Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/040103

Hermite-Hadamard Type Inequalities for *n*-Time Differentiable and GA-Convex Functions with Applications to Means

Shu-Hong wang[1,*] *and Xiao-Ting Shi*[2]

[1] College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China
[2] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China

Received: 2 Feb. 2015, Revised: 3 Oct. 2015, Accepted: 4 Nov. 2015
Published online: 1 Jan. 2016

Abstract: In the paper, by Hölder's integral inequality, the authors establish some Hermite-Hadamard type integral inequalities for n-time differentiable and GA-convex functions and apply these inequalities to construct several inequalities for special means.

Keywords: Hermite-Hadamard type inequality, GA-convex function, special Mean

1 Introduction

The following definition is well known in literature.

Let I be an interval on $\mathbb{R}=(-\infty,\infty)$. A function $f: I\to\mathbb{R}$ is said to be convex if

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y) \tag{1}$$

holds for $x,y\in I$ and $\lambda\in[0,1]$. If the inequality (1) reverses, then f is said to be concave on I.

One of the most famous inequalities for convex functions is Hermite-Hadamard's inequality.

Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function on an interval I of real numbers and $a,b\in I$ with $a<b$. Then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\le\frac{f(a)+f(b)}{2}. \tag{2}$$

If f is concave on I, then the inequality (2) is reversed.

On convex functions, there have been the following results.

Theorem 1.1.[[3]] Let $f: I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$. If $|f'(x)|$ is convex on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le\frac{(b-a)\big[|f'(a)|+|f'(b)|\big]}{8}. \tag{3}$$

Theorem 1.2.[[3]] Let $f: I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$. If $|f'(x)|^q$ for $q\ge 1$ is a convex function on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le\frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q} \tag{4}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le\frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}. \tag{5}$$

Theorem 1.3.[[4]] Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be differentiable on I° and $a,b\in I$ with $a<b$. If $|f'(x)|^{p/(p-1)}$ for $p>1$ is a

* Corresponding author e-mail: shuhong7682@163.com

convex function on $[a,b]$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\times\left\{\left[|f'(a)|^{p/(p-1)}+3|f'(b)|^{p/(p-1)}\right]^{1-1/p}+\left[3|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}\right]^{1-1/p}\right\}. \quad (6)$$

The concepts of geometrically convex function and GA-convex function were introduced as follows.

Definition 1.1. The function $f: I\subset\mathbb{R}_+=(0,\infty)\to\mathbb{R}_+$ is said to be geometrically convex on I if

$$f(x^\lambda y^{1-\lambda})\le[f(x)]^\lambda[f(y)]^{1-\lambda} \quad (7)$$

holds for $x,y\in I$ and $\lambda\in[0,1]$.

Definition 1.2.[[5]] The function $f: I\subset\mathbb{R}_+\to\mathbb{R}$ is said to be GA-convex on I if

$$f(x^\lambda y^{1-\lambda})\le\lambda f(x)+(1-\lambda)f(y) \quad (8)$$

holds for $x,y\in I$ and $\lambda\in[0,1]$.

Hermite-Hadamard type inequalities for geometrically convex functions and GA-convex functions were obtained as follows.

Theorem 1.4.[[18]] Let $f: I\subset\mathbb{R}_+\to\mathbb{R}_+$ be a differentiable function on I° and $a,b\in I^\circ$ with $a<b$. If $|f'(x)|$ is geometrically convex on $[a,b]$, then

$$\left|\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\mathrm{d}x-f(\sqrt{ab})\right|\le\frac{\ln b-\ln a}{4}\left\{L\left(\left[a|f'(a)|\right]^{1/2},\left[b|f'(b)|\right]^{1/2}\right)\right\}^2, \quad (9)$$

where $L(a,b)$ is the logarithmic mean defined by

$$L(a,b)=\begin{cases}\frac{b-a}{\ln b-\ln a}, & a\ne b,\\ a, & a=b.\end{cases} \quad (10)$$

Theorem 1.5.[[19]] Let $f: I\subset\mathbb{R}_+\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I$ with $a<b$, and $f'\in L([a,b])$. If $|f'(x)|^q$ is GA-convex on $[a,b]$ for $q\ge1$, then

$$\left|[bf(b)-af(a)]-\int_a^b f(x)\mathrm{d}x\right|\le\frac{[(b-a)A(a,b)]^{1-1/q}}{2^{1/q}}\left\{\left[L(a^2,b^2)-a^2\right]|f'(a)|^q+\left[b^2-L(a^2,b^2)\right]|f'(b)|^q\right\}^{1/q}, \quad (11)$$

where $L(u,v)$ is the logarithmic mean.

In recent years, some other kinds of Hermite -Hadamard type inequalities were generated. For more systematic information, please refer to papers and monographs [2], [6], [7], [13], [14], [15], [16], [17] and related references therein.

In what follows, we need some notions of means. For positive numbers $a>0$ and $b>0$, the quantities

$$A(a,b)=\frac{a+b}{2} \quad (12)$$

and

$$L_p(a,b)=\begin{cases}\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & a\ne b \text{ and } p\ne0,-1,\\ L(a,b), & p=-1,\\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & a\ne b\text{ and } p=0\end{cases} \quad (13)$$

are called the arithmetic mean and the generalized logarithmic mean of order $p\in\mathbb{R}$, respectively.

For more information on means, please refer to [1], [8], [9], [10] and a number of references therein.

In this paper, integral inequalities of Hermite -Hadamard type related to GA-convex functions are obtained and applied to means.

2 A lemma

In order to obtain our main results, we need the following lemma.

Lemma 2.1. For $n\in\mathbb{N}$ and $n\ge1$, let $f: I\subset\mathbb{R}_+\to\mathbb{R}$ be an n-time differentiable function on I° and $a,b\in I$ with $a<b$. If $f^{(n)}\in L([a,b])$, then

$$\sum_{k=1}^n\frac{(-1)^{k-1}}{k!}\left[b^kf^{(k-1)}(b)-a^kf^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x=\frac{(-1)^{n-1}(\ln b-\ln a)}{n!}\times\int_0^1 a^{(n+1)t}b^{(n+1)(1-t)}f^{(n)}(a^tb^{1-t})\mathrm{d}t. \quad (14)$$

Proof. When $n=1$, integrating by part and letting $x=a^tb^{1-t}$ for $0\le t\le1$ lead to

$$(\ln b-\ln a)\int_0^1 a^{2t}b^{2(1-t)}f'(a^tb^{1-t})\mathrm{d}t=\int_a^b xf'(x)\mathrm{d}x=xf(x)\big|_a^b-\int_a^b f(x)\mathrm{d}x=bf(b)-af(a)-\int_a^b f(x)\mathrm{d}x.$$

Hence, the identity (14) holds for $n=1$.

When $n=m-1$ and $m\ge2$, suppose that the identity (14) is valid.

When $n=m$, by the inductive hypothesis, integrating by part and letting $x=a^tb^{1-t}$ for $0\le t\le 1$ yield

$$\begin{aligned}
&\frac{(-1)^{m-1}(\ln b-\ln a)}{m!}\\
&\times\int_0^1 a^{(m+1)t}b^{(m+1)(1-t)}f^{(m)}(a^tb^{1-t})\mathrm{d}\,t\\
=&\frac{(-1)^{m-1}}{m!}\int_a^b x^m f^{(m)}(x)\mathrm{d}x\\
=&\frac{(-1)^{m-1}}{m!}\left[b^m f^{(m-1)}(b)-a^m f^{(m-1)}(a)\right]\\
&-\frac{(-1)^{m-1}}{(m-1)!}\int_a^b x^{m-1}f^{(m-1)}(x)\mathrm{d}x\\
=&\sum_{k=1}^{m}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x.
\end{aligned}$$

Therefore, when $n=m$, the identity (14) holds. By induction, the proof of Lemma 2.1. is complete.

Remark 2.1. Under the conditions of Lemma 2.1, taking $n=1$, we get

$$\begin{aligned}
&bf(b)-af(a)-\int_a^b f(x)\mathrm{d}x\\
&=(\ln b-\ln a)\int_0^1 a^{2t}b^{2(1-t)}f'(a^tb^{1-t})\mathrm{d}t,
\end{aligned}$$

which may be found in [19].

3 Hermite-Hadamard type inequalities for *n*-time differentiable and GA-convex functions

Now we start out to establish some new Hermite -Hadamard type inequalities for n-time differentiable and GA-convex functions.

Theorem 3.1. For $n\in\mathbb{N}$, suppose that $f:I\subset\mathbb{R}_+\to\mathbb{R}$ is an n-time differentiable function on I° and $f^{(n)}\in L([a,b])$ for $a,b\in I$ with $a<b$. If $\left|f^{(n)}\right|^q$ is a GA-convex function on $[a,b]$ for $q\ge 1$, then

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|\\
\le&\frac{(\ln b-\ln a)^{1-1/q}}{n!(n+1)^{1/q}}\left[L\left(a^{n+1},b^{n+1}\right)\right]^{1-1/q}\\
&\times\left\{\left[L\left(a^{n+1},b^{n+1}\right)-a^{n+1}\right]\left|f^{(n)}(a)\right|^q\right.\\
&\left.+\left[b^{n+1}-L\left(a^{n+1},b^{n+1}\right)\right]\left|f^{(n)}(b)\right|^q\right\}^{1/q}, \qquad (15)
\end{aligned}$$

where $L(u,v)$ is the logarithmic mean.

Proof. By GA-convexity of $\left|f^{(n)}\right|^q$, Lemma 2.1, and Hölder's inequality, one has

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|\\
\le&\frac{\ln b-\ln a}{n!}\int_0^1 a^{(n+1)t}b^{(n+1)(1-t)}\left|f^{(n)}\left(a^tb^{1-t}\right)\right|\mathrm{d}t\\
\le&\frac{\ln b-\ln a}{n!}\left[\int_0^1 a^{(n+1)t}b^{(n+1)(1-t)}\mathrm{d}t\right]^{1-1/q}\\
&\times\left\{\int_0^1 a^{(n+1)t}b^{(n+1)(1-t)}\left[t\left|f^{(n)}(a)\right|^q\right.\right.\\
&\left.\left.+(1-t)\left|f^{(n)}(b)\right|^q\right]\mathrm{d}t\right\}^{1/q}\\
=&\frac{(\ln b-\ln a)^{1-1/q}}{n!(n+1)^{1/q}}\left[L\left(a^{n+1},b^{n+1}\right)\right]^{1-1/q}\\
&\times\left\{\left[L\left(a^{n+1},b^{n+1}\right)-a^{n+1}\right]\left|f^{(n)}(a)\right|^q\right.\\
&\left.+\left[b^{n+1}-L\left(a^{n+1},b^{n+1}\right)\right]\left|f^{(n)}(b)\right|^q\right\}^{1/q}.
\end{aligned}$$

Theorem 3.1 is thus proved.

Corollary 3.1.1. Under the assumptions of Theorem 3.1, if $q=1$, we have

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|\\
\le&\frac{1}{(n+1)!}\left\{\left[L\left(a^{n+1},b^{n+1}\right)-a^{n+1}\right]\left|f^{(n)}(a)\right|\right.\\
&\left.+\left[b^{n+1}-L\left(a^{n+1},b^{n+1}\right)\right]\left|f^{(n)}(b)\right|\right\}, \qquad (16)
\end{aligned}$$

where $L(u,v)$ is the logarithmic mean.

Theorem 3.2. For $n\in\mathbb{N}$, suppose that $f:I\subset\mathbb{R}_+\to\mathbb{R}$ is an n-time differentiable function on I° and $f^{(n)}\in L([a,b])$ for $a,b\in I$ with $a<b$. If $\left|f^{(n)}\right|^q$ is a GA-convex function on $[a,b]$ for $q>1$, then

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|\\
\le&\frac{\ln b-\ln a}{n!}\left[L\left(a^{\frac{q(n+1)}{q-1}},b^{\frac{q(n+1)}{q-1}}\right)\right]^{1-1/q}\\
&\times\left[\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right]^{1/q}, \qquad (17)
\end{aligned}$$

where $L(u,v)$ is the logarithmic mean.

Proof. Since $\left|f^{(n)}\right|^q$ is a GA-convex function on $[a,b]$, from Lemma 2,1 and Hölder's inequality, we deduce that

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|\\
&\le \frac{\ln b-\ln a}{n!}\int_0^1 a^{(n+1)t}b^{(n+1)(1-t)}\left|f^{(n)}\left(a^t b^{1-t}\right)\right|\mathrm{d}t\\
&\le \frac{\ln b-\ln a}{n!}\left[\int_0^1 a^{q(n+1)t/(q-1)}b^{q(n+1)(1-t)/(q-1)}\mathrm{d}t\right]^{1-1/q}\\
&\quad\times\left\{\int_0^1\left[t\left|f^{(n)}(a)\right|^q+(1-t)\left|f^{(n)}(b)\right|^q\right]\mathrm{d}t\right\}^{1/q}\\
&=\frac{\ln b-\ln a}{n!}\left[L\left(a^{\frac{q(n+1)}{q-1}},b^{\frac{q(n+1)}{q-1}}\right)\right]^{1-1/q}\\
&\quad\times\left[\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right]^{1/q}.
\end{aligned}$$

Theorem 3.2 is thus proved.

Theorem 3.3. For $n\in\mathbb{N}$, suppose that $f:I\subset\mathbb{R}_+\to\mathbb{R}$ is an n-time differentiable function on I° and $f^{(n)}\in L([a,b])$ for $a,b\in I$ with $a<b$. If $\left|f^{(n)}\right|^q$ is a GA-convex function on $[a,b]$ for $q\ge 1$, then

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|\le\\
&\frac{(\ln b-\ln a)^{1-1/q}}{n!\left[q(n+1)\right]^{1/q}}\left\{\left[L\left(a^{q(n+1)},b^{q(n+1)}\right)-a^{q(n+1)}\right]\left|f^{(n)}(a)\right|^q\right.\\
&\left.+\left[b^{q(n+1)}-L\left(a^{q(n+1)},b^{q(n+1)}\right)\right]\left|f^{(n)}(b)\right|^q\right\}^{1/q},
\end{aligned}\tag{18}$$

where $L(u,v)$ is the logarithmic mean.

Proof. Using GA-convexity of $\left|f^{(n)}\right|^q$, Lemma 2.1, and Hölder's inequality turns out that

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|\\
&\le \frac{\ln b-\ln a}{n!}\int_0^1 a^{(n+1)t}b^{(n+1)(1-t)}\left|f^{(n)}\left(a^t b^{1-t}\right)\right|\mathrm{d}t\\
&\le \frac{\ln b-\ln a}{n!}\left(\int_0^1 1\mathrm{d}t\right)^{1-1/q}\left\{\int_0^1 a^{q(n+1)t}b^{q(n+1)(1-t)}\right.\\
&\quad\left.\times\left[t\left|f^{(n)}(a)\right|^q+(1-t)\left|f^{(n)}(b)\right|^q\right]\mathrm{d}t\right\}^{1/q}\\
&=\frac{(\ln b-\ln a)^{1-1/q}}{n!\left[q(n+1)\right]^{1/q}}\\
&\quad\times\left\{\left[L\left(a^{q(n+1)},b^{q(n+1)}\right)-a^{q(n+1)}\right]\left|f^{(n)}(a)\right|^q\right.\\
&\quad\left.+\left[b^{q(n+1)}-L\left(a^{q(n+1)},b^{q(n+1)}\right)\right]\left|f^{(n)}(b)\right|^q\right\}^{1/q},
\end{aligned}$$

which completes the proof of Theorem 3.3.

Corollary 3.3.1. Under the assumptions of Theorem 3.3, if $q=1$, we have

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|\\
&\le\frac{1}{(n+1)!}\left\{\left[L\left(a^{n+1},b^{n+1}\right)-a^{n+1}\right]\left|f^{(n)}(a)\right|\right.\\
&\quad\left.+\left[b^{n+1}-L\left(a^{n+1},b^{n+1}\right)\right]\left|f^{(n)}(b)\right|\right\},
\end{aligned}\tag{19}$$

where $L(u,v)$ is the logarithmic mean.

Theorem 3.4. For $n\in\mathbb{N}$, suppose that $f:I\subset\mathbb{R}_+\to\mathbb{R}$ is an n-time differentiable function on I° and $f^{(n)}\in L([a,b])$ for $a,b\in I$ with $a<b$. If $\left|f^{(n)}\right|^q$ is a GA-convex function on $[a,b]$ for $q>1$, then for $0\le m,r\le(n+1)q$,

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|\\
&\le\frac{\ln b-\ln a}{n!\left(m\ln a-r\ln b\right)^{1/q}}\left[L\left(a^{\frac{q(n+1)-m}{q-1}},b^{\frac{q(n+1)-r}{q-1}}\right)\right]^{1-1/q}\\
&\quad\times\left\{\left[a^m-L\left(a^m,b^r\right)\right]\left|f^{(n)}(a)\right|^q\right.\\
&\quad\left.+\left[L\left(a^m,b^r\right)-b^r\right]\left|f^{(n)}(b)\right|^q\right\}^{1/q},
\end{aligned}\tag{20}$$

where $L(u,v)$ is the logarithmic mean.

Proof. From the GA-convexity of $\left|f^{(n)}\right|^q$, Lemma 2.1, and Hölder's inequality, we write

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|\\
&\le \frac{\ln b-\ln a}{n!}\int_0^1 a^{(n+1)t}b^{(n+1)(1-t)}\left|f^{(n)}\left(a^t b^{1-t}\right)\right|\mathrm{d}t\\
&\le \frac{\ln b-\ln a}{n!}\left[\int_0^1 a^{[q(n+1)-m]t/(q-1)}\right.\\
&\quad\left.\times b^{[q(n+1)-r](1-t)/(q-1)}\mathrm{d}t\right]^{1-1/q}\left\{\int_0^1 a^{mt}b^{r(1-t)}\right.\\
&\quad\left.\times\left[t\left|f^{(n)}(a)\right|^q+(1-t)\left|f^{(n)}(b)\right|^q\right]\mathrm{d}t\right\}^{1/q}\\
&=\frac{\ln b-\ln a}{n!\left(m\ln a-r\ln b\right)^{1/q}}\left[L\left(a^{\frac{q(n+1)-m}{q-1}},b^{\frac{q(n+1)-r}{q-1}}\right)\right]^{1-1/q}\\
&\quad\times\left\{\left[a^m-L\left(a^m,b^r\right)\right]\left|f^{(n)}(a)\right|^q\right.\\
&\quad\left.+\left[L\left(a^m,b^r\right)-b^r\right]\left|f^{(n)}(b)\right|^q\right\}^{1/q}.
\end{aligned}$$

The proof of Theorem 3.4 is established.

Corollary 3.4.1. Under the assumptions of Theorem 3.4,
1. if $m=0$ and $r=q(n+1)$,

$$\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|$$
$$\le \frac{\ln b-\ln a}{n!\left[q(n+1)\ln b\right]^{1/q}}\left[L\left(a^{\frac{q(n+1)}{q-1}},1\right)\right]^{1-1/q}$$
$$\times\left\{\left[L\left(1,b^{q(n+1)}\right)-1\right]\left|f^{(n)}(a)\right|^q+\left[b^{q(n+1)}-L\left(1,b^{q(n+1)}\right)\right]\left|f^{(n)}(b)\right|^q\right\}^{1/q}; \quad (21)$$

2. if $m=n+1$ and $r=q(n+1)$,

$$\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|$$
$$\le \frac{\ln b-\ln a}{n!\left[(n+1)(\ln a-q\ln b)\right]^{1/q}}\left[L\left(a^{n+1},1\right)\right]^{1-1/q}$$
$$\times\left\{\left[a^{n+1}-L\left(a^{n+1},b^{q(n+1)}\right)\right]\left|f^{(n)}(a)\right|^q+\left[L\left(a^{n+1},b^{q(n+1)}\right)-b^{q(n+1)}\right]\left|f^{(n)}(b)\right|^q\right\}^{1/q}; \quad (22)$$

3. if $m=q(n+1)$ and $r=0$,

$$\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|$$
$$\le \frac{\ln b-\ln a}{n!\left[q(n+1)\ln a\right]^{1/q}}\left[L\left(1,b^{\frac{q(n+1)}{q-1}}\right)\right]^{1-1/q}$$
$$\times\left\{\left[a^{q(n+1)}-L\left(a^{q(n+1)},1\right)\right]\left|f^{(n)}(a)\right|^q+\left[L\left(a^{q(n+1)},1\right)-1\right]\left|f^{(n)}(b)\right|^q\right\}^{1/q}; \quad (23)$$

4. if $m=q(n+1)$ and $r=n+1$,

$$\left|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\left[b^k f^{(k-1)}(b)-a^k f^{(k-1)}(a)\right]-\int_a^b f(x)\mathrm{d}x\right|$$
$$\le \frac{\ln b-\ln a}{n!\left[(n+1)(q\ln a-\ln b)\right]^{1/q}}\left[L\left(1,b^{n+1}\right)\right]^{1-1/q}$$
$$\times\left\{\left[a^{q(n+1)}-L\left(a^{q(n+1)},b^{n+1}\right)\right]\left|f^{(n)}(a)\right|^q+\left[L\left(a^{q(n+1)},b^{n+1}\right)-b^{n+1}\right]\left|f^{(n)}(b)\right|^q\right\}^{1/q}, \quad (24)$$

where $L(u,v)$ is the logarithmic mean.

4 Applications in special means

Now using the results of Section 3, we get some inequalities for special means of real numbers.

For $n\in\mathbb{N}$, let $f(x)=\frac{\Gamma(s+1)x^{s+n}}{\Gamma(s+n+1)}$, $x\in\mathbb{R}_+$, $s>0$, then $\left|f^{(n)}(x)\right|^q=x^{sq}$ is GA-convex function on $\mathbb{R}_+$ for $q\ge1$. Taking $f(x)=\frac{\Gamma(s+1)x^{s+n}}{\Gamma(s+n+1)}$ in Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4, respectively, the following results are obtained.

Theorem 4.1. For $n\in\mathbb{N}$, if $0<a<b$, $s>0$ and $q\ge1$, then

$$\left|\frac{\Gamma(s+1)}{\Gamma(s+n)}+(s+n+1)\Gamma(s+1)\sum_{k=2}^{n}\frac{(-1)^{k-1}}{k!\Gamma(s+n+2-k)}\right|\times\left[L_{s+n}(a,b)\right]^{s+n}$$
$$\le\frac{\ln b-\ln a}{n!(b-a)(n+1)^{1/q}}\left[L\left(a^{n+1},b^{n+1}\right)\right]^{1-1/q}$$
$$\times\left[(sq+n+1)L\left(a^{sq+n+1},b^{sq+n+1}\right)-sqL\left(a^{n+1},b^{n+1}\right)L\left(a^{sq},b^{sq}\right)\right]^{1/q}, \quad (25)$$

where $L(u,v)$ and $L_p(u,v)$ are the logarithmic mean and the generalized logarithmic mean of order $p\in\mathbb{R}$, respectively.

Corollary 4.1.1. Under the assumptions of Theorem 4.1, if $q=1$,

$$\left|\frac{\Gamma(s+1)}{\Gamma(s+n)}+(s+n+1)\Gamma(s+1)\sum_{k=2}^{n}\frac{(-1)^{k-1}}{k!\Gamma(s+n+2-k)}\right|\times\left[L_{s+n}(a,b)\right]^{s+n}$$
$$\le\frac{\ln b-\ln a}{(n+1)!(b-a)}\left[(s+n+1)L\left(a^{s+n+1},b^{s+n+1}\right)-sL\left(a^{n+1},b^{n+1}\right)L\left(a^{s},b^{s}\right)\right]. \quad (26)$$

If $q=1$ and $n=1$,

$$\left[L_{s+1}(a,b)\right]^{s+1}\le\frac{\ln b-\ln a}{2(b-a)}\left[(s+2)L\left(a^{s+2},b^{s+2}\right)-sL\left(a^2,b^2\right)L\left(a^s,b^s\right)\right], \quad (27)$$

where $L(u,v)$ and $L_p(u,v)$ are the logarithmic mean and the generalized logarithmic mean of order $p\in\mathbb{R}$, respectively.

Theorem 4.2. For $n\in\mathbb{N}$, if $0<a<b$, $s>0$ and $q>1$, then

$$\left|\frac{\Gamma(s+1)}{\Gamma(s+n)}+(s+n+1)\Gamma(s+1)\sum_{k=2}^{n}\frac{(-1)^{k-1}}{k!\Gamma(s+n+2-k)}\right| \times\left[L_{s+n}(a,b)\right]^{s+n} \le\frac{\ln b-\ln a}{n!(b-a)}\left[L\left(a^{\frac{q(n+1)}{q-1}},b^{\frac{q(n+1)}{q-1}}\right)\right]^{1-1/q}\left[A\left(a^{sq},b^{sq}\right)\right]^{1/q}, \tag{28}$$

where $A(u,v)$, $L(u,v)$ and $L_p(u,v)$ are the arithmetic mean, the logarithmic mean and the generalized logarithmic mean of order $p\in\mathbb{R}$, respectively.

Corollary 4.2.1. Under the assumptions of Theorem 4.2, if $n=1$,

$$\left[L_{s+1}(a,b)\right]^{s+1}\le\frac{\ln b-\ln a}{b-a}\left[L\left(a^{\frac{2q}{q-1}},b^{\frac{2q}{q-1}}\right)\right]^{1-1/q} \times\left[A\left(a^{sq},b^{sq}\right)\right]^{1/q}, \tag{29}$$

where $A(u,v)$, $L(u,v)$ and $L_p(u,v)$ are the arithmetic mean, the logarithmic mean and the generalized logarithmic mean of order $p\in\mathbb{R}$, respectively.

Theorem 4.3. For $n\in\mathbb{N}$, if $0<a<b$, $s>0$ and $q\ge1$, then

$$\left|\frac{\Gamma(s+1)}{\Gamma(s+n)}+(s+n+1)\Gamma(s+1)\sum_{k=2}^{n}\frac{(-1)^{k-1}}{k!\Gamma(s+n+2-k)}\right| \times\left[L_{s+n}(a,b)\right]^{s+n} \le\frac{\ln b-\ln a}{n!(b-a)\left[q(n+1)\right]^{1/q}} \times\Big[q(n+1+s)L\left(a^{q(n+1+s)},b^{q(n+1+s)}\right) -sqL\left(a^{q(n+1)},b^{q(n+1)}\right)L\left(a^{sq},b^{sq}\right)\Big]^{1/q}, \tag{30}$$

where $L(u,v)$ and $L_p(u,v)$ are the logarithmic mean and the generalized logarithmic mean of order $p\in\mathbb{R}$, respectively.

Corollary 4.3.1. Under the assumptions of Theorem 4.3, if $q=1$, we have

$$\left|\frac{\Gamma(s+1)}{\Gamma(s+n)}+(s+n+1)\Gamma(s+1)\sum_{k=2}^{n}\frac{(-1)^{k-1}}{k!\Gamma(s+n+2-k)}\right| \times\left[L_{s+n}(a,b)\right]^{s+n} \le\frac{\ln b-\ln a}{(n+1)!(b-a)}\Big[(n+1+s)L\left(a^{n+1+s},b^{n+1+s}\right) -sL\left(a^{n+1},b^{n+1}\right)L\left(a^{s},b^{s}\right)\Big]. \tag{31}$$

In particular, when $n=1$ and $q=1$,

$$\left[L_{s+1}(a,b)\right]^{s+1}\le\frac{\ln b-\ln a}{2(b-a)}\Big[(2+s)L\left(a^{2+s},b^{2+s}\right) -sL\left(a^{2},b^{2}\right)L\left(a^{s},b^{s}\right)\Big], \tag{32}$$

where $L(u,v)$ and $L_p(u,v)$ are the logarithmic mean and the generalized logarithmic mean of order $p\in\mathbb{R}$, respectively.

Theorem 4.4. For $n\in\mathbb{N}$, if $0<a<b$, $s>0$ and $q>1$, then

$$\left|\frac{\Gamma(s+1)}{\Gamma(s+n)}+(s+n+1)\Gamma(s+1)\sum_{k=2}^{n}\frac{(-1)^{k-1}}{k!\Gamma(s+n+2-k)}\right| \times\left[L_{s+n}(a,b)\right]^{s+n} \le\frac{\ln b-\ln a}{n!(b-a)\left(m\ln a-r\ln b\right)^{1/q}} \times\left[L\left(a^{\frac{q(n+1)-m}{q-1}},b^{\frac{q(n+1)-r}{q-1}}\right)\right]^{1-1/q} \times\Big\{\big[(m+sq)\ln a-(r+sq)\ln b\big]L\left(a^{m+sq},b^{r+sq}\right) +sq\left(\ln b-\ln a\right)L\left(a^{m},b^{r}\right)L\left(a^{sq},b^{sq}\right)\Big\}^{1/q}, \tag{33}$$

where $L(u,v)$ and $L_p(u,v)$ are the logarithmic mean and the generalized logarithmic mean of order $p\in\mathbb{R}$, respectively.

Corollary 4.4.1. Under the assumptions of Theorem 4,4,
1. if $m=0$ and $r=q(n+1)$,

$$\left|\frac{\Gamma(s+1)}{\Gamma(s+n)}+(s+n+1)\Gamma(s+1) \times\sum_{k=2}^{n}\frac{(-1)^{k-1}}{k!\Gamma(s+n+2-k)}\right|\left[L_{s+n}(a,b)\right]^{s+n} \le\frac{\ln b-\ln a}{n!(b-a)\left[q(n+1)\ln b\right]^{1/q}}\left[L\left(a^{\frac{q(n+1)}{q-1}},1\right)\right]^{1-1/q} \times\Big\{\big[q(n+1+s)\ln b-sq\ln a\big]L\left(a^{sq},b^{q(n+1+s)}\right) +sq\left(\ln a-\ln b\right)L\left(1,b^{q(n+1)}\right)L\left(a^{sq},b^{sq}\right)\Big\}^{1/q}. \tag{34}$$

When $n=1$,

$$\left[L_{s+1}(a,b)\right]^{s+1} \le\frac{\ln b-\ln a}{(b-a)\left(2q\ln b\right)^{1/q}}\left[L\left(a^{\frac{2q}{q-1}},1\right)\right]^{1-1/q} \times\Big\{\big[q(2+s)\ln b-sq\ln a\big]L\left(a^{sq},b^{q(2+s)}\right) +sq\left(\ln a-\ln b\right)L\left(1,b^{2q}\right)L\left(a^{sq},b^{sq}\right)\Big\}^{1/q}; \tag{35}$$

2. if $m = n+1$ and $r = q(n+1)$,

$$\left|\frac{\Gamma(s+1)}{\Gamma(s+n)} + (s+n+1)\Gamma(s+1) \times \sum_{k=2}^{n} \frac{(-1)^{k-1}}{k!\Gamma(s+n+2-k)}\right| \left[L_{s+n}(a,b)\right]^{s+n}$$
$$\le \frac{(\ln b - \ln a)\left[L\left(a^{n+1},1\right)\right]^{1-1/q}}{n!(b-a)\left[(n+1)\left(\ln a - q\ln b\right)\right]^{1/q}} \times \Big\{\left[(n+1+sq)\ln a - q(n+1+s)\ln b\right] \times L\left(a^{n+1+sq}, b^{q(n+1+s)}\right) + sq\left(\ln b - \ln a\right) \times L\left(a^{n+1}, b^{q(n+1)}\right) L\left(a^{sq}, b^{sq}\right)\Big\}^{1/q}. \quad (36)$$

When $n = 1$,

$$\left[L_{s+1}(a,b)\right]^{s+1} \le \frac{(\ln b - \ln a)\left[L\left(a^{2},1\right)\right]^{1-1/q}}{(b-a)\left[2\left(\ln a - q\ln b\right)\right]^{1/q}} \times \Big\{\left[(2+sq)\ln a - q(2+s)\ln b\right] L\left(a^{2+sq}, b^{q(2+s)}\right) + sq\left(\ln b - \ln a\right) L\left(a^{2}, b^{2q}\right) L\left(a^{sq}, b^{sq}\right)\Big\}^{1/q}; \quad (37)$$

3. if $m = q(n+1)$ and $r = 0$,

$$\left|\frac{\Gamma(s+1)}{\Gamma(s+n)} + (s+n+1)\Gamma(s+1) \times \sum_{k=2}^{n} \frac{(-1)^{k-1}}{k!\Gamma(s+n+2-k)}\right| \left[L_{s+n}(a,b)\right]^{s+n}$$
$$\le \frac{\ln b - \ln a}{n!(b-a)\left[q(n+1)\ln a\right]^{1/q}} \left[L\left(1, b^{\frac{q(n+1)}{q-1}}\right)\right]^{1-1/q} \times \Big\{sq\left(\ln b - \ln a\right) L\left(a^{q(n+1)}, 1\right) L\left(a^{sq}, b^{sq}\right) + \left[q(n+1+s)\ln a - sq\ln b\right] L\left(a^{q(n+1+s)}, b^{sq}\right)\Big\}. \quad (38)$$

If $n = 1$,

$$\left[L_{s+1}(a,b)\right]^{s+1} \le \frac{\ln b - \ln a}{(b-a)\left(2q\ln a\right)^{1/q}} \left[L\left(1, b^{\frac{2q}{q-1}}\right)\right]^{1-1/q} \times \Big\{sq\left(\ln b - \ln a\right) L\left(a^{2q}, 1\right) L\left(a^{sq}, b^{sq}\right) + \left[q(2+s)\ln a - sq\ln b\right] L\left(a^{q(2+s)}, b^{sq}\right)\Big\}; \quad (39)$$

4. if $m = q(n+1)$ and $r = n+1$,

$$\left|\frac{\Gamma(s+1)}{\Gamma(s+n)} + (s+n+1)\Gamma(s+1) \times \sum_{k=2}^{n} \frac{(-1)^{k-1}}{k!\Gamma(s+n+2-k)}\right| \left[L_{s+n}(a,b)\right]^{s+n}$$
$$\le \frac{(\ln b - \ln a)\left[L\left(1, b^{n+1}\right)\right]^{1-1/q}}{n!(b-a)\left[(n+1)(q\ln a - \ln b)\right]^{1/q}} \times \Big\{sq\left(\ln b - \ln a\right) L\left(a^{q(n+1)}, b^{n+1}\right) L\left(a^{sq}, b^{sq}\right) + \left[q(n+1+s)\ln a - (n+1+sq)\ln b\right] \times L\left(a^{q(n+1+s)}, b^{n+1+sq}\right)\Big\}^{1/q}. \quad (40)$$

If $n = 1$,

$$\left[L_{s+1}(a,b)\right]^{s+1} \le \frac{(\ln b - \ln a)\left[L\left(1, b^{2}\right)\right]^{1-1/q}}{(b-a)\left[2(q\ln a - \ln b)\right]^{1/q}} \Big\{sq\left(\ln b - \ln a\right) \times L\left(a^{2q}, b^{2}\right) L\left(a^{sq}, b^{sq}\right) + \left[q(2+s)\ln a - (2+sq)\ln b\right] L\left(a^{q(2+s)}, b^{2+sq}\right)\Big\}^{1/q}, \quad (41)$$

where $L(u,v)$ and $L_p(u,v)$ are the logarithmic mean and the generalized logarithmic mean of order $p \in \mathbb{R}$, respectively.

Acknowledgement

This work was partially supported by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant No. NJZY14192, China and by the Science Research Funding of Inner Mongolia University for Nationalities under Grant No. NMD1301, China.
The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] P. S. Bullen, Handbook of Means and Their Inequalities, Mathematics and its Applications, Volume 560, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.
[2] P.-S. Bai, S.-H. Wang, and F. Qi, Some Hermite-Hadamard type inequalities for n times differentiable $(\alpha; m)$-convex functions, J. Inequal. Appl. **267**, 11 pages (2012), http://dx.doi.org/10.1186/1029-242X-2012-267.
[3] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett. **11** , no. 5, 91–95 (1998), http://dx.doi.org/10.1016/S0893-9659(98)00086-X.

[4] U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp. **147**, no. 1, 137–146 (2004), http://dx.doi.org/10.1016/S0096-3003(02)00657-4.
[5] C. P. Niculescu, Convexity according to the geometric mean, Math. Inequal. Appl. **3**, no. 2, 155–167 (2000), http://dx.doi.org/10.7153/mia-03-19.
[6] C. E. M. Pearce and J. E. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett. **13**, no. 2, 51–55 (2000), http://dx.doi.org/10.1016/S0893-9659(99)00164-0.
[7] J. E. Pečarić and Y. L. Tong, Convex Functions, Partial Ordering and Statistical Applications, Academic Press, New York, 1991.
[8] F. Qi, Generalized abstracted mean values, J. Inequal. Pure Appl. Math. **1**, no. 1, Art. 4 (2000), http://www.emis.de/journals/JIPAM/article97.html.
[9] F. Qi, Generalized weighted mean values with two parameters, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **454**, 2723–2732 (1998), http://dx.doi.org/10.1098/rspa.1998.0277.
[10] F. Qi and S.-Q. Zhang, Note on monotonicity of generalized weighted mean values, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **455**, 3259–3260 (1999), http://dx.doi.org/10.1098/rspa.1999.0449.
[11] M. Z. Sarikaya and N. Aktan, On the generalization some integral inequalities and their applications,arXiv:1005.2879v1 [math.CA] (2011).
[12] Y. Shuang, H.-P. Yin, and F. Qi, Hermite-Hadamard type integral inequalities for geometric-arithmetically s-convex functions, Analysis (Munich), **33** (2013), no. 2, 197-208; Available online at http://dx.doi.org/10.1524/anly.2013.1192..
[13] S.-H. Wang and F. Qi, Hermite-Hadamard type inequalities for convex function which n-times differentiable, Math. Inequal. Appl., **16**, No. 4, 1269–1278 (2013), http://dx.doi.org/doi:10.7153/mia-16-97.
[14] S.-H. Wang and F. Qi, Hermite-Hadamard type inequalities for n-times differentiable and preinvex functions, J. Inequal. Appl., **49**, 9 pages (2014), http://www.journalofinequalitiesandapplications.com/content/2014/1/49.
[15] S.-H. Wang, B.-Y. Xi, and F. Qi, On Hermite-Hadamard type inequalities for $(\alpha;m)$-convex functions, Int. J. Open Probl. Comput. Sci. Math., **5**, No. 4, 47–56 (2012).
[16] S.-H. Wang, B.-Y. Xi, and F. Qi, Some new inequalities of Hermite-Hadamard type for n-time differentiable functions which are m-convex, Analysis (Munich), **32**, No. 3, 247–262 (2012), http://dx.doi.org/10.1524/anly.2012.1167.
[17] B.-Y. Xi, R.-F. Bai, and F. Qi, Hermite-Hadamard type inequalities for the m and $(\alpha;m)$-geometrically convex functions, Aequationes Math., **84**, No. 3, 261–269 (2014), http://dx.doi.org/10.1007/s00010-011-0114-x.
[18] B.-Y. Xi and F. Qi, Hermite-Hadamard type inequalities for functions whose derivatives are of convexities,Nonlinear Functional Analysis and Applications, **18**, No. 2, 163–176 (2013), http://nfaa.kyungnam.ac.kr/jour-nfaa.htm.
[19] T.-Y. Zhang, A.-P. Ji, and F. Qi, Some inequalities of Hermite-Hadamard type for GA-convex functions with applications to means,Le Matematiche, LXVIII, Fasc. I, 229–239 (2013), http://dx.doi.org/10.4418/2013.68.1.17.

Shu-Hong Wang is an associate professor in mathematics at Inner Mongolia University for Nationalities in China. She received her PhD degree of Science in Mathematics from Dalian University of Technology in China. Her research interests are in the areas such as convex analysis of functions and mathematical inequalities.

Xiao-Ting Shi is being a graduate for the master degree of science in mathematics at Tianjin Polytechnic University of China. Her bachelor degree of science in mathematics was obtained in June 2014 from Changzhi University of Shanxi Province in China. Currently, her research interests are theory of special functions, mathematical inequalities and applications, analytic combinatorics, analytic number theory, and the like.