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Compound of Discrete Pareto and Kumaraswamy Distributions: An Advanced Discrete Model with Properties and Applications

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Abstract: In this article, we study a new discrete model which is obtained by compounding discrete Pareto distribution with Kumaraswamy distribution. We shall first study some basic distributional and moment properties of the new distribution. Structural properties of the distribution such as its unimodality, hazard rate behavior and index of dispersion are also discussed. Finally, real data sets are analyzed to investigate the suitability of the proposed distribution in modeling count.

Keywords: Weibull Discrete Pareto Distribution, Zero Inflated, Kumaraswamy Distribution, Compound Distribution.

1 Introduction

Statistical distributions play a prominent role in various fields like social sciences, medical sciences etc. Researchers obtainplethora of distribution for the sake of analyzing complex data from various fields. Lot of well known techniques are getting employed to serve the purpose of constructing new probability distributions. Some well known techniques like discretization [2,3,16], transmutation [5] and Compounding methodologies received utmost attention from researchers from past decade. Compound distribution arises when all or some parameters of a distribution known as parent distribution vary according to some probability distribution called the compounding distribution, for instance negative binomial distribution can be obtained from Poisson distribution when its parameter λ follows gamma distribution. If the parent distribution is discrete then resultant compound distribution will also be discrete and if the parent distribution is continuous then resultant compound distribution will also be continuous i,e. the support of the original (parent) distribution determines the support of compound distributions. In several research papers it has been found that compound distributions are very flexible and can be used efficiently to model different types of data sets. With this in mind many compound probability distributions have been constructed. Sankaran [1] introduced a compound of Poisson distribution with that of Lindley distribution for modeling count data. Zamani and Ismail [4] also constructed a new compound distribution by compounding negative binomial with one parameter Lindley distribution that provides good fit for count data where the probability at zero has a large value. Adil and Jan [15] constructed a new compound distribution as compound of size biased Geeta distribution with generalized beta distribution. Recently, Para and Jan [6] constructed compound of discrete inverse Weibull distribution with Minimax distribution as a new discrete model with applications in medical sciences.

In this paper we propose a new count data model by compounding two parameter discrete Pareto distribution with Kumaraswamy distribution, as there is a need to find more plausible discrete probability models or survival models in medical science and other fields, to fit to various discrete data sets. It is well known in general that a compound model is more flexible than the ordinary model and it is preferred by many data analysts in analyzing statistical data. Moreover, it presents beautiful mathematical exercises and broadened the scope of the concerned model being compounded.

2 Material and Methods

A discrete analogue of the continuous Pareto distribution

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was introduced by Krishna and Punder [6], and is defined by the probability mass function (pmf):

$$f_1(x;q) = q^{\log(1+x)} - q^{\log(2+x)}, \quad x = 0,1,2,...... \tag{1}$$

where $0<q<1$ is its parameter. The first and the second moments of the DP random variable X are given by

$$E(X) = \sum_{x=1}^{\infty} q^{\log(1+x)} \quad \text{for } x = 0,1,2,... \tag{2}$$

$$E(X^2) = 2\sum_{x=1}^{\infty} x q^{\log(1+x)} + E(X)$$

There are various types of life time models such as exponential, Pareto and Gamma that are used in reliability and life testing. Jones [7] studied two-parameter distribution on (0,1) which he has called the Kumaraswamy distribution, Kumaraswamy (α,β) ,where its two shape parameters α and β are positive. It has many of the same properties as the beta distribution

$$f(x;\gamma,\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}; \qquad 0<x<1; \alpha,\beta>0 \tag{3}$$

If we put $\alpha=\beta=1$, the equation (3) reduces to Uniform distribution.

Where $\alpha,\beta>0$ are shape parameters. The raw moments of Kumaraswamy distribution (KD) are given by

$$E(X^r) = \int_0^1 x^r f_2(X;\alpha,\beta)dx$$

$$E(X^r) = \frac{\beta\Gamma(1+\frac{1}{\alpha})\Gamma(\beta)}{\Gamma\left(1+\frac{1}{\alpha}+\beta\right)} \tag{4}$$

Kumaraswamy distribution is not very popular among statisticians because researchers have not analyzed and investigated it systematically in much detail. Kumaraswamy distribution is similar to the Beta distribution but unlike beta distribution it has a closed form of cumulative distribution function which makes it very simple to deal with. For more detailed properties one can see references [7, 8]

3 Definition of Proposed Model

If $X|q\sim$ DPD(q) where q is itself a random variable following Kumaraswamy distribution KD(α,β), then determining the distribution that results from marginalizing over q will be known as a compound of discrete Pareto distribution with that of Kumaraswamy distribution, which is denoted by DPKD (α,β). It may be noted that proposed model will be a discrete since the parent distribution DPD is discrete.

Theorem 3.1: The probability mass function of a compound of DPD (q) with KD (α,β) is given by

$$f_{DPKD}(X;\alpha,\beta) = \beta[B(\beta,\frac{\log(1+x)}{\alpha}+1) - B(\beta,\frac{\log(2+x)}{\alpha}+1)]$$

Where $x=0,1,2,......$ and $\alpha,\beta>0$

Proof: Using the definition (3), the pmf of a compound of DPD(q) with KD(α,β) can be obtained as

$$f_{DPKD}(X;\alpha,\beta) = \int_0^1 f_1(x|q) f_2(q)\, dq$$

$$f_{DPKD}(X;\alpha,\beta) = \alpha\beta\int_0^1 (q^{\log(1+x)} - q^{\log(2+x)}) q^{\alpha-1}(1-q^{\alpha})^{\beta-1}\, dq$$

$$f_{DPKD}(X;\alpha,\beta) = \beta\left[\frac{\Gamma(\beta)\Gamma\left(\frac{\log(1+x)}{\alpha}+1\right)}{\Gamma\left(\beta+\frac{\log(1+x)}{\alpha}+1\right)} - \frac{\Gamma(\beta)\Gamma\left(\frac{\log(2+x)}{\alpha}+1\right)}{\Gamma\left(\beta+\frac{\log(2+x)}{\alpha}+1\right)}\right]$$

$$f_{DPKD}(X;\alpha,\beta) = \beta[B(\beta,\frac{\log(1+x)}{\alpha}+1) - B(\beta,\frac{\log(2+x)}{\alpha}+1)] \tag{5}$$

Where $x=0,1,2,...$ and $\alpha,\beta>0$. From here a random variable X following a compound of DPD with KD will be symbolized by DPKD(α,β).

Fig.1(a) to fig.1(c) provides a pmf plot of the proposed model DPKD$(x;\alpha,\beta)$ for different values of parameters.The Cumulative distribution function of the DPKD (α,β) is given by $x=0,1,2,...$ and $(\alpha>0,\beta>0)$

Where, B (.) refers to the beta function defined by

$$B(a,b) = \frac{\Gamma a\ \Gamma b}{\Gamma(a+b)}$$

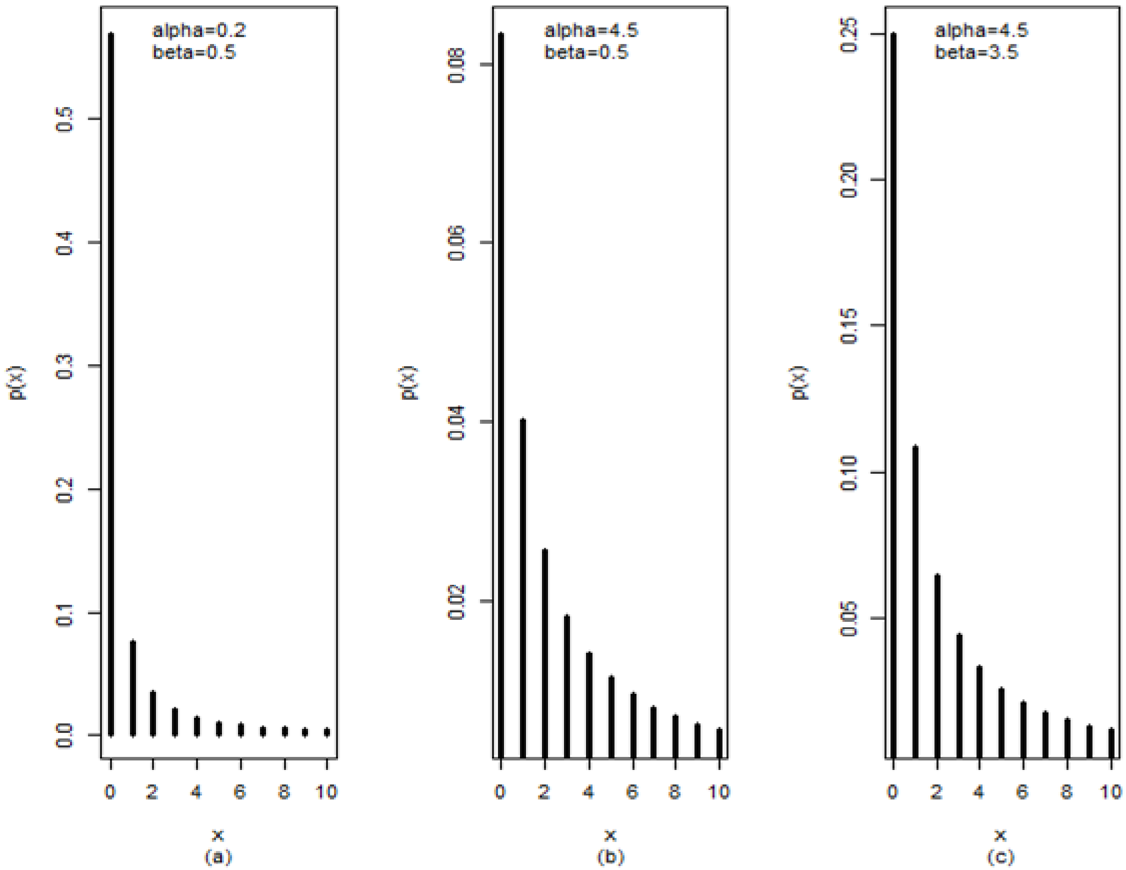

Fig 1. pmf plot of Discrete Pareto Kumaraswamy distribution

3.1 Random Data Generation from Discrete Pareto Kumaraswamy Distributions

For simulating random sample data of size n, $x_1, x_2,...., x_n$ of the discrete Pareto and Kumaraswamy random variable X with pmf

$$p(X = x_i) = p_i, \sum_{i=0}^{k} p_i = 1$$

and a cdf F(x), where k may be finite or infinite can be described as

Step1: Generate a random number u from uniform distribution U(0,1).

Step2: Generate random number x_i based on

$$if\ u \le p_0 = F(x_0)\ then\ X = x_0$$
$$if\ p_0 < u \le p_0 + p_1 = F(x_1)\ then\ X = x_1$$

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$$.if\ \sum_{j=0}^{k-1} p_j < u \le \sum_{j=0}^{k} p_j = F(x_k)\ then\ X = x_k$$

In order to generate n random numbers from Discrete ParetoKumaraswamy distribution, $x_1, x_2,...., x_n$, repeat step 1 to step 2 n times.

4 Nested Distributions

In this particular section we show that the proposed model can be nested to different models under specific parameter setting.

Proposition 4.1: If $X \sim DPKD(\alpha, \beta)$ then by setting $\alpha = \beta = 1$ we obtain a compound of DPD distribution with uniform distribution.

Proof: For $\alpha = \beta = 1$ in KD reduces to Uniform (0,1) distribution, therefore a compound DPD with uniform distribution is followed from (5) by simply putting $\alpha = \beta = 1$ in it.

$$f_{DPUD}(x) = B(1,\log(1+x)+1) - B(1,\log(2+x)+1)$$

For $x = 0,1,2,...,\gamma > 0$

Which is probability mass function of a compound of DPD with uniform distribution

5 Reliability Measures of Compound Discrete Pareto Kumaraswamy Distribution

If $X \sim DPMD(\alpha,\beta)$, then the various reliability measures of a random variable X are given by

5.1 Survival Function.

$$S(x) = \beta\, B\left(\beta, \frac{\log(1+x)}{\alpha}+1\right) \qquad (6)$$

Where $B\left(\beta, \frac{\log(1+x)}{\alpha}+1\right) = \frac{\Gamma(\beta)\,\Gamma\left(\frac{\log(1+x)}{\alpha}+1\right)}{\Gamma\left(\beta+\frac{\log(1+x)}{\alpha}+1\right)}$

5.2 Rate of Failure Function

$$r(x) = \frac{p(x)}{s(x)} = 1 - \frac{B(\beta, \frac{\log(2+x)}{\alpha}+1)}{B\left(\beta, \frac{\log(1+x)}{\alpha}+1\right)} \qquad (7)$$

5.3 Second Rate of Failure Function

$$h(x) = \log\left(\frac{s(x)}{s(x+1)}\right) = \log\left(\frac{B\left(\beta, \frac{\log(1+x)}{\alpha}+1\right)}{B\left(\beta, \frac{\log(2+x)}{\alpha}+1\right)}\right) \qquad (8)$$

$x = 0,1,2...$ And $\alpha > 0, \beta > 0, \gamma > 0$

Where, B (.) refers to the beta function defined by

$$B(a,b) = \frac{\Gamma a\ \Gamma b}{\Gamma(a+b)}$$

6 Statistical Properties of Pareto Kumaraswamy Distribution

6.1 Moments

The r th moment of the Compound discrete Pareto Kumaraswamy distribution is given as

$$E(X^r) = \mu_r' = \sum_{x=0}^{\infty} x^r p(x)$$

$$= \sum_{x=0}^{\infty} x^r \beta[B(\beta, \frac{\log(1+x)}{\alpha}+1) - B(\beta, \frac{\log(2+x)}{\alpha}+1)]$$

$$= \sum_{x=0}^{\infty} x^r \beta[\psi(1+x;\beta,\alpha) - \psi(2+x;\beta,\alpha)]$$

Where $\psi(x;\beta,\alpha) = B(\beta, \frac{\log x}{\alpha}+1)$

$$= \beta\left(0 + \psi(2;\beta,\alpha) + 2^r\psi(3;\beta,\alpha) + 3^r\psi(4;\beta,\alpha) + ... - \{\psi(3;\beta,\alpha) + 2^r\psi(4;\beta,\alpha) + 3^r\psi(5;\beta,\alpha) + ...\}\right)$$

$$= \beta\left(\psi(2;\beta,\alpha) + (2^r-1)\psi(3;\beta,\alpha) + (3^r-2^r)\psi(4;\beta,\alpha) + (4^r-3^r)\psi(5;\beta,\alpha) + ...\right)$$

$$= \beta\psi(2;\beta,\alpha) + \beta\sum_{x=2}^{\infty}(x^r-(x-1)^r)\psi(1+x;\beta,\alpha) \Rightarrow \mu_r' = \beta\sum_{x=1}^{\infty}(x^r-(x-1)^r)\psi(1+x;\beta,\alpha)$$

Substituting r= 1,2,3,4 we get first four moments

$$\mu_1' = \beta\sum_{x=1}^{\infty}\psi(1+x;\beta,\alpha)$$

$$\mu_2' = \beta\sum_{x=1}^{\infty}(2x-1)\ \psi(1+x;\beta,\alpha)$$

$$\mu_3' = \beta\sum_{x=1}^{\infty}(3x^2-3x+1)\ \psi(1+x;\beta,\alpha)$$

$$\mu_4' = \beta\sum_{x=1}^{\infty}(4x^3-6x^2+4x-1)\ \psi(1+x;\beta,\alpha)$$

6.2 Moment Generating of $DPMD(\alpha,\beta)$

The moment generating function of the Compound discrete ParetoKumaraswamy distribution is

$$M_x(t)=\sum_{x=0}^{\infty}e^{tx}p(x)$$

$$M_x(t)=\sum_{x=0}^{\infty}e^{tx}\beta[B(\beta,\frac{\log(1+x)}{\alpha}+1)-B(\beta,\frac{\log(2+x)}{\alpha}+1)]$$

$$M_x(t)=\sum_{x=0}^{\infty}e^{tx}\beta[\psi(1+x;\beta,\alpha)-\psi(2+x;\beta,\alpha)]$$

Where $\psi(x;\beta,\alpha)=B(\beta,\frac{\log x}{\alpha}+1)$

$$M_x(t)=\beta\left(\psi(1;\beta,\alpha)+e^{t}\psi(2;\beta,\alpha)+e^{2t}\psi(3;\beta,\alpha)+...-\{\psi(2;\beta,\alpha)+e^{t}\psi(3;\beta,\alpha)+e^{2t}\psi(4;\beta,\alpha)+......\}\right)$$

$$M_x(t)=\beta\left(\psi(1;\beta,\alpha)+(e^{t}-1)\psi(2;\beta,\alpha)+(e^{2t}-e^{t})\psi(3;\beta,\alpha)+(e^{3t}-e^{2t})\psi(4;\beta,\alpha)+...\right)$$

$$M_x(t)=1+\beta\sum_{x=1}^{\infty}(e^{xt}-e^{(x-1)t})\psi(1+x;\beta,\alpha)$$

Differentiating $M_x(t)$ r times with respect to t

$$M_x^{(r)}(t)=\beta\sum_{x=1}^{\infty}(x^{r}e^{xt}-(x-1)^{r}e^{(x-1)t})\psi(1+x;\beta,\alpha)$$

First four moments of the proposed model are given by

$$\mu_1^{'}=\beta\sum_{x=1}^{\infty}\psi(1+x;\beta,\alpha)$$

$$\mu_2^{'}=\beta\sum_{x=1}^{\infty}(2x-1)\psi(1+x;\beta,\alpha)$$

$$\mu_3^{'}=\beta\sum_{x=1}^{\infty}(3x^2-3x+1)\psi(1+x;\beta,\alpha)$$

$$\mu_4^{'}=\beta\sum_{x=1}^{\infty}(4x^3-6x^2+4x-1)\psi(1+x;\beta,\alpha)$$

7 Parameter Estimation

In this section the estimation of parameters of $DPKD(x;\gamma,\alpha,\beta)$ model will be discussed method of moments and maximum likelihood estimation

7.1 Moments Method of Estimation

In order estimate three unknown parameters of $DPKD(x;\alpha,\beta)$ model by the method of moments, we need to equate first three sample moments with their corresponding population moments.

$m_1=\gamma_1$; $m_2=\gamma_2$

Where γ_i is the ith sample moment and m_i is the ith corresponding population moment and the solution for $\hat{\alpha}$ and $\hat{\beta}$ may be obtained by solving above equations simultaneously through numerical methods.

7.2 Maximum Likelihood Method of Estimation

The estimation of parameters of $DPKD(x;\alpha,\beta)$ model via maximum likelihood estimation method requires the log likelihood function of $DPKD(x;\alpha,\beta)$

$$£(X;\gamma,\alpha,\beta)=\log\ L(X;\gamma,\alpha,\beta)=n\log\ \beta+\sum_{i=1}^{n}\log\left(B\left(\beta,\frac{\log(1+x)}{\alpha}+1\right)-B\left(\beta,\frac{\log(2+x)}{\alpha}+1\right)\right) \quad (9)$$

The maximum likelihood estimate of $\Theta=\left(\hat{\alpha},\hat{\beta}\right)^T$ can be obtained by differentiating (9) with respect unknown parameters α and β respectively and then equating them to zero.

$$\frac{\partial}{\partial\beta}£(X;\alpha,\beta)=\frac{n}{\beta}+\sum_{i=1}^{n}\left(\frac{\frac{\partial}{\partial\beta}\left(B\left(\beta,\frac{\log(1+x)}{\alpha}+1\right)-B\left(\beta,\frac{\log(2+x)}{\alpha}+1\right)\right)}{B\left(\beta,\frac{\log(1+x)}{\alpha}+1\right)-B\left(\beta,\frac{\log(2+x)}{\alpha}+1\right)}\right) \quad (10)$$

$$\frac{\partial}{\partial\alpha}£(X;\gamma,\alpha,\beta)=\sum_{i=1}^{n}\left(\frac{\frac{\partial}{\partial\beta}\left(B\left(\beta,\frac{\log(1+x)}{\alpha}+1\right)-B\left(\beta,\frac{\log(2+x)}{\alpha}+1\right)\right)}{B\left(\beta,\frac{\log(1+x)}{\alpha}+1\right)-B\left(\beta,\frac{\log(2+x)}{\alpha}+1\right)}\right) \quad (11)$$

These two derivative equations cannot be solved analytically, therefore $\hat{\alpha}$ and $\hat{\beta}$ will be obtained by maximizing the log likelihood function numerically using Newton-Raphson method which is a powerful technique for solving equations iteratively and numerically.

8 Application of DPKD (Discrete Pareto Kumaraswamy Distribution)

In this section, we present a real data set to examine the fit of the proposed model. MLE based on the likelihood Eqs. (10) and (11) was used to obtain the parameter estimates of the proposed distribution, although it is also possible to perform a direct search of the maximum likelihood function to obtain the maximum likelihood estimators. This can be done using appropriate software such as R Studio statistical software. In this section an attempt has been made to fit to data relating to automobile claims as given in table 1 (Automobile claims frequencies data in Willmot [10]), using discrete Pareto Kumaraswamy distribution (DPKD) in comparison with some compound discrete models like, Poisson Akasha distribution (PAD) [9], Poisson Lindley distribution (PLD) [12] , Poisson Sujatha distribution (PSD) [11] and other classical discrete models.

The p-value of Pearson's Chi-square statistic is 0.375 for discrete Pareto Kumaraswamy distribution and <0.01for Poisson, discrete Rayleigh, Poisson Lindley, Poisson Akasha, and Poisson Sujatha distributions, respectively (see Table 3).

This reveals that Poisson,discrete Rayleigh, PLD, PAD and PSD distributions are not good fit at all, whereas discrete Pareto Kumaraswamy model being the significant model for automobile claim data. The null hypothesis that data come from discrete inverse Pareto Kumaraswamy distribution is accepted.

We have compared discrete Pareto Kumaraswamy distribution with Poisson, discrete Rayleigh, Poisson Lindley, Poisson Akasha, and Poisson Sujatha distributions using the Akaike information criterion (AIC), given by Akaike [13] and the Bayesian information criterion (BIC), given by Schwarz [14] . Generic function calculating Akaike's 'An Information Criterion' for one or several fitted model objects for which a log-likelihood value can be obtained, according to the formula -2*log-likelihood + k*npar, where npar represents the number of parameters in the fitted model, and k = 2 for the usual AIC, or k = log(n) (n being the number of observations) for the so called BIC or SBC (Schwarz's Bayesian criterion). From table 4, Comparing the fits using AIC and BIC criterion, it is obvious that AIC and BIC criterion favors discrete Pareto Kumaraswamy distribution in comparison with the Poisson, discrete Rayleigh, Poisson Lindley, Poisson Akasha, and Poisson Sujatha distributions, in the case of Counts of Automobile claim data.

Table 1. Automobile claim data studied by Willmot [7]

Count	0	1	2	3	4	5
Observed	3719	212	38	7	3	1

The ML estimates provided by the fitdistr procedure in R studio are given in the table 2.

Table 2. Estimated parameters by ML method for fitted distributions for Counts of Automobile claim data studied by Willmot [10]

Distribution	parameter Estimates	Model function
Discrete Pareto Kumaraswamy	$\beta = 400$ $\alpha = 1.57$	$\beta[B(\beta, \frac{\log(1+x)}{\alpha}+1) - B(\beta, \frac{\log(2+x)}{\alpha}+1)]$ $x = 0,1,2,...$ for $\beta > 0, \alpha > 0$
Poisson	$\lambda = 0.08$,	$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $\lambda > 0;$ $x = 0,1,2,.....$
Poisson Akasha	$\theta = 12.48$,	$p(x) = \frac{\theta^3(x^2+3x+(\theta^2+2\theta+2))}{(\theta^2+2)(\theta+1)^{x+3}}$ $x = 0,1,2,....$ $\theta > 0$
Poisson Lindley	$\theta = 13.08$,	$p(x) = \frac{\theta^2(x+\theta+2)}{(\theta+1)^{x+3}}$ $x = 0,1,2,....$ $\theta > 0$
Poisson Sujatha	$\theta = 13.30$,	$p(x) = \frac{\theta^3}{\theta^2+\theta+2} \frac{x^2+(\theta+4)x+(\theta^2+3\theta+4)}{(\theta+1)^{x+3}}$ $x = 0,1,2,....$ $\theta > 0$
Discrete Rayleigh	$q = 0.12$,	$p(x) = q^{x^2} - q^{(x+1)^2}$ $0 < q < 1; x = 0,1,2,.....$

Table 3. Table for goodness of fit for Counts of Automobile claim data (Willmot [7]).

Observed	DPKD	Poisson	DRayleigh	PLD	PAD	PSD
3719	3725.62	3667.00	3509.20	3678.71	3678.13	3678.71
212	202.06	300.36	470.03	278.44	278.75	278.44
38	35.09	12.30	0.78	21.11	21.34	21.11
7	9.91	0.34	0.00	1.60	1.65	1.60
3	3.67	0.01	0.00	0.12	0.13	0.12
1	3.66	0.00	0.00	0.01	0.01	0.01
P-values	0.375	<0.01	<0.01	0.0003	0.000001	0.000071

Table 4. AIC, BIC and log likelihood values for fitted distributions

Criterion	DPKD	Poisson	D Rayleigh	PLD	PAD	PSD
Log likelihood Value	-1128.03	-1194.9	-1535.85	-1154.92	-1154.13	-1154.61
AIC	2260.058	2391.802	3073.705	2311.842	2310.258	2311.217
BIC	2272.636	2398.091	3079.994	2318.131	2316.547	2317.506

4 Conclusions

In this paper, a new model is proposed by compounding discrete Pareto distribution (DPD) with Kumaraswamy distribution (KD) and it has been shown that proposed model can be nested to different compound distributions. Some important probabilistic properties and the problem of estimation of its parameters are studied. The proposed model is well competitive of some well known compound and classical discrete distributions.

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