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On Abundancy Index of Some Special Class of Numbers

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Abstract: For any positive integer n, abundancy index $I(n)$ is defined as $I(n)=\frac{\sigma(n)}{n}$, where $\sigma(n)$ is the sum of all positive divisors of n. In this paper, we have discussed non trivial lower and upper bounds of $I(n)$ for some special class of numbers like Quasi perfect numbers, Super hyperperfect numbers, Near perfect numbers and Hyperperfect numbers.

Keywords: Quasi perfect number, Super hyperperfect number, Near perfect number, Hyperperfect number

1 Introduction

We known divisor function $\sigma(n)$ is the sum of all positive divisors of n , including 1 and n itself. The abundancy index $I(n)$ for any positive integer n is associated with the divisor function $\sigma(n)$ and is defined as

$$I(n)=\frac{\sigma(n)}{n}$$

Since perfect numbers are solutions of the equation $\sigma(n)=2n$, therefore $I(n)=2$. For abundant numbers, $I(n)>2$ and for deficient numbers, $I(n)<2$.

We first recall some well known definitions.

Definition 1.1. A positive integer n, which is a solution of the functional equation $\sigma(n)=2n+1$, is called quasi perfect number. No single example for quasi perfect number has been found, nor has a proof for their non existence been established. It is still an open question to determine existence (or otherwise) for a quasi perfect number. Interest to this problem has produced many analogous notions. If a quasi perfect number exists, then it must be an odd square number and have at least seven distinct prime factors and also greater than 35^{10} [4]. For more details on quasi perfect numbers see [1,3].

Definition 1.2. P. Pollack and V. Shevelev [7] introduced the concept of near perfect numbers and classified all the near perfect numbers with two distinct prime factors. Near perfect numbers are solutions of the equation $\sigma(n)=2n+d$, where d is a proper divisor (divisors excluding 1 and n itself) of n, known as redundant divisor of n. If $m=2^{p-1}M_p$ is an even perfect number, where both p and $M_p=2^p-1$ are primes, then $n=2m$, $n=2^pm$ and $n=(2^p-1)m$ are near perfect numbers. There are finitely many near perfect numbers other than these three shapes. The primes of the form $M_p=2^p-1$ are called Mersenne primes.

Definition 1.3. If a positive integer n is a $k-$ hyperperfect number, then

$$n=1+k[\sigma(n)-n-1]$$

for some positive integer k. This equation can also be written as

$$\sigma(n)=\frac{k+1}{k}n+\frac{k-1}{k}$$

D. Minoli and R. Bear [6] introduced the notion of $k-$hyperperfect numbers. They conjectured that there are $k-$hyperperfect numbers for every natural number k. Perfect numbers are $1-$hyperperfect numbers. In 2000, J. S. McCranie [5] computed all hyperperfect numbers less than 11^{10}. If n is a $2-$hyperperfect number, then

$$\sigma(n)=\frac{3}{2}n+\frac{1}{2}$$

A. Bege and K. Fogarasi [2] have given list of $k-$ hyperperfect numbers for some positive integer k and also they conjectured that all $2-$hyperperfect numbers are of the form $n=3^{k-1}(3^k-2)$, where 3^k-2 are primes.

Definition 1.4. A positive integer n, which satisfies the equation

$$\sigma(\sigma(n))=\frac{3}{2}n+\frac{1}{2}$$

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is called super hyperperfect number. A. Bege and K. Fogarasi [2] conjectured that solutions of this equation are of the form 3^{p-1}, where both p and $\frac{3^p-1}{2}$ are primes.

In this paper, we have determined non trivial lower and upper bounds of $I(n)$ for quasi perfect, near perfect, super hyperperfect and $k-$hyperperfect numbers. Moreover, quasi perfect, near perfect and $k-$hyperperfect numbers can also be defined in terms of abundancy index $I(n)$.

2 Main Result

Proposition 2.1. If n is a quasi perfect number, then

$$2 < I(n) < \frac{2n+4}{n+1}$$

Proof. If n is a quasi perfect number, then $\sigma(n) = 2n+1$. Therefore abundancy index $I(n) = \frac{\sigma(n)}{n} = 2+\frac{1}{n} > 2$. Next we prove $I(n) < \frac{2n+4}{n+1}$. On the contrary suppose that $I(n) \geq \frac{2n+4}{n+1}$, then from definition of $I(n)$, one obtains $\sigma(n) \geq \frac{2n^2+4n}{n+1}$. But $\frac{2n^2+4n}{n+1} = \frac{2n(n+1)+2n}{n+1} = 2n+\frac{2n}{n+1} = 2n+2-\frac{2}{n+1}$. Therefore our assuming inequality becomes $\sigma(n) \geq 2n+2-\frac{2}{n+1}$. Since n is a quasi perfect number, therefore $2n+1 = \sigma(n) \geq 2n+2-\frac{2}{n+1}$, this is clearly a contradiction for any positive integer $n > 1$. This obstacle implies that $I(n) < \frac{2n+4}{n+1}$.

Remark 2.1. Converse of the proposition 2.1 is also true, i.e., the inequality obtained in proposition 2.1 for n, is also necessary for n to be a quasi perfect number.

Proposition 2.2. If n is a positive integer which satisfies the inequality

$$2 < I(n) < \frac{2n+4}{n+1}$$

, then n is a quasi perfect number.

Proof. Let n be a positive integer, and suppose that n satisfies the inequality $2 < I(n) < \frac{2n+4}{n+1}$, then from definition of $I(n)$, one obtains $2n < \sigma(n) < \frac{2n^2+4n}{n+1}$. But $\frac{2n^2+4n}{n+1} = 2n+2-\frac{2}{n+1}$, therefore the inequality becomes $2n < \sigma(n) < 2n+2-\frac{2}{n+1}$. Since divisor function $\sigma(n)$ is always a positive integer for any $n > 1$, therefore the last inequality strictly implies that $\sigma(n) = 2n+1$. Hence n is a quasi perfect number.

Remark 2.2. From proposition 2.1 and proposition 2.2, we can define quasi perfect numbers in terms of abundancy index $I(n)$. We can say that positive integer n is a quasi perfect number if and only if the inequality $2 < I(n) < \frac{2n+4}{n+1}$ holds. The number 2 is the lower bound of $I(n)$, when n is a quasi perfect number. Moreover, this lower bound of $I(n)$ can be expressed in terms of rational function of n. Following proposition gives the more improve lower bound for $I(n)$.

Proposition 2.3. A positive integer n is quasi perfect number if and only if the following inequality holds:

$$\frac{2n+3}{n+1} < I(n) < \frac{2n+4}{n+1}$$

Proof. First we assume that n is a quasi perfect number, then from the proposition 2.2, we can write $I(n) < \frac{2n+4}{n+1}$. Next we prove $\frac{2n+3}{n+1} < I(n)$. On the contrary suppose that $\frac{2n+3}{n+1} \geq I(n)$, then $\frac{2n^2+3n}{n+1} \geq \sigma(n)$. But $\frac{2n^2+3n}{n+1} = \frac{2n(n+1)+n}{n+1} = 2n+\frac{n}{n+1} = 2n+1-\frac{1}{n+1}$. Therefore the last inequality becomes $2n+1-\frac{1}{n+1} \geq \sigma(n) = 2n+1$, which is clearly a contradiction. This contradiction strictly implies that $\frac{2n+3}{n+1} < I(n)$. Hence if n is a quasi perfect number, then $\frac{2n+3}{n+1} < I(n) < \frac{2n+4}{n+1}$ holds.

Conversely, suppose that n is a positive integer which satisfies the following inequality

$$\frac{2n+3}{n+1} < I(n) < \frac{2n+4}{n+1}$$

Then from definition of $I(n)$, we can write $\frac{2n^2+3n}{n+1} < \sigma(n) < \frac{2n^2+4n}{n+1}$. This inequality implies that $2n+1-\frac{1}{n+1} < \sigma(n) < 2n+2-\frac{2}{n+1}$. Since $\sigma(n)$ is always a positive integer, therefore the last inequality implies that $\sigma(n) = 2n+1$. Hence n is a quasi perfect number.

Proposition 2.4. If n is a super hyperperfect number, then

$$1 < I(n) < \frac{3n+2}{2(n+1)}$$

Proof. If n is a super hyperperfect number, the n is a solution of the equation $\sigma(\sigma(n)) = \frac{3}{2}n+\frac{1}{2}$. If 3^{p-1} are the only solutions [2] of this equation, where both p and $\frac{3^p-1}{2}$ are primes, then for super hyperperfect numbers, $\sigma(n)$ are always primes and therefore $\sigma(\sigma(n)) = \sigma(n)+1$. From the last equation, we obtain $\sigma(n) = \frac{3n}{2}-\frac{1}{2}$. Therefore abundancy index, $I(n) = \frac{3}{2}-\frac{1}{2n}$. Clearly $I(n) > 1$. Moreover $\frac{3n+2}{2(n+1)} = \frac{3}{2}-\frac{1}{2(n+1)} = \frac{3}{2}-\frac{1}{2}(1-\frac{n}{n+1}) > I(n)$.

For super hyperperfect number, the lower of $I(n)$ can also be improved.

Proposition 2.5. Let n be a positive integer. Then n is a super hyperperfect number if

$$\frac{3n+1}{2(n+1)} < I(n) < \frac{3n+2}{2(n+1)}$$

Remark 2.3. The converse of proposition 2.4 and proposition 2.5 are not true in general if n is a super hyperperfect number. Any numbers of the form 3^k, where $k \geq 1$, satisfy the equation $\sigma(n) = \frac{3n}{2}-\frac{1}{2}$, but all such type of numbers do not satisfy the equation $\sigma(\sigma(n)) = \frac{3}{2}n+\frac{1}{2}$.

Proposition 2.6. If n is a near perfect number with redundant divisor d, then n satisfies the following inequality
$$2 < I(n) < \frac{2n+d+3}{n+1}$$

Proof. If n is a near perfect number with redundant divisor d, then $\sigma(n) = 2n+d$. Therefore abundancy index $I(n) = 2 + \frac{d}{n} > 2$. Next we prove $I(n) < \frac{2n+d+3}{n+1}$. On the contrary suppose that $I(n) \geq \frac{2n+d+3}{n+1}$, then from definition of $I(n)$, one obtains $\sigma(n) \geq \frac{2n^2+(d+3)n}{n+1}$. But $\frac{2n^2+(d+3)n}{n+1} = \frac{2n(n+1)+(n+1)(d+1)-(d+1)}{n+1} = 2n+(d+1) - \frac{d+1}{n+1}$.

Since d is a proper divisor of n, so $\frac{d+1}{n+1} < 1$. Therefore our assuming inequality becomes $2n+d = \sigma(n) \geq 2n+(d+1) - \frac{d+1}{n+1}$, which is clearly a contradiction. This contradiction strictly implies that $I(n) < \frac{2n+d+3}{n+1}$.

Remark 2.4. Lower bound of $I(n)$ for near perfect numbers can also be improved. Following proposition gives the improve lower bound of $I(n)$ in terms of rational function of n.

Proposition 2.7. If n is a near perfect number with redundant divisor d, then n satisfies the following inequality
$$\frac{2n+d+2}{n+1} < I(n) < \frac{2n+d+3}{n+1}$$

Proof. Since $\frac{2n^2+(d+2)n}{n+1} = \frac{2n(n+1)+d(n+1)-d}{n+1} = 2n+d - \frac{d}{n+1} < 2n+d = \sigma(n)$, therefore $\frac{2n+d+2}{n+1} < I(n)$. Combining this inequality with the right hand side inequality of the proposition 2.6, the result follows.

Remark 2.5. From definition of $I(n)$, the inequality $\frac{2n+d+2}{n+1} < I(n) < \frac{2n+d+3}{n+1}$ can be written as $\frac{2n^2+(d+2)n}{n+1} < \sigma(n) < \frac{2n^2+(d+3)n}{n+1}$, which implies that $2n+d - \frac{d}{n+1} < \sigma(n) < 2n+(d+1) - \frac{d+1}{n+1}$. But $\sigma(n)$ is always a positive integer and so from the last inequality, we must have $\sigma(n) = 2n+d$. Therefore the inequality obtains in the proposition 2.7 is also sufficient condition for n to be a near perfect number, provided d is a proper divisor of n.

We now have the following proposition.

Proposition 2.8. Let d be a proper divisor of a positive integer n. The positive integer n is near perfect number with redundant divisor d if and only if n satisfies the following inequality:
$$\frac{2n+d+2}{n+1} < I(n) < \frac{2n+d+3}{n+1}$$
.

Remark 2.6. In fact, proposition 2.8 can be used to define particular families of integers being near perfect numbers.

Proposition 2.9. Let n be a positive integer, then n is $2-$hyperperfect number if and only if the following inequality holds:
$$\frac{3n+4}{2(n+1)} < I(n) < \frac{3n+5}{2(n+1)}$$
.

Remark 2.7. It is trivial to verify that the all known $2-$hyperperfect numbers of the form $n = 3^{k-1}(3^k-2)$, where 3^k-2 are primes, satisfy the inequality obtained in proposition 2.9. In general, for any integer $k > 1$, one can determine lower and upper bounds of $I(n)$ for $k-$hyperperfect numbers.

Proposition 2.10. Let $k > 1$ be any integer, then a positive integer n is $k-$ hyperperfect number if and only if the following inequality holds:
$$\frac{n(k+1)+2k}{k(n+1)} < I(n) < \frac{n(k+1)+3k-1}{k(n+1)}$$
.

Proof. Let n be a positive integer, which satisfies the inequality
$$\frac{n(k+1)+2k}{k(n+1)} < I(n) < \frac{n(k+1)+3k-1}{k(n+1)}$$
, then
$$\frac{n^2(k+1)+2kn}{k(n+1)} < \sigma(n) < \frac{n^2(k+1)+3nk-n}{k(n+1)}$$
But
$\frac{n^2(k+1)+2kn}{k(n+1)} = \frac{n(k+1)(n+1)+nk-n}{k(n+1)} = \frac{n(k+1)}{k} + \frac{n(k-1)}{k(n+1)} = \frac{(k+1)n}{k} + \frac{k-1}{k}(1 - \frac{1}{n+1})$

Also
$\frac{n^2(k+1)+3nk-n}{k(n+1)} = \frac{n(k+1)(n+1)+2nk-2n}{k(n+1)} = \frac{n(k+1)}{k} + \frac{2n(k-1)}{k(n+1)} = \frac{(k+1)n}{k} + \frac{k-1}{k}(2 - \frac{2}{n+1})$

Therefore, we obtain $\frac{(k+1)n}{k} + \frac{k-1}{k}(1 - \frac{1}{n+1}) < \sigma(n) < \frac{(k+1)n}{k} + \frac{k-1}{k}(2 - \frac{2}{n+1})$
Since $\sigma(n)$ is a positive integer, so the last inequality strictly implies that $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k}$, i.e., n is a $k-$hyperperfect number.

Conversely, suppose that n is a $k-$hyperperfect number, then abundancy index $I(n) = \frac{k+1}{k} + \frac{k-1}{kn}$.

Clearly
$\frac{n(k+1)+2k}{k(n+1)} = \frac{(k+1)(n+1)+k-1}{k(n+1)} = \frac{k+1}{k} + \frac{k-1}{k(n+1)} < I(n)$
and
$\frac{n(k+1)+3k-1}{k(n+1)} = \frac{(k+1)(n+1)+2(k-1)}{k(n+1)} = \frac{k+1}{k} + \frac{2(k-1)}{k(n+1)} > I(n)$

Remark 2.8. In fact, proposition 2.10 can be used to define particular families of integers being $k-$hyperperfect numbers.

3 Conclusion

We considered abundancy index of positive integers and determined lower and upper bounds of abundancy index for quasi perfect, super hyperperfect, near perfect and $k-$hyperperfect numbers. Finally, we proved that quasi perfect, near perfect and $k-$hyperperfect numbers can be defined in terms of abundancy index.

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