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Characterization for Generalized Exponential Distribution

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Abstract: In this work, a characterization of the probability density function of the general progressively Type-II right censored (GPTIIRCOS) exponential distribution is established and recurrence relations for the generalized exponential distribution are established (GED). Furthermore, the resulted equation has a very specialized trend to the progressively Type-II right censored ordered statistics (PTIIRCOS).

Keywords: Characterization; Generalized Exponential Distribution; General Progressively Type-II; Recurrence Relations.

1 Introduction

Many three-parameter models are used to analyze lifetime data or skewed data. The most preferable distributions are the three-parameter gamma and the three-parameter Weibull distributions since their scale and shape parameters have flexibility to resolve different types of life time data. Moreover they have increasing and decreasing hazard rate depending as the shape parameter. However, due to some deficiencies the generalized exponential model was firstly proposed to overcome their in adequacies by [1]. Yet, in reliability and life testing experiments, the data are oftentimes censored. Many schemes were introduced to model these types of data see for instance the look by [2.3]. For some recent references one can refer to [4,5] and see the references cited therein. In this paper we consider the GPTIIRCOS scheme, where it is possible to withdraw live it ems during the experiment.

In this paper we are interested to establish and characterize the GED based on GPTIIRCOS, which is conducted as follows: The times of failure for the first p failures are not observed, and the $(p+1)^{th}$,..., $(q)^{th}$ failures are observed at which times $D_{p+1}, D_{p+2}, \dots, D_q = n - q - \sum_{i=1}^{q-1} D_i$ surviving units are censored. The resulting $(q-p)$ ordered failure times are referred to as GPTIIRCOS scheme, see [6].

If we have a lifetime failures say $X_{p+1:m:,n}, X_{p+2:m:,n}, \dots, X_{m:m:,n}$, that follows a specific distribution with cumulative distribution function (cdf) F with and have probability density function (pdf) f, then we can have the joint probability density function of GPTIIRCOS, as follows

$$f_{X_{p+1:q:,n}, X_{p+2:q:,n}, \dots, X_{q:q:,n}}(x_{p+1}, x_{p+2}, \dots, x_q) = C_{(n,q-1)} \times$$

$$[F(x_{p+1}, \theta)]^k \prod_{i=p+1}^{q} f(x_i, \theta)[1 - F(x_i, \theta)]^{R_i},$$

$$x_{p+1} < x_{p+2} < \dots < x_q, \qquad (1)$$

where,

$$C_{(n,q-1)} = \frac{n!}{p!\,(n-p-1)!}\left(\prod_{j=p+2}^{q} n - \sum_{i=p+1}^{j-1} D_i - j + 1\right).$$

[7] derived characterization for generalized power function distribution using recurrence relations based on GPTIIRCOS.

This paper brings about an introduction of Recurrence Relations among single and Product moment (SPM) that are from the GPTIIRCOS. Characterization of GED using a recurrence relation between SPM of GPTIIRCOS is obtained.

Let $X_{p+1:q:n}^{(D_{p+1}, D_{p+2}, \dots, D_q)} < X_{p+2:q:n}^{(D_{p+1}, \dots, D_q)} < \dots < X_{q:q:n}^{(D_{p+1}, \dots, D_q)}$

When the random variables in a sample of size n-p are ordered and observed, the expected failure times in block of

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size n-p under GPTIIRCOS scheme are given by.

$$f(x,\alpha,\mu,\lambda)=\lambda\alpha e^{-\lambda(x-\mu)}\left[1-e^{-\lambda(x-\mu)}\right]^{\alpha-1},\quad 0<\mu\leq x,\quad \alpha,\lambda>0. \tag{2}$$

To find the cumulative distribution function (cdf), we first need the following

$$F(x,\alpha,\mu,\lambda)=\left[1-e^{-\lambda(x-\mu)}\right]^{\alpha}. \tag{3}$$

From (2) and (3) it can be seen that the relation between the pdf and the corresponding cdf is given by.

$$\{e^{\lambda(x-\mu)}-1\}f(x)=\lambda\alpha F(x). \tag{4}$$

For such a continuous distribution, the probability of the i^{th} possible moment can be computed using the GPTIIRCOS statistics in view of Eq. (1) is defined as .

$$\mu_{p:q:n}^{(D_{p+1},D_{p+2},\ldots,D_q)^{(i)}}=E\left[X_{p:q:n}^{(D_{p+1},D_{p+2},\ldots,D_q)}\right]^i=$$
$$C_{(n,q-1)}\iint\cdots\int_{0<x_{p+1}<\cdots<x_q<\infty}x_q^i\left[F(x_{p+1})\right]^p f(x_{p+1})\times$$
$$\left[1-F(x_{p+1})\right]^{D_{p+1}}f(x_{p+2})\left[1-F(x_{p+2})\right]^{D_{p+2}}\ldots\times$$
$$f(x_q)\left[1-F(x_q)\right]^{D_q}dx_{p+1}\ldots dx_q, \tag{5}$$

and the i^{th} and j^{th} product moments as

$$\mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\ldots,D_q)^{(i,j)}}=E\left[X_{u:q:n}^{(D_{p+1},\ldots,D_q)^i}X_{v:q:n}^{(D_{p+1},\ldots,D_q)^j}\right]=$$
$$C_{(n,q-1)}\iint\cdots\int_{0<x_{p+1}<\cdots<x_q<\infty}x_u^i x_v^j\left[F(x_{p+1})\right]^p f(x_{p+1})\times$$
$$\left[1-F(x_{p+1})\right]^{D_{p+1}}f(x_{p+2})\left[1-F(x_{p+2})\right]^{D_{p+2}}\ldots\times$$
$$f(x_q)\left[1-F(x_q)\right]^{D_q}dx_{p+1}\ldots dx_q. \tag{6}$$

2 The Recurrence Relations of SPM

In this section we introduce the recurrence relations for SPM based on GPTIIRCOS.
Next, we will derive various mathematical relations for a single moment from GPTIIRCOS.

Theorem 1

Let $X_{p+1:n}\leq X_{p+2:n}\leq\ldots\leq X_{n:n}$ be a sample of failure times from specific order sample with size of $(n-p)$ following GED, for $p+2\leq \mathrm{u}\leq q-1,\ q\leq n\ and\ i\geq 0$, then

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\ldots,D_q)^{(i+k)}}-\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\ldots,D_q)^{(i)}}=$$
$$\frac{\lambda\alpha(R_{\mathrm{u}})}{i+1}\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\ldots,D_{\mathrm{u}-1},(D_{\mathrm{u}}-1),D_{\mathrm{u}+1},\ldots,D_q)^{(i+1)}}$$
$$-\frac{\lambda\alpha(D_{\mathrm{u}}+1)}{i+1}\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\ldots,D_q)^{(i+1)}}$$
$$-\frac{\lambda\alpha}{i+1}\left(n-D_{p+1}-\cdots-D_{\mathrm{u}-1}-\mathrm{u}+1\right)\times$$
$$\mu_{\mathrm{u}-1:q-1:n}^{(D_{p+1},\ldots,D_{\mathrm{u}-2},(D_{\mathrm{u}-1}+D_{\mathrm{u}}),D_{\mathrm{u}+1},\ldots,D_q)^{(i+1)}}$$
$$+\frac{\lambda\alpha}{i+1}\left(n-D_{p+1}-D_{p+2}-\cdots-D_{\mathrm{u}}-\mathrm{u}\right)\times$$
$$\mu_{\mathrm{u}:q-1:n}^{(D_{p+1},\ldots,D_{\mathrm{u}-1},(D_{\mathrm{u}}+D_{\mathrm{u}+1}),D_{\mathrm{u}+2},\ldots,D_q)^{(i+1)}}$$
$$+\frac{\lambda\alpha}{i+1}\left(n-D_{p+1}-\cdots-D_{\mathrm{u}-1}-\mathrm{u}+1\right)\times$$
$$\mu_{\mathrm{u}-1:q-1:n}^{(D_{p+1},\ldots,D_{\mathrm{u}-2},(D_{\mathrm{u}-1}+D_{\mathrm{u}}+1),D_{\mathrm{u}+1},\ldots,D_q)^{(i+1)}}$$
$$-\frac{\lambda\alpha}{i+1}\left(n-D_{p+1}-D_{p+2}-\cdots-D_{\mathrm{u}}-\mathrm{u}\right)\times$$
$$\mu_{\mathrm{u}:q-1:n}^{(D_{p+1},\ldots,D_{\mathrm{u}-1},(D_{\mathrm{u}}+D_{\mathrm{u}+1}+1),D_{\mathrm{u}+2},\ldots,D_q)^{(i+1)}}. \tag{7}$$

Proof

From Eq. (4) and Eq. (5), we get

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\ldots,D_q)^{(i+k)}}-\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\ldots,D_q)^{(i)}}=$$
$$C_{(n,q-1)}\iint\ldots\int_{0<x_{p+1}<\cdots<x_{\mathrm{u}-1}<x_{\mathrm{u}+1}<\cdots<x_q<\infty}\left[F(x_{p+1})\right]^p\times$$
$$K_1(x_{\mathrm{u}-1},x_{\mathrm{u}+1})\ldots f(x_{p+1})\left[1-F(x_{p+1})\right]^{D_{p+1}}\ldots f(x_{\mathrm{u}-1})\times$$
$$[1-F(x_{\mathrm{u}-1})]^{D_{\mathrm{u}-1}}f(x_{\mathrm{u}+1})[1-F(x_{\mathrm{u}+1})]^{D_{\mathrm{u}+1}}\ldots\times$$
$$f(x_q)\left[1-F(x_q)\right]^{D_q}dx_{p+1}dx_{p+2}\ldots dx_{\mathrm{u}-1}dx_{\mathrm{u}+1}\ldots dx_q, \tag{8}$$

where

$$K_1(x_{\mathrm{u}-1},x_{\mathrm{u}+1})=\lambda\alpha\int_{x_{u-1}}^{x_{\mathrm{u}+1}}x_{\mathrm{u}}^i\times\{[1-F(x_{\mathrm{u}})]^{D_{\mathrm{u}}}-[1-F(x_{\mathrm{u}})]^{D_u+1}\}\,dx_{\mathrm{u}}. \tag{9}$$

Now, integrating by parts gives

$$K_1(x_{u-1},x_{u+1})=\frac{\lambda\alpha x_{u+1}^{i+1}[1-F(x_{u+1})]^{D_u}-\lambda\alpha x_{u-1}^{i+1}[1-F(x_{u-1})]^{D_u}}{i+1}$$
$$+\lambda\alpha\left(\frac{D_u}{i+1}\right)\int_{x_{u-1}}^{x_{u+1}}x_u^{i+1}f(x_u)[1-F(x_u)]^{D_u-1}dx_u$$

$$+\frac{\lambda\alpha x_{u-1}^{i+1}[1-F(x_{u-1})]^{D_u+1}-\lambda\alpha x_{u+1}^{i+1}[1-F(x_{u+1})]^{D_u+1}}{i+1}$$

$$-\lambda\alpha\left(\frac{D_u+1}{i+1}\right)\int_{x_{u-1}}^{x_{u+1}} x_u^{i+1} f(x_u)[1-F(x_u)]^{D_u} dx_u. \quad (10)$$

Now substituting for the resultant expression of $K_1(x_{u-1},x_{u+1})$ from Eq. (10) in Eq. (8) and simplifying, yields Eq. (7). This completes the proof.

Special case

Theorem (1) will be valid for the PTIIRCOS as a special case from the GPTIIRCOS when $p=0$,

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mu_{u:q:n}^{(D_1,D_2,\dots,D_q)^{(i+k)}}-\mu_{u:q:n}^{(D_1,D_2,\dots,D_q)^{(i)}}=$$
$$\frac{\lambda\alpha(D_u)}{i+1}\mu_{u:m:n}^{(D_1,D_2,\dots,D_{u-1},(D_u-1),D_{u+1},\dots,D_q)^{(i+1)}}$$
$$-\frac{\lambda\alpha(D_u+1)}{i+1}\mu_{u:q:n}^{(D_1,D_2,\dots,D_q)^{(i+1)}}$$
$$-\frac{\lambda\alpha}{i+1}(n-D_1-\dots-D_{u-1}-u+1)\times$$
$$\mu_{u-1:q-1:n}^{(D_1,\dots,D_{u-2},(D_{u-1}+D_u),D_{u+1},\dots,D_q)^{(i+1)}}$$
$$+\frac{\lambda\alpha}{i+1}(n-D_1-D_2-\dots-D_u-u)\times$$
$$\mu_{u:q-1:n}^{(D_1,\dots,D_{u-1},(D_u+D_{u+1}),D_{u+2},\dots,D_q)^{(i+1)}}$$
$$+\frac{\lambda\alpha}{i+1}(n-D_1-\dots-D_{u-1}-u+1)\times$$
$$\mu_{u-1:q-1:n}^{(D_1,\dots,D_{u-2},(D_{u-1}+D_u+1),D_{u+1},\dots,D_q)^{(i+1)}}$$
$$-\frac{\lambda\alpha}{i+1}(n-D_1-D_2-\dots-D_u-u)\times$$
$$\mu_{u:q-1:n}^{(D_1,\dots,D_{u-1},(D_u+D_{u+1}+1),D_{u+2},\dots,D_q)^{(i+1)}}.$$

In the next two theorems we introduce recurrence relations for product moments based on GPTIIRCOS

Theorem 2

Let $X_{p+1:n}\le X_{p+2:n}\le\dots\le X_{n:n}$ be the order stasistics of a random sample of size $(n-p)$ following GED, for $p+1\le u<v\le m-1, m\le n$ then *and* $i,j\ge 0$,

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i+k,j)}}-\mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i,j)}}=$$
$$\frac{\lambda\alpha(D_u)}{i+1}\mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\dots,D_{u-1},(D_u-1),D_{u+1},\dots,D_q)^{(i+1,j)}}$$
$$-\frac{\lambda\alpha(D_u+1)}{i+1}\mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i+1,j)}}$$
$$-\frac{\lambda\alpha}{i+1}(n-D_{p+1}-\dots-D_{u-1}-u+1)\times$$
$$\mu_{u-1,v-1:q-1:n}^{(D_{p+1},\dots,D_{u-2},(D_{u-1}+D_u),D_{u+1},\dots,D_q)^{(i+1,j)}}$$
$$+\frac{\lambda\alpha}{i+1}(n-D_{p+1}-D_{p+2}-\dots-D_u-u)\times$$
$$\mu_{u,v-1:q-1:n}^{(D_{p+1},\dots,D_{u-1},(D_u+D_{u+1}),D_{u+2},\dots,D_q)^{(i+1,j)}}$$
$$+\frac{\lambda\alpha}{i+1}(n-D_{p+1}-\dots-D_{u-1}-u+1)\times$$
$$\mu_{u-1,v-1:q-1:n}^{(D_{p+1},\dots,D_{u-2},(D_{u-1}+D_u+1),D_{u+1},\dots,D_q)^{(i+1,j)}}$$
$$-\frac{\lambda\alpha}{i+1}(n-D_{p+1}-D_{p+2}-\dots-D_u-u)\times$$
$$\mu_{u,v-1:q-1:n}^{(D_{p+1},\dots,D_{u-1},(D_u+D_{u+1}+1),D_{u+2},\dots,D_q)^{(i+1,j)}}. \quad (11)$$

Proof

From Eq. (6), we get

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mu_{u,v-1:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i+k,j)}}-\mu_{u,v-1:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i,j)}}=$$
$$C_{(n,q-1)}\iint\dots\int_{0<x_{p+1}<\dots<x_{u-1}<x_{u+1}<\dots<x_q<\infty} x_v^j\left[F(x_{p+1})\right]^p$$
$$K_1(x_{u-1},x_{u+1})f(x_{p+1})\left[1-F(x_{p+1})\right]^{D_{p+1}}\dots f(x_{u-1})\times$$
$$[1-F(x_{u-1})]^{D_{u-1}}f(x_{u+1})[1-F(x_{u+1})]^{D_{u+1}}\dots f(x_q)\times$$
$$\left[1-F(x_q)\right]^{D_q}dx_{p+1}dx_{p+2}\dots dx_{u-1}dx_{u+1}\dots dx_q, \quad (12)$$

Substituting the resultant expression of $K_1(x_{u-1},x_{u+1})$ from Eq. (9) in Eq. (12) and simplifying, yields Eq. (11). This completes the proof.

Theorem 3

Let $X_{p+1:n}\le X_{p+2:n}\le\dots\le X_{n:n}$ be the order stasistics of a random sample of size $(n-p)$ following GED, for $p+1\le u<v\le q-1$, $m\le n$ *and* $i,j\ge 0$, then

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i,j+k)}}-\mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i,j)}}=$$
$$\frac{\lambda\alpha(D_v)}{i+1}\mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\dots,D_{v-1},(D_v-1),D_{v+1},\dots,D_q)^{(i,j+1)}}$$
$$-\frac{\lambda\alpha(D_v+1)}{i+1}\mu_{u,vq:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i,j+1)}}$$
$$-\frac{\lambda\alpha}{j+1}(n-D_{p+1}-\dots-D_{v-1}-v+1)\times$$
$$\mu_{u,v-1:q-1:n}^{(D_{p+1},\dots,R_{v-2},(R_{v-1}+R_v),R_{v+1},\dots,R_q)^{(i,j+1)}}$$
$$+\frac{\lambda\alpha}{j+1}(n-D_{p+1}-D_{p+2}-\dots-D_v-v)\times$$
$$\mu_{u,v:q-1:n}^{(D_{p+1},\dots,D_{v-1},(D_v+D_{v+1}),D_{v+2},\dots,D_q)^{(i,j+1)}}$$
$$+\frac{\lambda\alpha}{j+1}(n-D_{p+1}-\dots-D_{v-1}-v+1)\times$$
$$\mu_{u,v-1:q-1:n}^{(D_{p+1},\dots,D_{v-2},(D_{v-1}+D_v+1),D_{v+1},\dots,D_q)^{(i,j+1)}}$$
$$-\frac{\lambda\alpha}{j+1}(n-D_{p+1}-D_{p+2}-\dots-D_v-v)\times$$

$$\mu_{u,v:q-1:n}^{(D_{p+1},\dots,D_{v-1},(D_v+D_{v+1}+1),D_{v+2},\dots,D_q)^{(i,j+1)}}. \quad (13)$$

Proof
Similarly as proved in theorem 1.

3 The Characterizations

In this section we introduce the characterizations of the GED using the relation between pdf and cdf and using recurrence relations for SPM based on GPTIIRCOS.

3.1 Characterization Via Differential Equation for GED

In the next theorem we introduce the characterization of the GED using relation between pdf and cdf.

Theorem 4
Let X be a continuous random variable with pdf $f(\cdot)$ and cdf $F(\cdot)$. Then X has GED iff

$$\{e^{\lambda(x-\mu)}-1\}f(x)=\lambda\alpha[F(x)]. \qquad x\geq 0 \quad (14)$$

Proof
Necessity:
From Eq. (2) and Eq. (3) we can easily obtain Eq. (14).
Sufficiency:
Suppose that X is a continuous random variable with pdf $f(\cdot)$ and cdf $F(\cdot)$. Suppose, also, that Eq. (14) is true. Then we have:

$$\frac{d[F(x)]}{F(x)}=\frac{\lambda\alpha}{e^{\lambda(x-\mu)}-1}dx$$
$$=\frac{\lambda\alpha e^{-\lambda(x-\mu)}}{1-e^{-\lambda(x-\mu)}}dx.$$

On integrating, we get

$$\ln|F(x)|=\alpha\ln|1-e^{-\lambda(x-\mu)}|+C,$$

where C is an arbitrary constant.

Now, since $[F(\infty)]=1$, then putting $x\to\infty$ in this equation, we get $C=0$.

Therefore,

$$\ln|F(x)|=\ln\{1-e^{-\lambda(x-\mu)}\}^{\alpha}.$$

Hence,

$$F(x)=\{1-e^{-\lambda(x-\mu)}\}^{\alpha}.$$

That is the distribution function of GED. This completes the proof.

3.2 Characterization Via Recurrence Relations For Single Moments

In the next theorem we introduce the characterization of the GED using recurrence relations for single moment based on GPTIIRCOS has introduced in the following theorems.

Theorem 5
Let $X_{p+1:n}\leq X_{p+2:n}\leq\dots\leq X_{n:n}$ be the order stasistics of a random sample of size $(n-p)$. Then X has GED iff, for $p+2\leq u\leq q-1,\ q\leq n\ and\ i\geq 0,$

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i+k)}}-\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i)}}=$$
$$\frac{\lambda\alpha(R_{\mathrm{u}})}{i+1}\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\dots,D_{\mathrm{u}-1},(D_{\mathrm{u}}-1),D_{\mathrm{u}+1},\dots,D_q)^{(i+1)}}$$
$$-\frac{\lambda\alpha(D_{\mathrm{u}}+1)}{i+1}\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i+1)}}$$
$$-\frac{\lambda\alpha}{i+1}\left(n-D_{p+1}-\dots-D_{\mathrm{u}-1}-\mathrm{u}+1\right)\times$$
$$\mu_{\mathrm{u}-1:q-1:n}^{(D_{p+1},\dots,D_{\mathrm{u}-2},(D_{\mathrm{u}-1}+D_{\mathrm{u}}),D_{\mathrm{u}+1},\dots,D_q)^{(i+1)}}$$
$$+\frac{\lambda\alpha}{i+1}\left(n-D_{p+1}-D_{p+2}-\dots-D_{\mathrm{u}}-\mathrm{u}\right)\times$$
$$\mu_{\mathrm{u}:q-1:n}^{(D_{p+1},\dots,D_{\mathrm{u}-1},(D_{\mathrm{u}}+D_{\mathrm{u}+1}),D_{\mathrm{u}+2},\dots,D_q)^{(i+1)}}$$
$$+\frac{\lambda\alpha}{i+1}\left(n-D_{p+1}-\dots-D_{\mathrm{u}-1}-\mathrm{u}+1\right)\times$$
$$\mu_{\mathrm{u}-1:q-1:n}^{(D_{p+1},\dots,D_{\mathrm{u}-2},(D_{\mathrm{u}-1}+D_{\mathrm{u}}+1),D_{\mathrm{u}+1},\dots,D_q)^{(i+1)}}$$
$$-\frac{\lambda\alpha}{i+1}\left(n-D_{p+1}-D_{p+2}-\dots-D_{\mathrm{u}}-\mathrm{u}\right)\times$$
$$\mu_{\mathrm{u}:q-1:n}^{(D_{p+1},\dots,D_{\mathrm{u}-1},(D_{\mathrm{u}}+D_{\mathrm{u}+1}+1),D_{\mathrm{u}+2},\dots,D_q)^{(i+1)}}. \quad (15)$$

Proof
Necessity:
Theorem 1 proved the necessary part of this theorem
Sufficiency:
Suppose that X is a continuous random variable with pdf $f(\cdot)$ and cdf $F(\cdot)$. Assuming that equation (15) holds, then we have:

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i+k)}}-\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i)}}=$$
$$\frac{\lambda\alpha(R_{\mathrm{u}})}{i+1}\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\dots,D_{\mathrm{u}-1},(D_{\mathrm{u}}-1),D_{\mathrm{u}+1},\dots,D_q)^{(i+1)}}$$
$$-\frac{\lambda\alpha(D_{\mathrm{u}}+1)}{i+1}\mu_{\mathrm{u}:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i+1)}}$$
$$-\frac{\lambda\alpha}{i+1}\left(n-D_{p+1}-\dots-D_{\mathrm{u}-1}-\mathrm{u}+1\right)\times$$
$$\mu_{\mathrm{u}-1:q-1:n}^{(D_{p+1},\dots,D_{\mathrm{u}-2},(D_{\mathrm{u}-1}+D_{\mathrm{u}}),D_{\mathrm{u}+1},\dots,D_q)^{(i+1)}}$$
$$+\frac{\lambda\alpha}{i+1}\left(n-D_{p+1}-D_{p+2}-\dots-D_{\mathrm{u}}-\mathrm{u}\right)\times$$
$$\mu_{\mathrm{u}:q-1:n}^{(D_{p+1},\dots,D_{\mathrm{u}-1},(D_{\mathrm{u}}+D_{\mathrm{u}+1}),D_{\mathrm{u}+2},\dots,D_q)^{(i+1)}}$$

$$+\frac{\lambda\alpha}{i+1}\left(n-D_{p+1}-\cdots-D_{u-1}-u+1\right)\times \mu_{u-1:q-1:n}^{(D_{p+1},\ldots,D_{u-2},(D_{u-1}+D_u+1),D_{u+1},\ldots,D_q)^{(i+1)}}$$
$$-\frac{\lambda\alpha}{i+1}\left(n-D_{p+1}-D_{p+2}-\cdots-D_u-u\right)\times \mu_{u:q-1:n}^{(D_{p+1},\ldots,D_{u-1},(D_u+D_{u+1}+1),D_{u+2},\ldots,D_q)^{(i+1)}}, \quad (16)$$

where,

$$\mu_{u:q:n}^{(D_{p+1},\ldots,D_q)^{(i+1)}}= C_{(n,q-1)}\iint\cdots\int_{0<x_{p+1}<\cdots<x_{u-1}<x_{u+1}<\cdots<x_q<\infty}[F(x_{p+1})]^p\times K_2(x_{u-1},x_{u+1})f(x_{p+1})[1-F(x_{p+1})]^{D_{p+1}}\ldots f(x_{u-1}) [1-F(x_{u-1})]^{D_{u-1}}f(x_{u+1})[1-F(x_{u+1})]^{D_{u+1}}\ldots\times f(x_q)[1-F(x_q)]^{D_m}dx_{p+1}\ldots dx_{u-1}dx_{u+1}\ldots dx_q, \quad (17)$$

where

$$K_2(x_{u-1},x_{u+1})=\int_{x_{u-1}}^{x_{u+1}}x_u^{i+1}f(x_u)[1-F(x_u)]^{D_u}dx_u. \quad (18)$$

Now, integrating by parts gives

$$K_2(x_{u-1},x_{u+1})=\frac{-1}{D_u+1}x_{u+1}^{i+1}[1-F(x_{u+1})]^{D_u+1} +\frac{1}{D_u+1}x_{u-1}^{i+1}[1-F(x_{u-1})]^{D_u+1} +\frac{i+1}{D_u+1}\int_{x_{u-1}}^{x_{u+1}}x_u^{i}[1-F(x_u)]^{D_u+1}dx_u. \quad (19)$$

Now by substituting in Eq. (17), we get

$$\mu_{u:q:n}^{(D_{p+1},\ldots,D_q)^{(i+1)}}= \frac{i+1}{D_u+1}C_{(n,q-1)}\iint\cdots\int_{0<x_{p+1}<\cdots<x_{u-1}<x_{u+1}<\cdots<x_q<\infty}[F(x_{p+1})]^p f(x_{p+1})[1-F(x_{p+1})]^{D_{p+1}}\ldots\times \int_{x_{u-1}}^{x_{u+1}}x_u^{i}[1-F(x_u)]^{D_u+1}dx_u\, f(x_{u-1})[1-F(x_{u-1})]^{D_{u-1}}\times f(x_{u+1})[1-F(x_{u+1})]^{D_{u+1}}\ldots f(x_q)[1-F(x_q)]^{D_q}\times dx_{p+1}\ldots dx_{u-1}dx_{u+1}\ldots dx_q$$
$$+\frac{C_{(n,q-1)}}{D_u+1}\iint\cdots\int_{0<x_{p+1}<\cdots<x_{u-1}<x_{u+1}<\cdots<x_q<\infty}x_{u-1}^{i+1}\times [F(x_{p+1})]^p f(x_{p+1})[1-F(x_{p+1})]^{D_{p+1}}\ldots f(x_{u-1})\times [1-F(x_{u-1})]^{1+D_u+D_{u-1}}f(x_{u+1})[1-F(x_{u+1})]^{D_{u+1}}\ldots\times f(x_q)[1-F(x_q)]^{D_m}dx_{p+1}\ldots dx_{u-1}dx_{u+1}\ldots dx_q$$
$$-\frac{C_{(n,q-1)}}{D_u+1}\iint\cdots\int_{0<x_{p+1}<\cdots<x_{u-1}<x_{u+1}<\cdots<x_q<\infty}x_{u+1}^{i+1}\times [F(x_{p+1})]^p f(x_{p+1})[1-F(x_{p+1})]^{D_{p+1}}\ldots f(x_{u-1})\times [1-F(x_{u-1})]^{D_{u-1}}f(x_{u+1})[1-F(x_{u+1})]^{1+D_u+D_{u+1}}\ldots\times f(x_q)[1-F(x_q)]^{D_q}dx_{p+1}\ldots dx_{u-1}dx_{u+1}\ldots dx_q$$
$$= C_{(n,q-1)}\frac{i+1}{D_u+1}\times\iint\cdots\int_{0<x_{p+1}<\cdots<x_{u-1}<x_{u+1}<\cdots<x_q<\infty}[F(x_{p+1})]^p f(x_{p+1}) [1-F(x_{p+1})]^{D_{p+1}}\ldots\int_{x_{u-1}}^{x_{u+1}}x_u^{i}[1-F(x_u)]^{D_u+1}dx_u$$
$$f(x_{u-1})[1-F(x_{u-1})]^{D_{u-1}}f(x_{u+1})[1-F(x_{u+1})]^{D_{u+1}}\times \ldots f(x_q)[1-F(x_q)]^{D_u}dx_{p+1}\ldots dx_{u-1}dx_{u+1}\ldots dx_q$$
$$+\frac{(n-D_{p+1}-\cdots-D_u-u)}{D_u+1}\times \mu_{u:q-1:n}^{(D_{p+1},\ldots D_{u-1},(D_u+D_{u+1}+1),D_{u+2},\ldots,D_u)^{(i+1)}} -\frac{(n-D_{p+1}-\cdots-D_{u-1}-u+1)}{D_u+1}\times \mu_{u-1:q-1:n}^{(D_{p+1},\ldots D_{u-2},(D_{u-1}+D_u+1),D_{u+1},\ldots,D_q)^{(i+1)}}, \quad (20)$$

and

$$\mu_{u:q:n}^{(D_{p+1},D_{p+2},\ldots,D_{u-1},(D_u-1),D_{u+1},\ldots,D_q)^{(i+1)}}= C_{(n,q-1)}\frac{i+1}{D_u}\iint\cdots\int_{0<x_{p+1}<\cdots<x_{u-1}<x_{u+1}<\cdots<x_q<\infty}[F(x_{p+1})]^p f(x_{p+1})[1-F(x_{p+1})]^{D_{p+1}}\ldots\times \int_{x_{u-1}}^{x_{u+1}}x_u^{i}[1-F(x_u)]^{D_u}dx_u\, f(x_{u-1})[1-F(x_{u-1})]^{D_{u-1}}\times f(x_{u+1})[1-F(x_{u+1})]^{D_{u+1}}\ldots\times f(x_q)[1-F(x_q)]^{D_q}dx_{p+1}\ldots dx_{u-1}dx_{u+1}\ldots dx_q$$
$$+\frac{(n-D_{p+1}-\cdots-D_u-u)}{D_u}\times \mu_{u:q-1:n}^{(D_{p+1},\ldots D_{u-1},(D_u+D_{u+1})D_{u+2},\ldots,D_q)^{(i+1)}} -\frac{(n-D_{p+1}-\cdots-D_{u-1}-u+1)}{D_u}\times \mu_{u-1:q-1:n}^{(D_{p+1},\ldots D_{u-2},(D_{u-1}+D_u),D_{u+1},\ldots,D_q)^{(i+1)}}. \quad (21)$$

Now by substituting for $\mu_{u:q:n}^{(D_{p+1},D_{p+2},\ldots,D_q)^{(i+1)}}$ *and* $\mu_{u:q:n}^{(D_{p+1},D_{p+2},\ldots,D_{u-1},(D_u-1),D_{u+1},\ldots,D_q)^{(i+1)}}$ from Eq. (20) and Eq. (21) in Eq. (316), we get

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mu_{u:q:n}^{(D_{p+1},D_{p+2},\ldots,D_q)^{(i+k)}}-\mu_{u:q:n}^{(D_{p+1},D_{p+2},\ldots,D_q)^{(i)}}= \lambda\alpha C_{(n,q-1)}\iint\cdots\int_{0<x_{p+1}<\cdots<x_{u-1}<x_{u+1}<\cdots<x_q<\infty}[F(x_{p+1})]^p f(x_{p+1})[1-F(x_{p+1})]^{D_{p+1}}\ldots\times \int_{x_{u-1}}^{x_{u+1}}x_u^{i}[1-F(x_u)]^{D_u+1}dx_u\, f(x_{u-1})[1-F(x_{u-1})]^{D_{u-1}}$$

$$f(x_{u+1})[1-F(x_{u+1})]^{D_{u+1}} \dots \times$$
$$f(x_q)[1-F(x_q)]^{D_q} dx_{p+1} \dots dx_{u-1} dx_{u+1} \dots dx_q$$
$$-\lambda\alpha C_{(n,q-1)} \iint \dots \int_{0<x_{p+1}<\cdots<x_{u-1}<x_{u+1}<\cdots<x_q<\infty} [F(x_{p+1})]^p$$
$$f(x_{p+1})[1-F(x_{p+1})]^{D_{p+1}} \dots \times$$
$$\int_{x_{u-1}}^{x_{u+1}} x_u^i [1-F(x_u)]^{D_u} dx_u\, f(x_{u-1})[1-F(x_{u-1})]^{D_{u-1}}$$
$$f(x_{u+1})[1-F(x_{u+1})]^{D_{u+1}} \dots \times$$
$$f(x_q)[1-F(x_q)]^{D_q} dx_{p+1} \dots dx_{u-1} dx_{u+1} \dots dx_q. \qquad (22)$$

We get

$$C_{(n,q-1)} \iint \dots \int_{0<x_{p+1}<\cdots<x_q<\infty} x_u^i \times$$
$$\left[\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(x_u^k) - 1\right] f(x_u)[1-F(x_u)]^{D_u} \times$$
$$[F(x_{p+1})]^p f(x_{p+1})[1-F(x_{p+1})]^{D_{p+1}} \dots \times$$
$$f(x_{u-1})[1-F(x_{u-1})]^{D_{u-1}} \times$$
$$f(x_{u+1})[1-F(x_{u+1})]^{D_{u+1}} \dots \times$$
$$f(x_q)[1-F(x_q)]^{D_q} dx_{p+1} \dots dx_q$$
$$= \lambda\alpha C_{(n,q-1)} \times$$
$$\iint \dots \int_{0<x_{p+1}<\cdots<x_{u-1}<x_{u+1}<\cdots<x_q<\infty} [F(x_{p+1})]^p \times$$
$$[1-F(x_u)]^{D_u} F(x_u) \dots f(x_{p+1})[1-F(x_{p+1})]^{D_{p+1}} \dots \times$$
$$f(x_{u-1})[1-F(x_{u-1})]^{D_{u-1}} f(x_{u+1})[1-F(x_{u+1})]^{D_{u+1}} \dots \times$$
$$f(x_q)[1-F(x_q)]^{D_q} dx_{p+1} dx_{p+2} \dots dx_{u-1} dx_{u+1} \dots dx_q.$$

We get

$$C_{(n,q-1)} \iint \dots \int_{0<x_{p+1}<\cdots<x_q<\infty} x_u^i \times$$
$$f(x_u)[1-F(x_u)]^{D_u} [F(x_{p+1})]^p \times$$
$$\left[\left\{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(x_u^k) - 1\right\} f(x_u) - \lambda\alpha F(x_u)\right] \times$$
$$f(x_{p+1})[1-F(x_{p+1})]^{D_{p+1}} \dots \times$$
$$f(x_{u-1})[1-F(x_{u-1})]^{D_{u-1}} f(x_{u+1})[1-F(x_{u+1})]^{D_{u+1}}$$
$$\times \dots f(x_q)[1-F(x_q)]^{D_q} dx_{p+1} \dots dx_q = 0$$

Using Muntz-Szasz theorem, [8], we get

$$\left\{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(x_u^k) - 1\right\} f(x_u) = \lambda\alpha F(x_u).$$

Using Theorem 4, we get

$$F(x) = \left[1 - e^{-\lambda(x-\mu)}\right]^{\alpha}.$$

That is the distribution function of GED. This completes the proof.

3.3 Characterization Via Recurrence Relations For Product Moments

In this section we characterize the GED using recurrence relation for product moments based on GPTIIRCOS.

Theorem 6

Let $X_{p+1:n} \leq \dots \leq X_{n:n}$ be the order stasistics of a random sample of size $(n-p)$. Then X has GED iff, for $p+1 \leq u < v \leq q-1, \quad q \leq n \quad and \quad i,j \geq 0,$

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mu_{u.v:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i+k,j)}} - \mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i,j)}} =$$
$$\frac{\lambda\alpha(D_u)}{i+1} \mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\dots,D_{u-1},(D_u-1),D_{u+1},\dots,D_q)^{(i+1,j)}}$$
$$-\frac{\lambda\alpha(D_u+1)}{i+1} \mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i+1,j)}}$$
$$-\frac{\lambda\alpha}{i+1}\left(n - D_{p+1} - \cdots - D_{u-1} - u + 1\right) \times$$
$$\mu_{u-1,v-1:q-1:n}^{(D_{p+1},\dots,D_{u-2},(D_{u-1}+D_u),D_{u+1},\dots,D_q)^{(i+1,j)}}$$
$$+\frac{\lambda\alpha}{i+1}\left(n - D_{p+1} - D_{p+2} - \cdots - D_u - u\right) \times$$
$$\mu_{u,v-1:q-1:n}^{(D_{p+1},\dots,D_{u-1},(D_u+D_{u+1}),D_{u+2},\dots,D_q)^{(i+1,j)}}$$
$$+\frac{\lambda\alpha}{i+1}\left(n - D_{p+1} - \cdots - D_{u-1} - u + 1\right) \times$$
$$\mu_{u-1,v-1:q-1:n}^{(D_{p+1},\dots,D_{u-2},(D_{u-1}+D_u+1),D_{u+1},\dots,D_q)^{(i+1,j)}}$$
$$-\frac{\lambda\alpha}{i+1}\left(n - D_{p+1} - D_{p+2} - \cdots - D_u - u\right) \times$$
$$\mu_{u,v-1:q-1:n}^{(D_{p+1},\dots,D_{u-1},(D_u+D_{u+1}+1),D_{u+2},\dots,D_q)^{(i+1,j)}}. \qquad (23)$$

Proof

Necessity:

Theorem 2 proved the necessary part of this theorem

Similarly as proved in theorem 5 we obtain the distribution function of GED given by

$$F(x) = \left[1 - e^{-\lambda(x-\mu)}\right]^{\alpha}.$$

That is the distribution function of GED. This completes the proof.

Theorem 7

Let $X_{p+1:n} \leq X_{p+2:n} \leq \dots \leq X_{n:n}$ be the order stasistics of a random sample of size $(n-p)$. Then X has GED iff, for $p+1 \leq u < v \leq q-1\,, \quad q \leq n \quad and \quad i,j \geq 0,$

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i,j+k)}} - \mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i,j)}} =$$
$$\frac{\lambda\alpha(D_v)}{i+1} \mu_{u,v:q:n}^{(D_{p+1},D_{p+2},\dots,D_{v-1},(D_v-1),D_{v+1},\dots,D_q)^{(i,j+1)}}$$
$$-\frac{\lambda\alpha(D_v+1)}{i+1} \mu_{u,vq:n}^{(D_{p+1},D_{p+2},\dots,D_q)^{(i,j+1)}}$$

$$-\frac{\lambda\alpha}{j+1}\left(n-D_{p+1}-\cdots-D_{v-1}-v+1\right)\times$$
$$\mu_{u,v-1:q-1:n}^{(D_{p+1},\ldots,R_{v-2},(R_{v-1}+R_v),R_{v+1},\ldots,R_q)^{(i,j+1)}}$$
$$+\frac{\lambda\alpha}{j+1}\left(n-D_{p+1}-D_{p+2}-\cdots-D_v-v\right)\times$$
$$\mu_{u,v:q-1:n}^{(D_{p+1},\ldots,D_{v-1},(D_v+D_{v+1}),D_{v+2},\ldots,D_q)^{(i,j+1)}}$$
$$+\frac{\lambda\alpha}{j+1}\left(n-D_{p+1}-\cdots-D_{v-1}-v+1\right)\times$$
$$\mu_{u,v-1:q-1:n}^{(D_{p+1},\ldots,D_{v-2},(D_{v-1}+D_v+1),D_{v+1},\ldots,D_q)^{(i,j+1)}}$$
$$-\frac{\lambda\alpha}{j+1}\left(n-D_{p+1}-D_{p+2}-\cdots-D_v-v\right)\times$$
$$\mu_{u,v:q-1:n}^{(D_{p+1},\ldots,D_{v-1},(D_v+D_{v+1}+1),D_{v+2},\ldots,D_q)^{(i,j+1)}}. \qquad (24)$$

Proof

Necessity:

Theorem 6 proved the necessary part of this theorem

Sufficiency:

Similarly as proved in theorem 5 we obtain the distribution function of GED given by

$$F(x)=\left[1-e^{-\lambda(x-\mu)}\right]^{\alpha}.$$

That is the distribution function of GED. This completes the proof.

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