Applied Mathematics & Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/amisl/060203

Congruences on Near Left Almost Rings

Fawad Hussain[1,*] *Samiya Firdous*[2] *and Nazia Sadiq* [2]

[1] Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad, Pakistan
[2] Department of Mathematics, Hazara University, Mansehra, Pakistan

Received: 8 Mar. 2016, Revised: 5 May 2016, Accepted: 10 Jan. 2018
Published online: 1 May 2018

Abstract: In this paper we generalize the concept of congruences from left almost rings to near left almost rings. We show that from every homomorphism, we can get a congruence relation on near left almost rings and then at the end we prove analogues of the isomorphism theorems.

Keywords: Homomorphisms, Congruences, Analogues of the isomorphism theorems

1 Introduction

In 1972, Kazim and Naseeruddin [2] introduced braces on the left of the equation $xyz = zyx$, and get a new pseudo associative law, that is $(xy)z = (zy)x$. It is known as left invertive law. A groupoid is called a left almost semigroup, abbreviated as LA-semigroup, if it satisfies the left invertive law. It corresponds to a semigroup and is basically the generalization of a commutative semigroup. In [4], LA-semigroup is also known as an Abel-Grassmanns groupoid (AG-groupoid) after the name of Abel-Grassmann. In 1993, Kamran [3] extended the concept of LA-semigroups to left almost groups, abbreviated as LA-groups. An LA-group corresponds to a group. It is a non-associative structure and the generalization of commutative groups. Let $(S, *)$ be a groupoid, then it is called a left almost group if it satisfies (i), (ii) and (iii):

(i) Elements of S must satisfy the left invertive law. That is $a*(b*c) = (c*b)*a \; \forall \; a,b,c \in S$.
(ii) There exists an element $f \in S \ni f*s = s \; \forall \; s \in S$. That is left identity element exists in S.
(iii) $\forall \; d \in S \; \exists \; d^{-1} \in S \ni d*d^{-1} = d^{-1}*d = f$. That is left inverse of each element of S exists in S.

In [3], the author proved some interesting and elegant results about LA-groups. Particularly the author discussed substructures of LA-groups and then quotient structures. In 2006, Yusuf [9] extended the concept of an LA-group to a non-associative structure called left almost ring, abbreviated as LA-ring. LA-rings basically correspond to rings. A left almost ring is a set $R \neq \emptyset$ with the binary operations "$+$" and "$\cdot$" which satisfies the conditions (i), (ii) and (iii):

(i) $(R,+)$ is an LA-group,
(ii) $(R,\cdot)$ is an LA-semigroup,
(iii) Distributive laws of multiplication over addition hold in R, i.e. $\forall \; c, \; l, \; e \in R$; $c\cdot(l+e) = c\cdot l + c\cdot e$ and $(c+l)\cdot e = c\cdot e + l\cdot e$.

Further different peoples in [[1], [5], [7],[8]] worked on LA-rings and explored many interesting and useful properties of LA-rings. In 2011, Shah, Rehman and Raees [6] gave the notion of a near left almost ring, abbreviated as nLA-ring which is basically the generalization of an LA-ring. A set $R \neq \emptyset$ with the binary operations of addition (denoted by "$+$") and multiplication (denoted by "$\cdot$") is known as a near left almost ring (shortly represented by nLA-ring) if and only if $(R,+)$ is an LA-group, $(R,\cdot)$ is an LA-semigroup, Left distributive law of multiplication over addition holds, i.e. $c\cdot(d+e) = c\cdot d + c\cdot e$ for every $c, \; d, \; e \in R$. They discussed many useful properties of nLA-rings. In particular they discussed substructures and homomorphism of nLA-rings and then proved well known isomorphism theorems. In 2015, Hussain and Khan [1] extended the concept of congruences from semigroups to LA-rings. In this study we generalize the concept of congruences from LA-rings to to nLA-rings. We show that every homomorphism defines a congruence relation on nLA-rings.

* Corresponding author e-mail: fawad.hussain30@gmail.com

2 Congruence Relations

In this section we discuss congruence relations on nLA-rings. It is important to describe compatible relations in order to describe congruence relations. The idea of this portion has come from the paper [1] in which the authors have done similar calculations for LA-rings.

Definition 2.1. Let ρ be a relation on a near left almost ring $(R,+,\cdot)$. Then ρ is said to be left compatible if $\forall\ c,\ l,\ e \in R \ni (c,\ l) \in \rho \implies (e+c,\ e+l) \in \rho$ and $(e\cdot c,\ e\cdot l) \in \rho$. ρ is called right compatible if for all $c,\ l,\ e \in R \ni (c,\ l) \in \rho \implies (c+e,\ l+e) \in \rho$ and $(c\cdot e,\ l\cdot e) \in \rho$. ρ is known as compatible if for $c,\ l,\ e,\ g \in R \ni (c,\ l) \in \rho$ and $(e,\ g) \in \rho \implies (c+e,\ l+g) \in \rho$ and $(c\cdot e,\ l\cdot g) \in \rho$.

An equivalence relation which is left compatible as well is called a left congruence relation and an equivalence relation which is right compatible as well is called a right congruence relation. An equivalence relation which is compatible as well is known as a congruence relation. Here we provide an example in order to understand the above mentioned concept.

Example 2.2. Let $R = \{x,\ y,\ z,\ w,\ q\}$ and define "+" and "·" in the following tables:

$+$	x	y	z	w	q
x	x	y	z	w	q
y	q	x	y	z	w
z	w	q	x	y	z
w	z	w	q	x	y
q	y	z	w	q	x

$\cdot$	x	y	z	w	q
x	x	x	x	x	x
y	x	y	z	w	q
z	x	w	y	q	z
w	x	z	q	y	w
q	x	q	w	z	y

Then according to [6], $(R,+,\cdot)$ is an nLA-ring. Now let $\rho = \{(x,y) : x = y\}$ be a relation on R, then one can easily show that ρ is a congruence relation on R.

Let us present some properties. The following result is important because it gives equivalent conditions for congruence relations.

Theorem 2.3. Let $(R,+,\cdot)$ be an nLA-ring and ρ a relation on R. Then ρ is a congruence relation if and only if ρ is left as well as right congruence relation.

Proof. Let ρ be a congruence relation on R. We prove that ρ is left as well as right congruence relation. Let $c,\ x,\ e \in R \ni (c,\ x) \in \rho$, then $(e+c,\ e+x) \in \rho$ and $(e\cdot c,\ e\cdot x) \in \rho$, as $(e,\ e) \in \rho$. Thus, ρ is left compatible. Hence, ρ is a left congruence relation. Let $c,\ x,\ e \in R$ such that $(c,\ x) \in \rho$, then $(c+e,\ x+e) \in \rho$ and $(c\cdot e,\ x\cdot e) \in \rho$, as $(e,\ e) \in \rho$. Thus, ρ is right compatible. Hence, ρ is a right congruence relation.

On the other hand, suppose ρ is both left and right congruence relation. Let $c_1,\ c_2,\ x_1,\ x_2 \in R$ such that $(c_1,\ c_2) \in \rho$ and $(x_1,\ x_2) \in \rho$. Then by left compatibility we have

$$(c_1+x_1,\ c_1+x_2) \in \rho \text{ and } (c_1\cdot x_1,\ c_1\cdot x_2) \in \rho.$$

Similarly by right compatibility we get

$$(c_1+x_2,\ c_2+x_2) \in \rho \text{ and } (c_1\cdot x_2,\ c_2\cdot x_2) \in \rho.$$

Thus by transitivity it follows that $(c_1+x_1,\ c_2+x_2) \in \rho$ and $(c_1\cdot x_1,\ c_2\cdot x_2) \in \rho$. Hence ρ is a congruence relation.

3 Homomorphism of Near Left Almost Rings

In this portion, we define homomorphism of nLA-rings which is taken from [6]. At the end we prove a result which says that every homomorphism defines a congruence relation on nLA-rings.

Definition 3.1. Let $(R,+,\cdot)$ and $(S,\oplus,\circ)$ be two nLA-rings. A map $\phi : R \to S$ is known as an nLA-ring homomorphism or simply a homomorphism if for all $c, \in R \ni$

$$(c+l)\phi = (c)\phi \oplus (l)\phi \text{ and } (c\cdot l)\phi = (c)\phi \circ (l)\phi.$$

It should be noted that

(i) A near left almost ring homomorphism is called a monomorphism if it is one-one.
(ii) A near left almost ring homomorphism is called an epimorphism if it is onto.

A near left almost ring homomorphism is called an isomorphism if it is both one-one and onto.

The below mentioned result indicates that from every nLA-ring homomorphism we may get a congruence relation.

Theorem 3.1. Assume $(R,+,\cdot)$ and $(S,\oplus,\circ)$ are nLA-rings. Let $\omega : R \to S$ be a homomorphism, then from ω, we may get a congruence relation ρ on R given by

$$(l_1,l_2) \in \rho \text{ if and only if } (l_1)\omega = (l_2)\omega, \forall\ l_1,l_2 \in R.$$

Proof. First we prove that ρ is an equivalence relation.

Reflexive:

As $(l)\omega = (l)\omega$ for all $l \in R$, therefore by definition, $(l,l) \in \rho$. Thus, ρ is reflexive.

Symmetric:

Choose $l_1,l_2 \in R$ such that $(l_1,l_2) \in \rho$, then by definition, $(l_1)\omega = (l_2)\omega \implies (l_2)\omega = (l_1)\omega$. Therefore again by definition, $(l_2,l_1) \in \rho$. Thus, ρ is symmetric.

Transitive:

Choose $l_1,l_2,l_3 \in R$ such that $(l_1,l_2) \in \rho$ and $(l_2,l_3) \in \rho$. Then by definition, $(l_1)\omega = (l_2)\omega$ and $(l_2)\omega = (l_3)\omega \implies (l_1)\omega = (l_3)\omega$. Therefore again by definition, $(l_1,l_3) \in \rho$. Thus, ρ is transitive.

We now prove that ρ is compatible.

Compatibility:

Choose $l_1, l_2, l_3, l_4 \in R$ such that $(l_1, l_2) \in \rho$ and $(l_3, l_4) \in \rho$. Then by definition, $(l_1)\omega = (l_2)\omega$ and $(l_3)\omega = (l_4)\omega$.

As

$(l_1 + l_3)\omega = (l_1)\omega \oplus (l_3)\omega$

$(\because \omega$ *is an nLA-ring homomorphism*$)$

$= (l_2)\omega \oplus (l_4)\omega$

$= (l_2 + l_4)\omega$

$(\because \omega$ *is an nLA-ring homomorphism*$)$

Thus by definition, $(l_1 + l_3, l_2 + l_4) \in \rho$.

In the same way as above

$(l_1 \cdot l_3)\omega = (l_1)\omega \circ (l_3)\omega$

$(\because \omega$ *is an nLA-ring homomorphism*$)$

$= (l_2)\omega \circ (l_4)\omega$

$= (l_2 \cdot l_4)\omega$

$(\because \omega$ *is an nLA-ring homomorphism*$)$

Thus, by definition, $(l_1 \cdot l_3, l_2 \cdot l_4) \in \rho$.

Now let R be an nLA-ring and ρ a congruence relation on R. We define $R/\rho = \{l\rho : l \in R\}$. In other words, R/ρ is the set which consists of all congruence classes corresponding to the elements of R.

Let $l_1\rho$, $l_2\rho$ be two congruence classes corresponding to the elements $l_1, l_2 \in R$. Then the below binary operations show that R/ρ becomes an nLA-ring.

$$l_1\rho + l_2\rho = (l_1 + l_2)\rho \text{ and } l_1\rho \cdot l_2\rho = (l_1 \cdot l_2)\rho.$$

We start by showing that the above binary operations are well-defined.

Choose $l_1, l_2, l_3, l_4 \in R$ such that $l_1\rho = l_3\rho$ and $l_2\rho = l_4\rho$, then obviously $(l_1, l_3) \in \rho$ and $(l_2, l_4) \in \rho$ $\Longrightarrow (l_1 + l_2, l_3 + l_4) \in \rho$ and $(l_1 \cdot l_2, l_3 \cdot l_4) \in \rho$, as ρ is compatible. Thus $(l_1 + l_2)\rho = (l_3 + l_4)\rho$ and $(l_1 \cdot l_2)\rho = (l_3 \cdot l_4)\rho$.

We now show $(R/\rho, +)$ is an LA-group.

(i) Closure property with respect to "$+$"

Follows from the above definition.

(ii) Left invertive law with respect to "$+$"

Let $l_1\rho, l_2\rho, l_3\rho \in R/\rho$, then

$(l_1\rho + l_2\rho) + l_3\rho$

$= (l_1 + l_2)\rho + l_3\rho$

$= ((l_1 + l_2) + l_3))\rho$

$= ((l_3 + l_2) + l_1))\rho$

$= (l_3 + l_2)\rho + l_1\rho$

$= (l_3\rho + l_2\rho) + l_1\rho.$

(iii) Left additive identity

As the left additive identity $0 \in R \Longrightarrow 0\rho \in R/\rho$. Now let $l\rho \in R/\rho$, then $0\rho + l\rho = (0 + l)\rho = l\rho$. It follows that 0ρ is the left additive identity element of R/ρ.

(iv) Left additive inverse

Let $l\rho \in R/\rho \Longrightarrow l \in R \Longrightarrow -l \in R$, so it follows that $-l\rho \in R/\rho$.

Now

$$-l\rho + l\rho = (-l + l)\rho = 0\rho.$$

Similarly

$$l\rho - l\rho = (l - l)\rho = 0\rho.$$

It follows that $-l\rho$ is the left additive inverse of $l\rho$.

Hence $(R/\rho, +)$ is an LA-group.

We now show that $(R/\rho, \cdot)$ is an LA-semigroup. For this

(v) Closure property with respect to "$\cdot$"

Follows from the above definition.

(vi) Left invertive law with respect to "$\cdot$"

Let $l_1\rho, l_2\rho, l_3\rho \in R/\rho$, then

$(l_1\rho \cdot l_2\rho) \cdot l_3\rho$

$= (l_1 \cdot l_2)\rho \cdot l_3\rho$

$= ((l_1 \cdot l_2) \cdot l_3))\rho$

$= ((l_3 \cdot l_2) \cdot l_1))\rho$

$= (l_3 \cdot l_2)\rho \cdot l_1\rho$

$= (l_3\rho \cdot l_2\rho) \cdot l_1\rho.$

Thus R/ρ satisfies the left invertible law with respect to multiplication.

Hence $(R/\rho, \cdot)$ is an LA-semigroup.

Further we prove

(vii) Left distributive law

Let $l_1\rho, l_2\rho, l_3\rho \in R/\rho$, then

$l_1\rho \cdot (l_2\rho + l_3\rho)$

$= l_1\rho \cdot (l_2 + l_3)\rho$

$= (l_1 \cdot (l_2 + l_3))\rho$

$= (l_1 \cdot l_2 + l_1 \cdot l_3)\rho$

$= (l_1 \cdot l_2)\rho + (l_1 \cdot l_3)\rho$

$= l_1\rho \cdot l_2\rho + l_1\rho \cdot l_3\rho.$

Thus, R/ρ satisfies left distributive law. Hence, R/ρ is an nLA-ring which is called quotient nLA-ring.

4 Analogues of the Isomorphism Theorems

In this portion, we state and prove the analogues of the first, second and third isomorphism theorems for nLA-rings. The equivalent of LA-rings is discussed in [1].

Theorem 4.1. (First Isomorphism Theorem)
Let $(R,+,\cdot)$ be an nLA-ring and ρ a congruence relation on R, then R/ρ is an nLA-ring with respect to the following operations of addition and multiplication:
$s_1\rho + s_2\rho = (s_1 + s_2)\rho$ and $s_1\rho \cdot s_2\rho = (s_1 \cdot s_2)\rho$ for all $s_1\rho,\ s_2\rho \in R/\rho$.
The mapping $\rho^{\#} : R \to R/\rho$ such that $(s)\rho^{\#} = s\rho$ for every $s \in R$ is an onto homomorphism. Let $\phi : R \to S$ be a homomorphism where $(R,+,\cdot)$ and $(S,\oplus,\circ)$ are two near left almost rings. Then

$$Ker\phi = \{(s_1,s_2) \in R \times R : (s_1)\phi = (s_2)\phi\}$$

is a congruence relation on R and there is an nLA-ring monomorphism $\alpha : R/Ker\phi \to S \ni ran\alpha = ran\phi$ and the diagram

$$\begin{array}{ccc} & \phi & \\ R & \longrightarrow & S \\ (ker\phi)^{\#}\downarrow & \nearrow \alpha & \\ R/ker\phi & & \end{array}$$

commutes.

Proof. To establish the first portion of the theorem, we follow it from the above discussion. $Ker\phi$ is a congruence relation by Theorem 3.1. We now show that $\rho^{\#}$ is an epimorphism.

Let $l_1, l_2 \in R$, then

$$(l_1 + l_2)\rho^{\#} = (l_1 + l_2)\rho$$
$$= (l_1)\rho + (l_2)\rho$$
$$= (l_1)\rho^{\#} + (l_2)\rho^{\#}$$

and

$$(l_1 \cdot l_2)\rho^{\#} = (l_1 \cdot l_2)\rho$$
$$= (l_1)\rho \cdot (l_2)\rho$$
$$= (l_1)\rho^{\#} \cdot (l_2)\rho^{\#}.$$

Thus, ρ is a homomorphism. From the definition it is obvious that ρ is onto. Now define

$\alpha : R/Ker\phi \to S$ by $(sKer\phi)\alpha = (s)\phi$ for all $sKer\phi \in R/Ker\phi$.

Then α is well-defined and one-one. For this let $s_1Ker\phi, s_2Ker\phi \in R/Ker\phi$ such that

$$s_1Ker\phi = s_2Ker\phi \iff (s_1,s_2) \in Ker\phi \iff (s_1)\phi = (s_2)\phi$$

$$\iff (s_1Ker\phi)\alpha = (s_2Ker\phi)\alpha.$$

Now let $s_1Ker\phi, s_2Ker\phi \in R/Ker\phi$, then

$$(s_1Ker\phi + s_2Ker\phi)\alpha = [(s_1 + s_2)Ker\phi]\alpha$$
$$= (s_1 + s_2)\phi$$
$$= (s_1)\phi \oplus (s_2)\phi$$
$$= (s_1Ker\phi)\alpha \oplus (s_2Ker\phi)\alpha.$$

Also

$$(s_1Ker\phi \cdot s_2Ker\phi)\alpha = [(s_1 \cdot s_2)Ker\phi]\alpha$$
$$= (s_1 \cdot s_2)\phi$$
$$= (s_1)\phi \circ (s_2)\phi$$
$$= (s_1Ker\phi)\alpha \circ (s_2Ker\phi)\alpha.$$

Clearly $ran\alpha = ran\phi$. From the definition it is clear that

$$(s)(Ker\phi)^{\#}\alpha = ((s)(Ker\phi)^{\#})\alpha$$

$$= (sKer\phi)\alpha = (s)\phi \text{ for all } s \in R.$$

This completes the proof.

Theorem 4.2. (Second Isomorphism Theorem)
Assume that $(R,+,\cdot)$ and $(S,\oplus,\circ)$ are nLA-rings and ρ a congruence relation on R. If $\theta : R \to S$ is an nLA-ring

homomorphism with $\rho \subseteq Ker\theta$, then there is one and only one an nLA-ring homomorphism $\psi : R/\rho \to S \ni ran\theta = ran\psi$ and the diagram

$$\begin{array}{ccc} & \theta & \\ R & \longrightarrow & S \\ \rho^{\#}\downarrow & \nearrow \psi & \\ R/\rho & & \end{array}$$

commutes.

Proof. Define a map $\psi : R/\rho \to S$ by $(r\rho)\psi = (r)\theta$ for all $r\rho \in R/\rho$. Choose $r_1\rho,\ r_2\rho \in R/\rho$ such that $r_1\rho = r_2\rho$, then obviously $(r_1,\ r_2) \in \rho \subseteq Ker\theta$. Thus $(r_1)\theta = (r_2)\theta$. It shows that the map ψ is well-defined.

Now let $l_1\rho,\ l_2\rho \in R/\rho$, then

$$\begin{aligned}(l_1\rho + l_2\rho)\psi &= ((l_1 + l_2)\rho)\psi \\ &= (l_1 + l_2)\theta \\ &= (l_1)\theta \oplus (l_2)\theta \\ &= (l_1\rho)\psi \oplus (l_2\rho)\psi\end{aligned}$$

and

$$\begin{aligned}(l_1\rho \cdot l_2\rho)\psi &= ((l_1 \cdot l_2)\rho)\psi \\ &= (l_1 \cdot l_2)\theta \\ &= (l_1)\theta \circ (l_2)\theta \\ &= (l_1\rho)\psi \circ (l_2\rho)\psi.\end{aligned}$$

Showing that ψ satisfies the conditions of the homomorphism.

Now $(r)\rho^{\#}\psi = ((r)\rho^{\#})\psi = (r\rho)\psi = (r)\theta$ showing the diagram commutes. It is obvious that $ran\psi = ran\theta$. Now let $\psi_1 : R/\rho \to S$ be homomorphism satisfying $\rho^{\#}\psi_1 = \theta$. Then for $r \in R$,
$(r)\rho^{\#}\psi_1 = ((r)\rho^{\#})\psi_1 = (r\rho)\psi_1 = (r)\theta = (r)\rho^{\#}\psi = ((r)\rho^{\#})\psi = (r\rho)\psi \implies (r\rho)\psi_1 = (r\rho)\psi \implies \psi = \psi_1$.

Theorem 4.3. (Third Isomorphism Theorem)
Let ρ and σ be congruence relations on an nLA-ring R such that $\rho \subseteq \sigma$. Then

$$\sigma/\rho = \{(r_1\rho,\ r_2\rho) \in R/\rho \times R/\rho : (r_1,\ r_2) \in \sigma\}$$

is a congruence relation on R/ρ and $R/\rho/\sigma/\rho \cong R/\sigma$.

Proof. First we show that

$$\sigma/\rho = \{(r_1\rho,\ r_2\rho) \in R/\rho \times R/\rho : (r_1,\ r_2) \in \sigma\}$$

is a congruence relation on R/ρ.

Reflexive:

Since $(r,\ r) \in \sigma$ for all $r \in R$, thus $(r\rho,\ r\rho) \in \sigma/\rho$ which implies that σ/ρ is reflexive.

Symmetric:

Choose $r_1,\ r_2 \in R$ such that $(r_1\rho,\ r_2\rho) \in \sigma/\rho$, then $(r_1,\ r_2) \in \sigma$. Since σ is symmetric so $(r_2,\ r_1) \in \sigma$. Therefore by definition $(r_2\rho,\ r_1\rho) \in \sigma/\rho$. Hence σ/ρ is symmetric.

Transitive:

Choose $l_1,\ l_2,\ l_3 \in R$ such that $(l_1\rho,\ l_2\rho) \in \sigma/\rho$ and $(l_2\rho,\ l_3\rho) \in \sigma/\rho$ then by definition, $(l_1,\ l_2) \in \sigma$ and $(l_2,\ l_3) \in \sigma$. Since σ is transitive therefore $(l_1,\ l_3) \in \sigma$. Thus again by definition, $(l_1\rho,\ l_3\rho) \in \sigma/\rho$. Hence σ/ρ is transitive. Showing that σ/ρ satisfies the three conditions of an equivalence relation.

Compatibility:

Now choose $l_1,\ l_2,\ l_3,\ l_4 \in R$ such that $(l_1\rho,\ l_2\rho) \in \sigma/\rho$ and $(l_3\rho,\ l_4\rho) \in \sigma/\rho$. Then by definition, $(l_1,\ l_2) \in \sigma$ and $(l_3,\ l_4) \in \sigma$ but σ is compatible therefore, $(l_1 + l_3,\ l_2 + l_4) \in \sigma$ and $(l_1 \cdot l_3,\ l_2 \cdot l_4) \in \sigma$. Therefore again by definition $((l_1 + l_3)\rho, (l_2 + l_4)\rho) \in \sigma/\rho \implies (l_1\rho + l_3\rho,\ l_2\rho + l_4\rho) \in \sigma/\rho$ and $((l_1 \cdot l_3)\rho,\ (l_2 \cdot l_4)\rho) \in \sigma/\rho \implies (l_1\rho \cdot l_3\rho,\ l_2\rho \cdot l_4\rho) \in \sigma/\rho$. Hence σ/ρ is compatible.

Hence σ/ρ is a congruence relation on R/ρ.

Next define a mapping $\theta : R/\rho \to R/\sigma$ by $(r\rho)\theta = r\sigma$. We start by showing that the map θ is well-defined. Choose $l_1\rho,\ l_2\rho \in R/\rho$ such that $l_1\rho = l_2\rho$, then obviously $(l_1,\ l_2) \in \rho \subseteq \sigma \implies (l_1,\ l_2) \in \sigma$. Thus $l_1\sigma = l_2\sigma \implies (l_1\rho)\theta = (l_2\rho)\theta$.

Now

$$\begin{aligned}(l_1\rho + l_2\rho)\theta &= ((l_1 + l_2)\rho)\theta \\ &= (l_1 + l_2)\sigma \\ &= l_1\sigma + l_2\sigma \\ &= (l_1\rho) + (l_2\rho)\theta.\end{aligned}$$

Also

$$\begin{aligned}(l_1\rho \cdot l_2\rho)\theta &= ((l_1 \cdot l_2)\rho)\theta \\ &= (l_1 \cdot l_2)\sigma \\ &= l_1\sigma \cdot l_2\sigma\end{aligned}$$

$$= (l_1\rho)\cdot(l_2\rho)\theta.$$

Thus from Theorem 4.1, it follows that there is a monomorphism $\alpha : R/\rho/Ker\theta \to R/\sigma$ defined by $((r\rho)Ker\theta)\alpha = r\sigma$. Clearly it is onto, because for $r\sigma \in R/\sigma$ there exists $(r\rho)Ker\theta \in R/\rho/Ker\theta$ such that $((r\rho)Ker\theta)\alpha = r\sigma$. Hence $R/\rho/Ker\theta \cong R/\sigma$.

Now

$$Ker\theta = \{(r_1\rho, r_2\rho) \in R/\rho \times R/\rho : (r_1\rho)\theta = (r_2\rho)\theta\}$$

$$= \{(r_1\rho,\ r_2\rho) \in R/\rho \times R/\rho : r_1\sigma = r_2\sigma\}$$

$$= \{(r_1\rho,\ r_2\rho) \in R/\rho \times R/\rho : (r_1,\ r_2) \in \sigma\}$$

$$= \sigma/\rho.$$

Hence, $R/\rho/\sigma/\rho \cong R/\sigma$.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] F. Hussain and W. Khan, Congruences on left almost rings. International Journal of Algebra and Statistics, **4(1)**. 1-6 (2015).
[2] M. Kazim and M. Naseeruddin, On almost semigroups. The Alig. Bull. Math, **2**. 1-7 (1972).
[3] M. S. Kamran, Conditions for LA-semigroups to resemble associative structures. Ph.D. Thesis. Quaid-i-Azam University, Islamabad, 1993. Available at http://eprints.hec.gov.pk/2370/1/2225.htm.
[4] P. V. Protic and N. Stevanovi, AG-test and some general properties of Abel Grassmanns groupiod. P. U. M. A, **6(4)**. 371-383 (1995).
[5] I. Rehman, On generalized commutative ring and related structures. PhD. Thesis. Quaid-i-Azam University, Islamabad, 2012.
[6] T. Shah, F. Rehman and M. Raees, On near left almost rings. International Mathematical Forum, **6(23)**. 1103-1111 (2011).
[7] T. Shah and I. Rehman, On LA-rings of finitely non-zero functions. Int. J. Contemp. Math. Sciences, **5(5)**. 209-222 (2010).
[8] M. Shah and T. Shah, Some basic properties of LA-rings. International Mathematical Forum, **6(44)**: 2195-2199 (2011).
[9] S. M. Yusuf, On left almost ring. Proc. of 7th International Pure Math. Conference, 2006.

Fawad Hussain is Lecturer in the Department of Mathematics, Abbottabad University of Science and Technology, Pakistan. Before he was Lecturer in the Department of Mathematics, Hazara University Mansehra, Pakistan. He got PhD degree in Mathematics from the University of Glasgow, Scotland, UK. His research interest is in the area of algebra. Particularly he is working in group theory, ring theory, invariant theory, and different types of algebra. He has published several research articles in local as well as in international journals of good repute.

Samiya Firdous received MPhil degree in Mathematics from Hazara University Mansehra, Pakistan. She did MPhil in non-associative ring theory. Particularly she worked in near left almost rings and explored some interesting and elegant properties. She has published some research articles in good reputed international journals.

Nazia Sadiq is Lecturer in the Department of Mathematics, Hazara University, Mansehra Pakistan. She got MPhil degree in Mathematics from Hazara University Mansehra, Pakistan. She did MPhil in functional analysis. Especially she worked in normed almost linear spaces. She is interested in algebra as well as in functional analysis and the connection between the two areas. She has published some research papers in international journals of good repute.