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# A Uniqueness Theorem for a Sturm-Liouville Equation with Spectral Parameter in Boundary Conditions

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**Abstract:** In this work a Sturm-Liouville operator with discontinuous coefficient and a spectral parameter in boundary conditions is considered. The orthogonality of the eigenfunctions, realness and simplicity of the eigenvalues are investigated. It is shown that the eigenfunctions form a complete system and expansion formula with respect to eigenfunctions is obtained. Also, the evolution of the Weyl solution and Weyl function is discussed. Uniqueness theorem for the solution of the inverse problem with Weyl function is proved.

**Keywords:** Sturm-Liouville operator, expansion formula, inverse problem, Weyl function

## 1 Introduction

We consider the boundary value problem

$$-y'' + q(x)y = \lambda^2 \rho(x) y, \quad 0 < x < \pi, \tag{1}$$

$$U(y) := \lambda^2 \left( y'(0) - hy(0) \right) - h_1 y'(0) + h_2 y(0) = 0, \tag{2}$$

$$V(y) := \lambda^2 \left( y'(\pi) + Hy(\pi) \right) - H_1 y'(\pi) - H_2 y(\pi) = 0, \tag{3}$$

where $q(x) \in L_2(0,\pi)$ is a real valued function, $\lambda$ is a complex parameter, $h, h_1, h_2, H, H_1, H_2$ are real numbers and

$$\rho(x) = \begin{cases} 1, & 0 \le x < a, \\ \alpha^2, & a < x \le \pi, \end{cases}$$

where $0 < \alpha \neq 1$.

Inverse problems of spectral analysis for different spectral datas are examined in [1,2,3,4,5,6] and in other monographs. The vibration problem of a homogeneous string with one end is fixed, the other end is eqqupied with a mass and other numerous problems of mathematical physics can be reduced to boundary value problems with spectral parameter in boundary condition (see [7,8,9],etc.).

In [10] and [8] operator-theoretic formulation for this type of problems is given. These problems are studied in Hilbert spaces $L_2(0,\pi) \oplus \mathbb{C}^k$, where $k$ is the number of eigenvalue containing boundary conditions.

Uniqueness theorems for inverse problems are proven in [11,12,13,14,15,16,17,18]. In [19,20,21,22] almost isospectral maps between classes of Sturm-Liouville problems are produced the generalization of norming constants is given. The inverse problem has been analyzed by zeros of the eigenfunctions in [23]. They showed that the potential and the asymptotic boundary conditions in such a problem are uniquely determined by a required dense set of nodal points of eigenfunctions.

Numerical methods for inverse problem of Sturm-Liouville operators with spectral parameter in boundary conditions are given in [16].

Uniqueness of the solutions of inverse problem by Weyl function for Sturm-Liouville operators with spectral parameter in boundary conditions is dealed in [24,25,26,27]. Necessary and sufficient conditions for the solution of the inverse problem for Sturm-Liouville equation with discontinuous coefficient and boundary conditions which doesn't contain eigenvalue parameter is obtained in [28].

In this work we discussed a Sturm-Liouville equation with piece-wise continuous coefficient and spectral parameter in both boundary conditions. Operator-theoretic formulation for the problem is given in $L_2(0,\pi) \oplus \mathbb{C}^2$ Hilbert space. Simplicity and asymptotic formulae of the eigenvalues are shown, expansion formula with respect to eigenfunctions is obtained. Also, uniqueness of the solution of the inverse problem by Weyl function is proven.

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## 2 Preliminaries

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of equation (1) satisfying the initial conditions

$$\varphi(0,\lambda)=h_1-\lambda^2,\quad \varphi'(0,\lambda)=h_2-\lambda^2 h, \tag{4}$$

$$\psi(\pi,\lambda)=H_1-\lambda^2,\quad \psi'(\pi,\lambda)=\lambda^2 H-H_2. \tag{5}$$

For the solution of equation (1), the following integral representation as $\mu^{\pm}(x)=\pm x\sqrt{\rho(x)}+a\left(1\mp\sqrt{\rho(x)}\right)$ is obtained similar to [6] for all $\lambda$:

$$e(x,\lambda)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^{+}(x)}+$$
$$+\frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^{-}(x)}+\int_{-\mu^{+}(x)}^{\mu^{+}(x)}K(x,t)e^{i\lambda t}dt,$$

where $K(x,\cdot)\in L_1(-\mu^{+}(x),\mu^{+}(x))$. The kernel $K(x,t)$ has the partial derivative $K_x$ belonging to the space $L_1(-\mu^{+}(x),\mu^{+}(x))$ for every $x\in[0,\pi]$ and the properties below hold:

$$K(x,t)=0,\quad |t|>|x|,\quad 0\le x\le a, \tag{6}$$

$$K(x,t)=0,\quad |t|>\left|\mu^{+}(x)\right|,\quad a\le x\le\pi, \tag{7}$$

$$K(x,-x)=0, \tag{8}$$

$$\frac{d}{dx}K\left(x,\mu^{+}(x)\right)=\frac{1}{4\sqrt{\rho(x)}}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)q(x), \tag{9}$$

$$\frac{d}{dx}K\left(x,\mu^{-}(x)+0\right)-\frac{d}{dx}K\left(x,\mu^{-}(x)-0\right)=$$
$$=\frac{1}{4\sqrt{\rho(x)}}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)q(x). \tag{10}$$

Using the representation of the solution $e(x,\lambda)$ and formula

$$\varphi(x,\lambda)=\frac{e(x,\lambda)-e(x,-\lambda)}{2i\lambda}$$

we obtain the integral representation of the solution $\varphi(x,\lambda)$:

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+ \tag{11}$$
$$+\left(h_1-\lambda^2\right)\int_0^{\mu^{+}(x)}A(x,t)cos\lambda t\,dt+$$
$$+\left(h_2-\lambda^2 h\right)\int_0^{\mu^{+}(x)}\tilde{A}(x,t)\frac{sin\lambda t}{\lambda}dt,$$

where

$$A(x,t)=K(x,t)-K(x,-t),$$

$$\tilde{A}(x,t)=K(x,t)+K(x,-t)$$

satisfy the properties (6) - (10).

We define

$$\Delta(\lambda):=\langle\varphi(x,\lambda),\psi(x,\lambda)\rangle=$$
$$=\varphi(x,\lambda)\psi'(x,\lambda)-\varphi'(x,\lambda)\psi(x,\lambda), \tag{12}$$

which is independent from $x\in[0,\pi]$. Substituting $x=0$ and $x=\pi$ into (12) we get,

$$\Delta(\lambda)=-U(\psi)=V(\varphi)$$

The function $\Delta(\lambda)$ is entire and has zeros at the eigenvalues of the problem (1)-(3).

In the Hilbert space $H_\rho=L_{2,\rho}(0,\pi)\oplus\mathbb{C}^2$ let an inner product be defined by

$$(F,G):=\int_0^\pi F_1(x)\overline{G_1(x)}\rho(x)dx+\frac{F_2\overline{G_2}}{\delta_1}+\frac{F_3\overline{G_3}}{\delta_2}$$

where

$$F=\begin{pmatrix}F_1(x)\\F_2\\F_3\end{pmatrix}\in H_\rho,\quad G=\begin{pmatrix}G_1(x)\\G_2\\G_3\end{pmatrix}\in H_\rho,$$

$$\delta_1:=hh_1-h_2>0,\quad \delta_2:=HH_1-H_2>0.$$

We define the operator

$$L(F):=\begin{pmatrix}-F_1''(x)+q(x)F_1(x)\\h_1F_1'(0)-h_2F_1(0)\\H_1F_1'(\pi)+H_2F_1(\pi)\end{pmatrix}$$

with

$$D(L)=\left\{\begin{array}{c}F\in H_\rho:F_1(x)\in W_2^2[0,\pi],\\F_2=F_1'(0)-hF_1(0),F_3=F_1'(\pi)+HF_1(\pi)\end{array}\right\},$$

where

$$l(F_1)=\frac{1}{\rho(x)}\left\{-F_1''+q(x)F_1\right\}.$$

The boundary value problem (1)-(3) is equivalent to the equation $LY=\lambda^2 Y$. When $\lambda=\lambda_n$ are the eigenvalues, the eigenfunctions of operator $L$ are in the form of

$$\Phi(x,\lambda_n)=\Phi_n:=\begin{pmatrix}\varphi(x,\lambda_n)\\\varphi'(0,\lambda_n)-h\varphi(0,\lambda_n)\\\varphi'(\pi,\lambda_n)+H\varphi(\pi,\lambda_n)\end{pmatrix},\quad n=1,2.$$

**Lemma 1.** *The operator $L$ is symmetric.*

*Proof.* Let $F,G\in D(L)$. Since,

$$(LF,G)-(F,LG)=\int_0^\pi LF_1(x)\overline{G_1(x)}\rho(x)dx+$$
$$+\frac{LF_2\overline{G_2}}{\delta_1}+\frac{LF_3\overline{G_3}}{\delta_2}$$
$$-\int_0^\pi F_1(x)\overline{LG_1(x)}\rho(x)dx-$$
$$-\frac{F_2\overline{LG_2}}{\delta_1}-\frac{F_3\overline{LG_3}}{\delta_2}$$

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by two partial integration, we get

$$(LF,G)-(F,LG)=\left[F_1(x)\overline{G_1'(x)}-F_1'(x)\overline{G_1(x)}\right]_{x=0}^{x=\pi}+$$
$$+\frac{1}{\delta_1}\left[-h_2\left(F_1(0)\overline{G_1'(0)}-F_1'(0)\overline{G_1(0)}\right)\right]-$$
$$-\frac{1}{\delta_1}\left[hh_1\left(F_1'(0)\overline{G_1(0)}-F_1(0)\overline{G_1'(0)}\right)\right]+$$
$$+\frac{1}{\delta_2}\left[HH_1\left(F_1'(\pi)\overline{G_1(\pi)}-F_1(\pi)\overline{G_1'(\pi)}\right)\right]+$$
$$+\frac{1}{\delta_2}\left[H_2\left(F_1(\pi)\overline{G_1'(\pi)}-F_1'(\pi)\overline{G_1(\pi)}\right)\right].$$

If we use the domain of the operator $L$ and $\delta_1>0$, $\delta_2>0$ we see that,

$$(LF,G)-(F,LG)=0.$$

So $L$ is symmetric. □

**Corollary 1.***The eigenfunctions $\Phi_1$ and $\Phi_2$ corresponding to different eigenvalues $\lambda_1\neq\lambda_2$ are orthogonal, i.e.*

$$\int_0^\pi\varphi(x,\lambda_1)\overline{\varphi(x,\lambda_2)}dx+$$
$$+\frac{1}{\delta_1}\left(\varphi'(0,\lambda_1)-h\varphi(0,\lambda_1)\right)\overline{\left(\varphi'(0,\lambda_2)-h\varphi(0,\lambda_2)\right)}+$$
$$+\frac{1}{\delta_2}\left(\varphi'(\pi,\lambda_1)+H\varphi(\pi,\lambda_1)\right)\overline{\left(\varphi'(\pi,\lambda_2)+H\varphi(\pi,\lambda_2)\right)}=0.$$

For any eigenvalue $\lambda_n$ the solutions (4), (5) satisfy the relation

$$\psi(x,\lambda_n)=k_n\varphi(x,\lambda_n),\tag{13}$$

where

$$k_n=-\frac{\psi'(0,\lambda_n)-h\psi(0,\lambda_n)}{\delta_1}=\frac{\delta_2}{H\varphi(\pi,\lambda_n)+\varphi'(\pi,\lambda_n)}\tag{14}$$

and the normalized numbers of the boundary value problem (1)-(3) are in the following form:

$$\alpha_n:=\int_0^\pi\varphi^2(x,\lambda_n)\rho(x)dx+\frac{1}{\delta_1}\left(\varphi'(0,\lambda_n)-h\varphi(0,\lambda_n)\right)^2+\frac{1}{\delta_2}\left(\varphi'(\pi,\lambda_n)+H\varphi(\pi,\lambda_n)\right)^2.\tag{15}$$

**Lemma 2.***The eigenvalues of the boundary value problem (1)-(3) are simple, i.e.*

$$\dot{\Delta}(\lambda)=2\lambda_nk_n\alpha_n.\tag{16}$$

*Proof.*Since

$$-\varphi''(x,\lambda_n)+q(x)\varphi(x,\lambda_n)=\lambda_n^2\rho(x)\varphi(x,\lambda_n),$$
$$-\psi''(x,\lambda)+q(x)\psi(x,\lambda)=\lambda^2\rho(x)\psi(x,\lambda),$$

we get

$$\frac{d}{dx}\left\langle\varphi(x,\lambda)\psi'(x,\lambda)-\varphi'(x,\lambda)\psi(x,\lambda)\right\rangle=$$
$$=\left(\lambda_n^2-\lambda^2\right)\rho(x)\varphi(x,\lambda_n)\psi(x,\lambda).$$

With the help of (2),(3) we get

$$\Delta(\lambda_n)-\Delta(\lambda)=\left(\lambda_n^2-\lambda^2\right)\int_0^\pi\varphi(x,\lambda_n)\psi(x,\lambda)\rho(x)dx.$$

Adding

$$\frac{\left(\lambda_n^2-\lambda^2\right)}{\delta_1}\left(\varphi'(0,\lambda_n)-h\varphi(0,\lambda_n)\right)\left(\psi'(0,\lambda)-h\psi(0,\lambda)\right)+$$
$$+\frac{\left(\lambda_n^2-\lambda^2\right)}{\delta_2}\left(\varphi'(\pi,\lambda_n)+H\varphi(\pi,\lambda_n)\right)\left(\psi'(\pi,\lambda)+H\psi(\pi,\lambda)\right)$$

both sides of the last equation and using the relations (13),(15) we have

$$\Delta(\lambda_n)-\Delta(\lambda)=(\lambda_n+\lambda)(\lambda_n-\lambda)k_n\alpha_n.$$

For $\lambda\to\lambda_n$, we reach (16) . □

## 3 Asymptotic Formulas of the Eigenvalues

The solution of the equation (1) satisfying the initial conditions (4) when $q(x)\equiv0$ is in the following form:

$$\varphi_0(x,\lambda)=(h_1-\lambda^2)cos\lambda x+(h_2-\lambda^2h)\frac{sin\lambda x}{\lambda}.\tag{17}$$

The eigenvalues $\lambda_n^0$ $(n=0,\mp1,\mp2,\cdots)$ of the boundary value problem (1)-(3) can be found by using the equation

$$\Delta_0(\lambda)=\left(\lambda^2H-H_2\right)\varphi_0(\pi,\lambda)-\left(H_1-\lambda^2\right)\varphi_0'(\pi,\lambda)=0$$

and can be represented in the following way

$$\lambda_n^0=n+\psi(n),\quad n=0,\mp1,\mp2,\cdots,\tag{18}$$

where $\sup_n|\psi(n)|<+\infty$.

The following lemma can be proved in a similar way to [4,29]:

**Lemma 3.***Roots $\lambda_n^0$ of the function $\Delta_0(\lambda)$ are seperated, i. e.,*

$$\inf_{n\neq k}\left|\lambda_n^0-\lambda_k^0\right|=\tau>0.$$

**Lemma 4.***The eigenvalues of the boundary value problem (1)-(3) are in the form of*

$$\lambda_n=\lambda_n^0+\frac{d_n}{\lambda_n^0}+\frac{\eta_n}{n},\quad\lambda_n>0,\tag{19}$$

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*where* $(d_n)$ *is a bounded sequence*

$$d_n = \frac{h_1-1}{4\lambda_n^0\dot{\Delta}(\lambda_n^0)}\int_0^\pi\left(1-\frac{1}{\sqrt{\rho(t)}}\right)\frac{q(t)sin\left(\lambda_n^0\mu^-(\pi)\right)}{\sqrt{\rho(t)}}dt-$$
$$-\frac{h_2-h}{4\lambda_n^0\dot{\Delta}(\lambda_n^0)}\int_0^\pi\left(1+\frac{1}{\sqrt{\rho(t)}}\right)\frac{q(t)cos\left(\lambda_n^0\mu^-(\pi)\right)}{\sqrt{\rho(t)}}dt$$

*and* $\{\eta_n\}\in l_2$.

*Proof.*From (11), it follows that

$$\varphi(\pi,\lambda)=\varphi_0(\pi,\lambda)+ \qquad (20)$$
$$+\left(h_1-\lambda^2\right)\int_0^{\mu^+(\pi)}A(\pi,t)cos\lambda t dt+$$
$$+\left(h_2-\lambda^2 h\right)\int_0^{\mu^+(\pi)}\tilde{A}(\pi,t)\frac{sin\lambda t}{\lambda}dt.$$

Expressions of $\Delta(\lambda)$ and $\Delta_0(\lambda)$ let us to calculate $\Delta(\lambda)-\Delta_0(\lambda)$ as,

$$\Delta(\lambda)-\Delta_0(\lambda)=$$
$$-\lambda^4\left(\alpha+\frac{\pi-1}{2\alpha}\right)A\left(\pi,\mu^+(\pi)\right)cos\lambda\mu^+(\pi)-$$
$$-\lambda^3 h\left(\alpha+\frac{\pi-1}{2\alpha}\right)\tilde{A}\left(\pi,\mu^+(\pi)\right)sin\lambda\mu^+(\pi)+$$
$$+I(\lambda)\lambda^4, \qquad (21)$$

where

$$I(\lambda)=H\int_0^{\mu^+(\pi)}A(\pi,t)cos\lambda t dt-$$
$$-\int_0^{\mu^+(\pi)}\frac{\partial}{\partial x}A(\pi,t)cos\lambda t dt+O\left(\frac{e^{|Im\lambda|\mu^+(\pi)}}{\lambda^3}\right).$$

Therefore, for sufficiently large $n$, on the contours

$$\Gamma_n=\left\{\lambda:|\lambda|=\left|\lambda_n^0\right|+\frac{\tau}{2}\right\},$$

we have

$$|\Delta(\lambda)-\Delta_0(\lambda)|<|\Delta_0(\lambda)|.$$

By the Rouche theorem, we obtain that, the number of zeros of the function

$$\{\Delta(\lambda)-\Delta_0(\lambda)\}+\Delta_0(\lambda)=\Delta(\lambda)$$

inside the contour $\Gamma_n$ coincides with the number of zeros of the function $\Delta_0(\lambda)$. Furthermore, applying the Rouche theorem to the circle $\gamma_n(\delta)=\left\{\lambda:\left|\lambda-\lambda_n^0\right|\le\delta\right\}$ we get that, for sufficiently large $n$ there exists one zero $\lambda_n$ of the function $\Delta(\lambda)$ in $\gamma_n(\delta)$. Owing to the arbitrariness of $\delta>0$ we have

$$\lambda_n=\lambda_n^0+\varepsilon_n,\quad \varepsilon_n=o(1),\quad n\to\infty. \qquad (22)$$

Substituting (22) into (20) we get,

$$\Delta(\lambda_n^0+\varepsilon_n)=\Delta_0(\lambda_n^0+\varepsilon_n)+$$
$$+\left[h_1-\left(\lambda_n^0+\varepsilon_n\right)^2\right]\int_0^{\mu^+(\pi)}A(\pi,t)cos\left(\lambda_n^0+\varepsilon_n\right)tdt+$$
$$+\left[h_2-\left(\lambda_n^0+\varepsilon_n\right)^2 h\right]\int_0^{\mu^+(\pi)}\tilde{A}(\pi,t)\frac{sin\left(\lambda_n^0+\varepsilon_n\right)t}{\lambda_n^0+\varepsilon_n}dt=0.$$

Hence, as $n\to\infty$ taking into the equality $\Delta_0(\lambda_n^0)=0$ and relations $sin\varepsilon_n\mu^+(\pi)\approx\varepsilon_n\mu^+(\pi)$, $cos\varepsilon_n\mu^+(\pi)\approx 1$ integrating by parts and using the properties (6)- (10) of the kernels $A(x,t)$ and $\tilde{A}(x,t)$ we have

$$\varepsilon_n\approx\frac{d_n}{\lambda_n^0+\varepsilon_n}+\frac{\eta_n}{\lambda_n^0}$$

where

$$\eta_n=(h_1-1)\int_0^{\mu^+(\pi)}A_t(\pi,t)sin\lambda_n^0 tdt+$$
$$+(h_2-h)\int_0^{\mu^+(\pi)}A_t(\pi,t)cos\lambda_n^0 tdt.$$

Let us show that $\eta_n\in l_2$. It is obvious that

$$(h_1-1)\int_0^{\mu^+(\pi)}A_t(\pi,t)sin\lambda_n^0 tdt+$$
$$+(h_2-h)\int_0^{\mu^+(\pi)}A_t(\pi,t)cos\lambda_n^0 tdt$$

can be reduced to

$$\int_{-\mu^+(\pi)}^{\mu^+(\pi)}R(t)e^{i\lambda t}dt,$$

where $R(t)\in L_2(-\mu^+(\pi),\mu^+(\pi))$. Now, take

$$\zeta(\lambda):=\int_{-\mu^+(\pi)}^{\mu^+(\pi)}R(t)e^{i\lambda t}dt.$$

It is clear from [6] (p. 66) that $\{\zeta_n\}=\zeta(\lambda_n)\in l_2$. By virtue of this we have $\{\eta_n\}\in l_2$. Lemma is proved. □

## 4 Expansion Formula with Respect to Eigenfunctions

Assume that $\lambda^2$ is not a spectrum point of the operator $L$. Then, there exists resolvent operator

$$R_{\lambda^2}(L)=(L-\lambda^2 I)^{-1}.$$

Let us find the expression of $R_{\lambda^2}(L)$.

**Lemma 5.***The resolvent* $R_{\lambda^2}(L)$ *is the integral operator with the kernel*

$$G(x,t;\lambda)=-\frac{1}{\Delta(\lambda)}\begin{cases}\varphi(t,\lambda)\psi(x,\lambda), & t\le x,\\ \psi(t,\lambda)\varphi(x,\lambda), & t\ge x.\end{cases} \qquad (23)$$

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*Proof.*To construct the resolvent operator of $L$, we need to solve the boundary value problem

$$-y'' + q(x)y = \lambda^2 \rho(x) y + \rho(x) f(x), \quad (24)$$

$$\lambda^2 \left(y'(0) - hy(0)\right) - h_1 y'(0) + h_2 y(0) = f_1, \quad (25)$$

$$\lambda^2 \left(y'(\pi) + Hy(\pi)\right) - H_1 y'(\pi) - H_2 y(\pi) = f_2, \quad (26)$$

where $f(x) \in D(L)$. By applying the method of variation of constants, we seek the solution of the problem (24)-(26) in the following form

$$y(x,\lambda) = c_1(x,\lambda)\psi(x,\lambda) + c_2(x,\lambda)\varphi(x,\lambda), \quad (27)$$

and we get the coefficients $c_1(x,\lambda)$ and $c_2(x,\lambda)$ as

$$c_1(x,\lambda) = c_1(0,\lambda) - \frac{1}{\Delta(\lambda)} \int_0^x \varphi(t,\lambda) f(t)\rho(t)dt, \quad (28)$$

$$c_2(x,\lambda) = c_2(\pi,\lambda) - \frac{1}{\Delta(\lambda)} \int_x^\pi \psi(t,\lambda) f(t)\rho(t)dt. \quad (29)$$

Substituting equations (28), (29) into (27) and taking into account the boundary conditions (25), (26) we have

$$y(x,\lambda) = \int_0^\pi G(x,t;\lambda) f(t)\rho(t)dt - \frac{f_1}{\Delta(\lambda)}\psi(x,\lambda) + \frac{f_2}{\Delta(\lambda)}\varphi(x,\lambda) \quad (30)$$

where $G(x,t;\lambda)$ is as in (23). □

**Theorem 1.***The eigenfunctions $\Phi(x,\lambda_n)$ of the boundary value problem (1)-(3) form a complete system in $L_{2,\rho}(0,\pi) \oplus \mathbb{C}^2$.*

*Proof.*With the help of (13) and (16), we can write

$$\psi(x,\lambda_n) = \frac{\dot{\Delta}(\lambda_n)}{2\lambda_n \alpha_n} \varphi(x,\lambda_n). \quad (31)$$

Using (23) and (30) we get

$$\underset{\lambda=\lambda_n}{Res}\, y(x,\lambda) = -\frac{1}{2\lambda_n\alpha_n}\varphi(x,\lambda_n)\int_0^\pi \varphi(t,\lambda_n) f(t)\rho(t)dt - \frac{1}{2\lambda_n\alpha_n}\varphi(x,\lambda_n)\left(f_1 - \frac{f_2}{k_n}\right). \quad (32)$$

Now let $f(x) \in L_{2,\rho}(0,\pi) \oplus \mathbb{C}^2$ and assume

$$(\Phi(x,\lambda_n), f(x)) = \int_0^\pi \varphi(x,\lambda_n)\overline{f_1(x)}\rho(x)dx + \frac{(\varphi'(0,\lambda_n) - h\varphi(0,\lambda_n))\overline{f_2}}{\delta_1} + \frac{(\varphi'(\pi,\lambda_n) + H\varphi(\pi,\lambda_n))\overline{f_3}}{\delta_2} = 0. \quad (33)$$

Then from (32), we have $\underset{\lambda=\lambda_n}{Res}\, y(x,\lambda) = 0$. Consequently, for fixed $x \in [0,\pi]$ the function $y(x,\lambda)$ is entire with respect to $\lambda$. Let us denote that

$$G_\delta := \left\{\lambda : \left|\lambda - \lambda_n^0\right| \geq \delta, \quad n = 0, \mp 1, \mp 2, \cdots\right\}$$

where $\delta$ is sufficiently small positive number. It is clear that the relation below holds:

$$|\Delta(\lambda)| \geq C|\lambda|^4 e^{|Im\lambda|\mu^+(\pi)}, \quad \lambda \in G_\delta, \quad C = cons. \quad (34)$$

From (30) it follows that for fixed $\delta > 0$ and sufficiently large $\lambda^* > 0$ we have

$$|y(x,\lambda)| \leq \frac{C}{|\lambda|^2}, \quad \lambda \in G_\delta, \quad |\lambda| \geq \lambda^* \quad C = cons.$$

Using maximum principle for module of analytic functions and Liouville theorem, we get $y(x,\lambda) \equiv 0$. From this and the expression of the boundary value problem (24)-(26) we obtain that $f(x) \equiv 0$ a.e. on $[0,\pi]$. Thus we reach the completeness of the eigenfunctions $\Phi(x,\lambda_n)$ in $L_{2,\rho}(0,\pi) \oplus \mathbb{C}^2$. □

**Theorem 2.***If $f(x) \in D(L)$, then the expansion formula*

$$f(x) = \sum_{n=1}^{\infty} a_n \varphi(x,\lambda_n) \quad (35)$$

*is valid, where*

$$a_n = \frac{1}{2\alpha_n}\int_0^\pi \varphi(t,\lambda_n) f(t)\rho(t)dt,$$

*and the series converge uniformly with respect to $x \in [0,\pi]$. For $f(x) \in L_{2,\rho}(0,\pi)$, the series converge in $L_{2,\rho}(0,\pi)$, moreover the Parseval equality holds:*

$$\int_0^\pi |f(x)|^2 \rho(x)dx = \sum_{n=1}^{\infty} \alpha_n |a_n|^2.$$

*Proof.*Since $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are the solutions of the boundary value problem (1)-(3), we have

$$y(x,\lambda) = -\frac{\psi(x,\lambda)}{\Delta(\lambda)}\left\{\int_0^\pi \frac{[-\varphi''(t,\lambda) + q(t)\varphi(t,\lambda)]f(t)}{\lambda^2}dt\right\} - \frac{\varphi(x,\lambda)}{\Delta(\lambda)}\left\{\int_\pi^x \frac{[-\psi''(t,\lambda) + q(t)\psi(t,\lambda)]f(t)}{\lambda^2}dt\right\} - \frac{f_1}{\Delta(\lambda)}\psi(x,\lambda) + \frac{f_2}{\Delta(\lambda)}\varphi(x,\lambda). \quad (36)$$

Integrating by parts and taking into account the boundary conditions (2), (3) we obtain

$$y(x,\lambda) = -\frac{1}{\lambda^2}f(x) - \frac{1}{\lambda^2}[Z_1(x,\lambda) + Z_2(x,\lambda)] - \frac{f_1}{\Delta(\lambda)}\psi(x,\lambda) + \frac{f_2}{\Delta(\lambda)}\varphi(x,\lambda), \quad (37)$$

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where

$$Z_1(x,\lambda) = \frac{1}{\Delta(\lambda)}\psi(x,\lambda)\int_0^x \varphi'(t,\lambda) f'(t)dt + \\ + \frac{1}{\Delta(\lambda)}\varphi(x,\lambda)\int_x^\pi \psi'(t,\lambda) f'(t)dt,$$

$$Z_2(x,\lambda) = \frac{1}{\Delta(\lambda)}\left[\left(h_2-\lambda^2 h\right)\psi(x,\lambda) f(0)\right] - \\ - \frac{1}{\Delta(\lambda)}\left[\left(\lambda^2 H - H_2\right)\varphi(x,\lambda) f(\pi)\right] + \\ + \frac{1}{\Delta(\lambda)}\psi(x,\lambda)\int_0^x \varphi(t,\lambda)q(t) f(t)dt + \\ + \frac{1}{\Delta(\lambda)}\varphi(x,\lambda)\int_x^\pi \psi(t,\lambda)q(t) f(t)dt.$$

If we consider the following contour integral where $\Gamma_n$ is a counter-clockwise oriented contour

$$I_n(x) = \frac{1}{2\pi i}\oint_{\Gamma_n}\lambda y(x,\lambda)d\lambda,$$

and then taking into consideration equation (32) we get

$$I_n(x) = \sum_{n=1}^{\infty} \underset{\lambda=\lambda_n}{Res}\left[\lambda y(x,\lambda)\right] = \\ = \sum_{n=1}^{\infty} a_n\varphi(x,\lambda_n) + \sum_{n=1}^{\infty}\frac{\lambda_n f_1}{\dot\Delta(\lambda_n)}\psi(x,\lambda_n) \\ - \sum_{n=1}^{\infty}\frac{\lambda_n f_2}{\dot\Delta(\lambda_n)}\varphi(x,\lambda_n), \tag{38}$$

where

$$a_n = \frac{1}{\alpha_n}\int_0^\pi \varphi(t,\lambda_n) f(t)\rho(t)dt.$$

On the other hand, with the help of (37) we get

$$I_n(x) = -f(x) - \frac{1}{2\pi i}\oint_{\Gamma_n}\left[Z_1(x,\lambda)+Z_2(x,\lambda)\right]d\lambda + \tag{39}$$
$$+ \sum_{n=1}^{\infty}\frac{\lambda_n f_1}{\dot\Delta(\lambda_n)}\psi(x,\lambda_n) - \sum_{n=1}^{\infty}\frac{\lambda_n f_2}{\dot\Delta(\lambda_n)}\varphi(x,\lambda_n).$$

Comparing (38) and (39) we obtain

$$\sum_{n=1}^{\infty} a_n\varphi(x,\lambda_n) = -f(x) + \varepsilon_n(x),$$

where

$$\varepsilon_n(x) = -\frac{1}{2\pi i}\oint_{\Gamma_n}\left[Z_1(x,\lambda)+Z_2(x,\lambda)\right]d\lambda.$$

The relations below hold for sufficiently large $\lambda^* > 0$

$$\underset{x\in[0,\pi]}{max}\left|Z_2(x,\lambda)\right| \le \frac{C_2}{|\lambda|^2}, \quad \lambda\in G_\delta, \quad |\lambda| \le \lambda^*, \tag{40}$$

$$\underset{x\in[0,\pi]}{max}\left|Z_1(x,\lambda)\right| \le \frac{C_1}{|\lambda|^2}, \quad \lambda\in G_\delta, \quad |\lambda| \le \lambda^*. \tag{41}$$

The validity of

$$\underset{n\to\infty}{lim}\ \underset{x\in[0,\pi]}{max}\left|\varepsilon_n(x)\right| = 0$$

can be easily seen from (40) and (41). The last equation gives us the expansion formula

$$f(x) = \sum_{n=1}^{\infty} a_n\varphi(x,\lambda_n).$$

Since the system of $\Phi(x,\lambda_n)$ is complete and orthogonal in $L_{2,\rho}(0,\pi)\oplus\mathbb{C}^2$ , the Parseval equality

$$\int_0^\pi |f(x)|^2\rho(x)dx = \sum_{n=1}^{\infty}\alpha_n|a_n|^2$$

holds. □

## 5 Weyl Solution, Weyl Function

We consider the statement of the inverse problem of the reconstruction of the boundary value problem (1)-(3) from the Weyl function.

Let the functions $c(x,\lambda)$ and $s(x,\lambda)$ denote the solutions of the equation (1) satisfying the conditions $c(0,\lambda)=1$, $c'(0,\lambda)=0$, $s(0,\lambda)=0$ and $s'(0,\lambda)=1$ respectively and $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of equation (1) under the initial conditions (4), (5).

Further, let the function $\Phi(x,\lambda)$ be the solution of (1) satisfying $U(\Phi)=1$ and $V(\Phi)=0$. We set

$$M(\lambda) := \frac{\psi(0,\lambda)}{\varphi(0,\lambda)\Delta(\lambda)}.$$

The functions $\Phi(x,\lambda)$ and $M(\lambda)$ are called the Weyl solution and the Weyl function for the boundary value problem (1)-(3).The Weyl function is a meromorphic function having simple poles at points $\lambda_n$ eigenvalues of the boundary value problem of (1)-(3).

The Wronskian

$$W(x) := \langle\Phi(x,\lambda),\varphi(x,\lambda)\rangle$$

does not depend on $x$. Taking $x=0$, we get

$$W(0) = \Phi(0,\lambda)\varphi'(0,\lambda) - \Phi'(0,\lambda)\varphi(0,\lambda) = 1.$$

Hence,

$$W(x) = \langle\Phi(x,\lambda),\varphi(x,\lambda)\rangle = 1. \tag{42}$$

In view of (4) and (5), we get for $\lambda\ne\lambda_n$

$$\Phi(x,\lambda) = -\frac{\psi(x,\lambda)}{\Delta(\lambda)}. \tag{43}$$

Using (43) we obtain

$$M(\lambda) = -\frac{\Delta^0(\lambda)}{\Delta(\lambda)},$$

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where $\Delta^0(\lambda) = -\psi(0,\lambda)$ is characteristic function of the boundary value problem $L_0$:

$$ly = \lambda^2 y, \quad 0 \le x \le \pi,$$
$$y(0) = 0, \quad V(y) = 0.$$

It is clear that

$$\Phi(x,\lambda) = -\frac{1}{\varphi(0,\lambda)}\left(s(x,\lambda) - M(\lambda)\varphi(x,\lambda)\right). \quad (44)$$

**Theorem 3.** *The boundary value problem of (1)-(3) is identically denoted by the Weyl function $M(\lambda)$.*

*Proof.* Let us denote the matrix $P(x,\lambda) = \left[P_{jk}(x,\lambda)\right]_{j,k=1,2}$ as

$$P(x,\lambda)\begin{pmatrix} \tilde{\varphi}(x,\lambda) & \tilde{\Phi}(x,\lambda) \\ \tilde{\varphi}'(x,\lambda) & \tilde{\Phi}'(x,\lambda) \end{pmatrix} = \begin{pmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{pmatrix}. \quad (45)$$

Then we have

$$\varphi(x,\lambda) = P_{11}(x,\lambda)\tilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda), \quad (46)$$
$$\Phi(x,\lambda) = P_{11}(x,\lambda)\tilde{\Phi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\Phi}'(x,\lambda)$$

or

$$P_{11}(x,\lambda) = \varphi(x,\lambda)\tilde{\Phi}'(x,\lambda) - \tilde{\varphi}'(x,\lambda)\Phi(x,\lambda), \quad (47)$$
$$P_{12}(x,\lambda) = \tilde{\varphi}(x,\lambda)\Phi(x,\lambda) - \varphi(x,\lambda)\tilde{\Phi}(x,\lambda).$$

Taking (43) into consideration in (47) we get

$$P_{11}(x,\lambda) = 1 + \frac{1}{\Delta(\lambda)}\psi(x,\lambda)\left[\varphi'(x,\lambda) - \tilde{\varphi}'(x,\lambda)\right] + $$
$$+ \frac{1}{\Delta(\lambda)}\varphi(x,\lambda)\left[\tilde{\psi}'(x,\lambda) - \psi'(x,\lambda)\right], \quad (48)$$
$$P_{12}(x,\lambda) = \frac{1}{\Delta(\lambda)}\left[\tilde{\varphi}(x,\lambda)\psi(x,\lambda) - \varphi(x,\lambda)\tilde{\psi}(x,\lambda)\right].$$

From the estimates as $|\lambda| \to \infty$

$$\left|\frac{\varphi'(x,\lambda) - \tilde{\varphi}'(x,\lambda)}{\Delta(\lambda)}\right| = O\left(\frac{1}{|\lambda|^2} e^{|Im\lambda|\mu^+(x)}\right),$$
$$\left|\frac{\psi'(x,\lambda) - \tilde{\psi}'(x,\lambda)}{\Delta(\lambda)}\right| = O\left(\frac{1}{|\lambda|^2} e^{|Im\lambda|\left(\mu^+(\pi) - \mu^+(x)\right)}\right),$$

we have from (48) that

$$\lim_{|\lambda|\to\infty} \max_{0\le x\le\pi} |P_{11}(x,\lambda) - 1| = \lim_{|\lambda|\to\infty} \max_{0\le x\le\pi} |P_{12}(x,\lambda)| = 0 \quad (49)$$

for $\lambda \in G_\delta$.

Now, if we take consideration equation (44) into (47), we have

$$P_{11}(x,\lambda) = \varphi(x,\lambda)\frac{\tilde{s}'(x,\lambda)}{\varphi(0,\lambda)} - \tilde{\varphi}'(x,\lambda)\frac{s(x,\lambda)}{\varphi(0,\lambda)} + $$
$$+ \frac{\tilde{\varphi}'(x,\lambda)\varphi(x,\lambda)}{\varphi(0,\lambda)}\left[\tilde{M}(\lambda) - M(\lambda)\right],$$
$$P_{12}(x,\lambda) = \varphi(x,\lambda)\frac{\tilde{s}(x,\lambda)}{\varphi(0,\lambda)} - \tilde{\varphi}(x,\lambda)\frac{s(x,\lambda)}{\varphi(0,\lambda)} + $$
$$+ \frac{\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)}{\varphi(0,\lambda)}\left[M(\lambda) - \tilde{M}(\lambda)\right].$$

Therefore if $M(\lambda) = \tilde{M}(\lambda)$, one has

$$P_{11}(x,\lambda) = c(x,\lambda)\tilde{s}'(x,\lambda) - s(x,\lambda)\tilde{c}'(x,\lambda),$$
$$P_{12}(x,\lambda) = c(x,\lambda)\tilde{s}(x,\lambda) - s(x,\lambda)\tilde{c}(x,\lambda).$$

Thus, for every fixed $x$ functions $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire functions for $\lambda$ . It can easily be seen from equation (48) that $P_{11}(x,\lambda) = 1$ and $P_{12}(x,\lambda) = 0$. Consequently, we get $\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda)$ and $\Phi(x,\lambda) \equiv \tilde{\Phi}(x,\lambda)$ for every $x$ and $\lambda$. Hence, we arrive at $q(x) \equiv \tilde{q}(x)$. □

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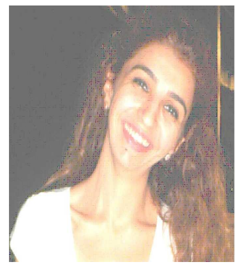

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