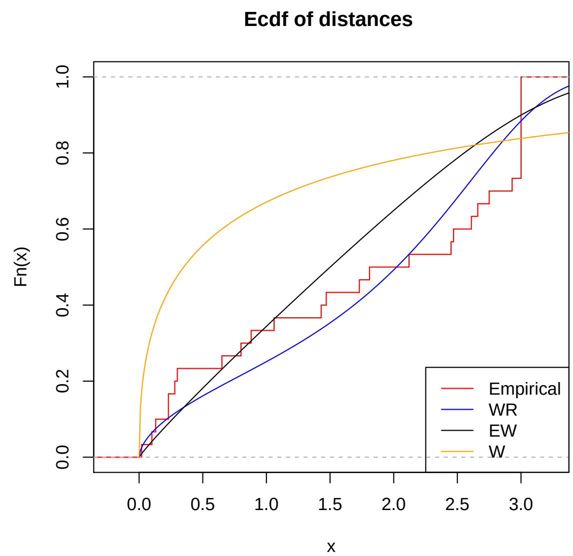

Fig. 8: Empirical, fitted WR, EW and Weibull cdf of the data set.

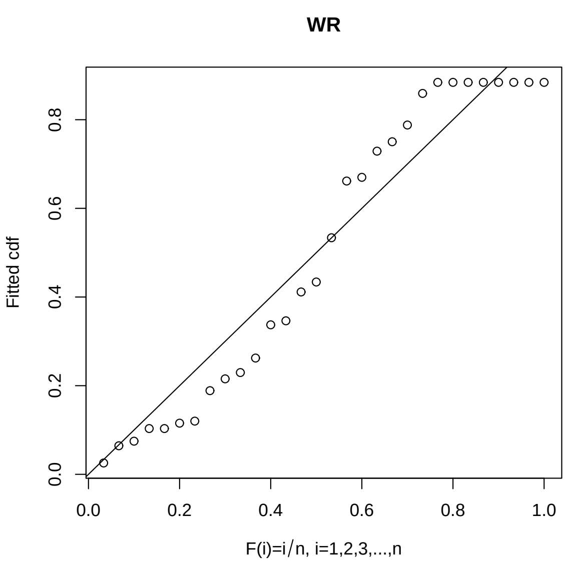

Fig. 7: P-P plots for fitted Weibull-Rayligh for data set.

Step 3.Set $j = 1$.
Step 4.Generate $U \sim Uniform(0,1)$.
Step 5.Update x^0 by using the Newton's formula such as
$$x^{\star} = x^0 - \left(\frac{F_{\Theta}(x)-U}{f_{\Theta}(x)}\right)\bigg|_{x=x^0}$$
Step 6.If $| x^0 - x^{\star} | \leq \varepsilon$, (very small, $\varepsilon > 0$ tolerance limit). Then, $x^{\star}$ will be the desired sample from $F(x)$.
Step 7.If $| x^0 - x^{\star} | > \varepsilon$, then, set $x^0 = x^{\star}$ and go to step 5.
Step 8.Repeat steps 4-7, for $j = 1,2,\ldots,n$ and obtained $x_1, x_2, \ldots, x_n$.

Using the above algorithm, we generated a sample of size 30 from WR distribution for arbitrary values of $\alpha = 0.1, \beta = 0.2$ and $\theta = 0.3$. The simulated sample is given by
0.01104553, 1.19916946, 1.72627166, 2.56539179 3.90112825, 6.59243391, 6.92991903, 7.03972818 7.14544040, 7.21187387, 7.41519680, 7.61834438 7.88554924, 8.00112815, 8.12042859, 8.25349064 8.91914020, 9.08167062, 9.13663588, 9.22711659 9.34932672, 9.52258078, 9.58550171, 9.74824898 9.83652106, 10.0859061, 10.4792932, 10.5071468 10.5493757, 11.4408300.

The maximum likelihood estimates with corresponding confidence intervals are calculated based on the simulated sample. The MLEs of (α, β, θ) are $(0.075,\ 0.206,\ 0.304)$ respectively. The asymptotic confidence intervals for (α, β, θ) are obtained as $(0.002 \sim 0.147)$, $(0 \sim 0.450)$, and $(0 \sim 0.679)$ respectively.

Empirical, fitted WR cdf and PP of the simulated data are given in (Fig. 9) and (Fig. 1)

9 Conclusion

Here, we propose a new model, the so-called the Weibull-Rayleigh distribution which extends the Weibull distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is

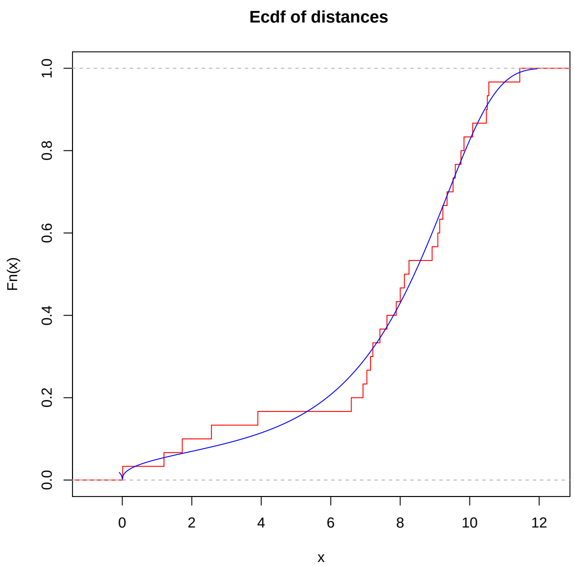

Fig. 9: Empirical, fitted WR cdf of the simulated data.

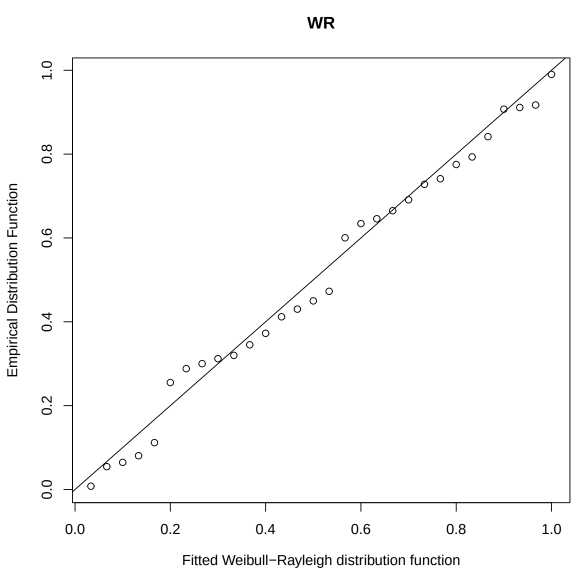

Fig. 10: PP of WR distribution for the simulated data.

because the generalized form provides larger flexibility in modelling real data. We derive expansions for the moments and for the moment generating function. The estimation of parameters is approached by the method of maximum likelihood and least squares estimator, also the information matrix is derived. We consider the likelihood ratio statistic to compare the model with its baseline model. An application of the Weibull-Rayleigh distribution to real data show that the new distribution can be used quite effectively to provide better fits than the Weibull distribution.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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