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Progressive First-Failure Censored Samples in Estimation and Prediction of NH Distribution

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Abstract: In this paper, we adopt the problem of estimation and prediction for Nadarajah and Haghighi (NH) distribution under the progressive first-failure censoring scheme. The obtained results can be specialized to the first-failure, progressive type-II, type-II, and complete data. The estimation results are formulated with maximum likelihood (ML) and Bayes methods of the unknown model parameters. The approximate confidence interval as well as Bayes highest posterior density (HPD) intervals are constructed with the help of MCMC method. Furthermore, two sample point and interval prediction of the sets of order and record samples are constructed. The estimation results are assessed and compared with the Monte Carlo study. The set of data are analyzed for illustration purposes. Finally, some brief comments are summarized.

Keywords: NH distribution; Progressive first-failure censored scheme; Maximum likelihood estimation; Bayesian estimation and prediction; MCMC.

1 Introduction

The phenomenon of censoring in life testing experiments is widely used for some time and cost restrictions. Form the first types of censoring schemes applied in different areas of life, testing experiments are type-I and type-II censoring schemes. In these types of censoring schemes, the experiment is terminated at some prior time or on the number failure units. For the availability of the removed units from the experiment other than the final point, two types are generalized in the progressive censoring scheme, see [1]. The progressive censoring scheme was discussed in the form of, type-I progressive censoring scheme, type-II progressive censoring scheme, and hybrid progressive censoring scheme. The experiment under high reliable products tested under the last censoring schemes can take a long period of time. One of the most significant solutions to this problem is grouping the test units into several sets with the same number of units and the first failure in each group is recorded, which is called first failure censoring scheme, see [2]. Several authors have reported some statistical inferences under this type of censoring, see [3], [4], and [5]. Under the first-failure censoring scheme, the experiment is terminated when recording the first failure in each set. The problem of the removed sets from the experiment before the final point was defined as a progressive first-failure censoring scheme, which was discussed and developed by [6]. Some properties of the progressive first-failure-censoring scheme were developed by [7], [8], [9] and [10].

A random sample is selected from the products with $n \times k$ size to be grouped into n sets and each set has k units to be put in a life testing experiment. When the failure time $T^{\mathbf{r}}_{i;m,n,k}$ is observed, r_i, $i = 1,2,\ldots,m$ sets and the set in which first failure is observed is randomly removed from the test. Then, the ordered sample $\underline{T}$=($T^{\mathbf{r}}_{1;m,n,k}$, $T^{\mathbf{r}}_{2;m,n,k}$, ..., $T^{\mathbf{r}}_{m;m,n,k}$) under scheme $\mathbf{r}$ = (r_1, r_2, ..., r_m) are called progressively first-failure censored sample which satisfies $n = m + \sum_{i=1}^{m} r_i$.

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For consideration, probability density function (PDF) $f(t)$, cumulative distribution function (CDF) $F(t)$ and observed progressively first-failure censored sample $\underline{t}$=($t^{\mathbf{r}}_{1;m,n,k}$, $t^{\mathbf{r}}_{2;m,n,k}$, ..., $t^{\mathbf{r}}_{m;m,n,k}$), the joint probability density function is given by

$$L_{1,2,...,m}(\underline{t}|\underline{\theta}) = Q\prod_{i=1}^{m} f(t_i)[1-F(t_i)]^{k(r_i+1)-1}, \tag{1}$$

where $\underline{\theta}$ is the parameters vector, $t_i = t^{\mathbf{r}}_{i;m,n,k}$, $0 < t_1 < t_2 < ... < t_m < \infty$, and

$$Q = nk^m \prod_{j=1}^{m-1}\left(n - \sum_{i=1}^{j} r_i - i\right). \tag{2}$$

The plan of the progressive first-failure censored scheme depends on using more units in the test but the only m units are failure to reduce test time. The progressively first-failure censored scheme is reduced to, first-failure censoring scheme at $\mathbf{r}=\underline{0}$, $m=n$ and $k \neq 1$, progressive type-II censoring scheme at $\mathbf{r} \neq \underline{0}$, and $k=1$,type-II at $\mathbf{r}=(0, 0, ..., 0, n-m)$ and $k=1$ and complete sample at $\mathbf{r}=\underline{0}$, $m=n$ and $k=1$.

Remark : If the progressive first-failure censored sample $\underline{T}$=($T^{\mathbf{r}}_{1;m,n,k}$, $T^{\mathbf{r}}_{2;m,n,k}$, ..., $T^{\mathbf{r}}_{m;m,n,k}$) is distributed with CDF $F(t)$, data are distributed as progressive type-II censoring sample with CDF given by

$$G(t) = 1-(1-F(t))^k. \tag{3}$$

For more details, see [11], and [12].

The NH distribution was introduced as it better fits for the data that contain zero values other than gamma, Weibull and the generalized exponential distributions. The lifetime random variable T is called NH random variable if T is distributed to the PDF and CDF given respectively by

$$f(t) = \alpha\lambda(1+\lambda t)^{\alpha-1}e^{(1-(1+\lambda t)^{\alpha})}, t>0, \alpha, \lambda>0, \tag{4}$$

and

$$F(x) = 1-e^{(1-(1+\lambda x)^{\alpha})}. \tag{5}$$

Also, the reliability function (RF) $R(t)$ and failure rate function (FRF) $H(t)$ are respectively given by

$$R(t) = e^{(1-(1+\lambda t)^{\alpha})}, \tag{6}$$

and

$$H(t) = \alpha\lambda(1+\lambda t)^{\alpha-1}, \tag{7}$$

where the parameters α and λ are called the shape and scale parameters, respectively.The NH distribution was introduced by [13] as a form of the extension exponential distribution. NH distribution is reduced to exponential distribution at $\alpha = 1$ with zero mode and has increasing, decreasing, or constant FRF. Different properties of the NH distribution were discussed by [14].

The development of estimation procedures for the parameters of the NH distribution under general censoring scheme is the main objective of this paper. All the developed results in this paper are specialized for the complete sample, type-II censoring sample, progressive type-II censoring sample, and first failure censoring sample. The classical maximum likelihood and Bayes with the help of the MCMC method are discussed for the constructed point and interval estimates of the parameters of the NH distribution. Also, as we see in [15], [16], [17], [18], the Bayesian prediction of future observation based on the observed progressively first-failure censored sample are adopted for future order statistic and future record values.

The paper is summarized into two parts. The first part is dealing with estimation problem as follows. The point and interval MLEs are constructed in Section 2. The point and interval Bayesian MCMC are constructed in Section 3. In the second part, the Bayesian prediction for future order statistic and upper record values are provided in Section 4. Monto Carlo simulation study to compare the ML and the Bayes estimators is provided in Section 5. The data set is analyzed for the estimation and prediction results in Section 6. Finally, we conclude the paper in Section 7.

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2 Maximum Likelihood Estimation

In this section, we discuss the process of constructing the point and interval MLEs of the unknown parameters of the NH distribution. Then, we suppose that $\underline{T}$=($T^{\mathbf{r}}_{1;m,n,k}$, $T^{\mathbf{r}}_{2;m,n,k}$, ..., $T^{\mathbf{r}}_{m;m,n,k}$) is the set of random progressive first-failure censoring from the NH distribution with the PDF and CDF given by (4) and (5), respectively. Then, the joint likelihood function given by (1) is reduced to

$$L(\alpha,\lambda|\underline{t}) = Q\alpha^m\lambda^m\left[\prod_{i=1}^{m}(1+\lambda t_i)^{\alpha-1}\right]e^{k\sum\limits_{i=1}^{m}(r_i+1)(1-(1+\lambda t_i)^{\alpha})}, \tag{8}$$

After that, the natural logarithms of the likelihood function without normalized constant is defined by

$$\ell(\alpha,\lambda|\underline{t}) = m\log\alpha + m\log\lambda + (\alpha-1)\sum_{i=1}^{m}\log(1+\lambda t_i) + k\sum_{i=1}^{m}(r_i+1)(1-(1+\lambda t_i)^{\alpha}). \tag{9}$$

2.1 MLEs

After taking the partial derivative of the log-likelihood function (9) with respect to α and λ, the likelihood equations are obtained under equating derivatives with zero, as follows

$$\frac{\partial\ell}{\partial\alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m}\log(1+\lambda t_i) - k\sum_{i=1}^{m}(r_i+1)(1+\lambda t_i)^{\alpha}\log(1+\lambda t_i) = 0, \tag{10}$$

and

$$\frac{\partial\ell}{\partial\lambda} = \frac{m}{\lambda} + (\alpha-1)\sum_{i=1}^{m}\frac{t_i}{1+\lambda t_i} - \alpha k\sum_{i=1}^{m}(r_i+1)t_i(1+\lambda t_i)^{\alpha-1} = 0. \tag{11}$$

Then, the likelihood equations are reduced to two non-linear equations. Hence, the MLEs of the unknown parameters can be obtained by using any iterative method, such as the Newton Raphson algorithm.

2.2 Asymptotic confidence intervals

The second derivative of the log-likelihood function given by (9) with respect to parameters α and λ is reduced to

$$\frac{\partial^2\ell(\alpha,\lambda|\underline{t})}{\partial\alpha^2} = -\frac{m}{\alpha^2} - k\sum_{i=1}^{m}(r_i+1)(1+\lambda t_i)^{\alpha}\log^2(1+\lambda t_i), \tag{12}$$

$$\frac{\partial^2\ell(\alpha,\lambda|\underline{t})}{\partial\lambda^2} = -\frac{m}{\lambda^2} - (\alpha-1)\sum_{i=1}^{m}\frac{t_i^2}{(1+\lambda t_i)^2} - k\alpha(\alpha-1)\sum_{i=1}^{m}(r_i+1)t_i^2(1+\lambda t_i)^{\alpha-2}, \tag{13}$$

and

$$\frac{\partial^2\ell(\alpha,\lambda|\underline{t})}{\partial\alpha\partial\lambda} = \sum_{i=1}^{m}\frac{t_i}{(1+\lambda t_i)} - k\sum_{i=1}^{m}t_i(r_i+1)(1+\lambda t_i)^{\alpha-1}(\alpha\log(1+\lambda t_i)+1). \tag{14}$$

The interval estimate of unknown parameters needs present the Fisher information matrix $\Phi(\alpha,\lambda)$, which is defined by taking the expectation of minus Eq. (12-14). The estimators $\hat{\alpha}$ and $\hat{\lambda}$ under some mild regularity conditions, are approximated with bivariate normal distribution with the mean $(\alpha,\ \lambda)$ and variance covariance matrix $\Phi_0^{-1}\left(\hat{\alpha},\hat{\lambda}\right)$, as follows

$$\left(\hat{\alpha},\ \hat{\lambda}\right) \to N\left((\alpha,\lambda),\Phi_0^{-1}\left(\hat{\alpha},\ \hat{\lambda}\right)\right), \tag{15}$$

where $\Phi_0^{-1}\left(\hat{\alpha},\ \hat{\lambda}\right)$ is the approximate information matrix.

$$\Phi_0^{-1}\left(\hat{\alpha},\ \hat{\lambda}\right) = \left[\begin{array}{cc} -\frac{\partial^2\ell(\alpha,\lambda|\underline{t})}{\partial\alpha^2} & -\frac{\partial^2\ell(\alpha,\lambda|\underline{t})}{\partial\alpha\partial\lambda} \\ -\frac{\partial^2\ell(\alpha,\lambda|\underline{t})}{\partial\alpha\partial\lambda} & -\frac{\partial^2\ell(\alpha,\lambda|\underline{v})}{\partial\lambda^2} \end{array}\right]^{-1}_{\text{at }(\hat{\alpha},\ \hat{\lambda})} = \left[\begin{array}{cc} v_{11} & v_{12} \\ v_{21} & v_{22} \end{array}\right]. \tag{16}$$

Then, the interval estimates of the unknown parameters under the normality distribution of the MLEs $\hat{\alpha}$ and $\hat{\lambda}$ with 100(1-γ)% confidence level are presented by

$$\begin{cases} \hat{\alpha} \mp z_{\frac{\gamma}{2}}\sqrt{v_{11}} \\ \hat{\lambda} \mp z_{\frac{\gamma}{2}}\sqrt{v_{22}} \end{cases}, \tag{17}$$

where v_{11} and v_{22} are the elements of the diagonal of variance covariance matrix $\Phi_0^{-1}\left(\hat{\alpha},\hat{\lambda}\right)$ with the tabulated value $z_{\frac{\gamma}{2}}$ of the standard normal distribution with a right-tail probability given by $\frac{\gamma}{2}$.

3 Bayesian Estimation

In this section, we adopted the Bayesian approach to estimate the parameter or any function of the parameters. The efficiency of this approach depending on the amount of information exist in the prior distributions about the unknown parameters and the information exists in data. Then, for the unknown NH parameters α and λ, the informative independent gamma prior densities are considered for each parameters, as follows

$$\begin{cases} \delta_1(\alpha) \propto \alpha^{a-1}e^{(-b\alpha)},\ \alpha > 0,\ a,b > 0 \\ \delta_2(\lambda) \propto \lambda^{c-1}e^{(-d\lambda)},\ \lambda > 0,\ c,d > 0 \end{cases}. \tag{18}$$

From (18), the joint prior density of α and λ can be written as

$$\delta(\alpha,\lambda) \propto \alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)},\ \alpha,\lambda > 0,\ a,b,c,d > 0. \tag{19}$$

The joint posterior distribution of α and λ is derived as

$$\pi(\alpha,\lambda|\underline{t}) = \frac{\delta(\alpha,\lambda)L(\alpha,\lambda|\underline{t})}{\int_0^\infty\int_0^\infty \delta(\alpha,\lambda)L(\alpha,\lambda|\underline{t})d\alpha d\lambda}, \tag{20}$$

Formulated under (19) and (8) by

$$\pi(\alpha,\lambda|\underline{t}) \propto \alpha^{m+a-1}\lambda^{m+c-1}e^{\left\{-b\alpha-d\lambda+(\alpha-1)\sum\limits_{i=1}^{m}\log(1+\lambda t_i)+k\sum\limits_{i=1}^{m}(R_i+1)[1-(1+\lambda t_i)^{\alpha}]\right\}}. \tag{21}$$

The Bayes estimators of function $\eta(\alpha,\lambda)$ under squared error (SE) loss function is given by

$$\hat{\eta}_{\mathrm{SE}} = \int_0^\infty\int_0^\infty \eta(\alpha,\lambda)\pi(\alpha,\lambda|\underline{t})d\alpha d\lambda. \tag{22}$$

Bayes estimators (21) with (22) show that the estimation is given in a ratio of two integrals, one from normalized constant and the other from the estimation properties, which in general can not be obtained in a closed form. Then, the approximate method is used. One of the important methods that can be used in this case is the MCMC method.

MCMC approximation

The joint posterior distribution (21) of α and λ given observed data $\underline{t}$ can be written as

$$\pi(\alpha,\lambda|\underline{t}) \propto G_1(\alpha|\lambda,\underline{t})G_2(\lambda|\underline{t})Z(\alpha,\lambda|\underline{t}), \tag{23}$$

where $G_1(\alpha|\lambda,\underline{t})$ is distributed as the probability gamma density with the shape parameter $(m+a)$ and scale parameter $b+2\sum\limits_{i=1}^{m}\log(1+\lambda t_i)$, and $G_2(\lambda|\underline{t})$ is a proper density function of λ given by

$$G_2(\lambda|\underline{t}) \propto \frac{e^{\left\{(m+c-1)\log\lambda-d\lambda-\sum\limits_{i=1}^{m}\log(1+\lambda t_i)\right\}}}{\left(b+2\sum\limits_{i=1}^{m}\log(1+\lambda t_i)\right)^{m+a}}, \tag{24}$$

and

$$Z(\alpha,\lambda|\underline{t}) \propto e^{\left\{3\alpha\sum\limits_{i=1}^{m}\log(1+\lambda t_i)-k\sum\limits_{i=1}^{m}(R_i+1)(1+\lambda t_i)^{\alpha}\right\}}. \tag{25}$$

Then, the importance sampling technique is applied to obtain the Bayes estimates, see [19] and the corresponding HPD intervals, as follows:

Algorithm

Step 1. Let η be any function of the parameters (α, λ). Then, with the initial guess vector $(\alpha^{(0)}, \lambda^{(0)})$=$(\hat{\alpha}, \hat{\lambda})$, the intial $\eta^{(0)} = \eta(\hat{\alpha}, \hat{\lambda})$
Step 2. Put $I = 1$.
Step 3. Generate $\alpha^{(I)}$ from gamma distribution with the shape parameter $(m+a)$, and the scale parameter $b+2\sum_{i=1}^{m}\log(1+\lambda^{(I-1)}t_i)$, and $\lambda^{(I)}$ from (24) with normal proposal distribution $N(\lambda^{(I-1)}, V_{22})$.
Step 4. Put $\eta^{(I)} = \eta(\alpha^{(I)}, \lambda^{(I)})$ and $I = I+1$.
Step 5. Repeat steps 3-4 for N times.
Step 6. Under SE loss function, the Bayes estimate of η is given by

$$\hat{\eta}_B = \frac{\frac{1}{N-M^*}\sum_{i=M^*+1}^{N}\eta(i)\left(\alpha^{(i)},\lambda^{(i)}\right)Z(\alpha^{(i)},\lambda^{(i)}|\underline{t})}{\frac{1}{N-M^*}\sum_{i=M^*+1}^{N}Z(\alpha^{(i)},\lambda^{(i)}|\underline{t})}. \tag{26}$$

The posterior variance of η is given by

$$V(\hat{\eta}_B) = \frac{\frac{1}{N-M^*}\sum_{i=M*+1}^{N}(\eta^{(i)}\left(\alpha^{(i)},\lambda^{(i)}\right)-\hat{\eta}_B)^2Z(\alpha^{(i)},\lambda^{(i)}|\underline{t})}{\frac{1}{N-M^*}\sum_{i=M*+1}^{N}Z(\alpha^{(i)},\lambda^{(i)}|\underline{t})}. \tag{27}$$

Step 7. The HPD credible intervals are adopted for the interval estimation. For more details see [19].

1-The sample $\eta^{(i)}, i = M^*+1, M^*+2, \ldots, N$ with the help of the importance sampling technique is obtained.
2-Sort the sample in the ascending order $\eta_{(i)}, i = 1, 2, \ldots, N-M^*$.
3-Compute the weighted function $w^{(i)}$

$$w^{(i)} = \frac{Z(\alpha^{(i)},\lambda^{(i)}|\underline{t})}{\sum_{j=M^*+1}^{N}Z(\alpha^{(i)},\lambda^{(i)}|\underline{t})}. \tag{28}$$

Then, the sort the weighted sample is in ascending order $w_{(i)}, i = 1, 2, \ldots, N-M^*$. Therefore the $i-$th value $\mathrm{w}_{(i)}$ corresponds to the the value $\eta_{(i)}$.
4-Estimate the $\gamma$the quantile of the marginal posterior of η by

$$\tilde{\eta}^{(\gamma)} = \begin{cases} \eta_{(1)}, \text{ if } \gamma = 0 \\ \eta_{(j)}, \text{ if } \sum_{i=1}^{j-1}w_{(i)} < \gamma < \sum_{i=1}^{j}w_{(i)} \end{cases}. \tag{29}$$

5-Compute the $100(1-\alpha\%)$ credible intervals of θ

$$(\theta^{(L/(N-M^*))}, \theta^{(\{L+[(1-\alpha)(N-M^*)]\}/(N-M^*))}) \tag{30}$$

where $L = 1, 2, \ldots, (N-M^*) - [(1-\alpha)(N-M^*)]$.
6-The 100 (1-α%) HPD interval is the one with the smallest interval width among all credible intervals.

4 Bayesian Prediction

In this section, we consider the Bayesian two-sample scheme to predict the future order statistic and record values under the observed progressive first-failure censored sample.

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4.1 Bayesian prediction for future order statistics

Suppose that $T^{\mathbf{r}}_{1:m:n:k}$, $T^{\mathbf{r}}_{2:m:n:k}, \ldots,$ $T^{\mathbf{r}}_{m:m:n:k}$ is m progressive first-failure censored sample is obtained from life testing experiment with n items whose lifetimes have the NH distribution with parameters α and λ. Moreover suppose Y_1, Y_2, ..., Y_s is an independent random sample of size s of future NH lifetimes random ordered values. Our objective is predicting the l-th ordered lifetime value from a future sample of size s, $1 < l < s$ known with two-sample Bayesian prediction technique.

The probability density function of the l-th order statistic [20] for given α, λ is given by

$$g_l(y_l|\alpha,\lambda) = D\left[1 - F(y_l|\alpha,\lambda)\right]^{(s-l)}\left[F(y_l|\alpha,\lambda)\right]^{l-1} f(y_l|\alpha,\lambda),\ \alpha,\lambda > 0, \tag{31}$$

where $D = l\binom{s}{l}$ is a normalizing constant satisfying $\int g_l(y_l|\alpha,\lambda)dy_l = 1$. From (4), (5), and (31), we get

$$g_l(y_l|\alpha,\lambda) = D\alpha\lambda\,(1+\lambda y_l)^{\alpha-1}\, e^{(s-l+1)\left(1-(1+\lambda y_l)^{\alpha}\right)}\left[1 - e^{\left\{1-(1+\lambda y_l)^{\alpha}\right\}}\right]^{l-1}, \tag{32}$$

Using the binomial expansion, the density function (32) takes the form

$$g_l(y_l|\alpha,\lambda) = D\alpha\lambda\,(1+\lambda y_l)^{\alpha-1}\sum_{j=0}^{l-1} a_j e^{(s-l+j+1)\left(1-(1+\lambda y_l)^{\alpha}\right)},\ y_l > 0, \tag{33}$$

where $a_j = (-1)^j\binom{l-1}{j}$. Then, the predictive posterior density of future ordered values under the progressive first-failure sample can be obtained by (21) and (33).

$$g_l^*(y_l|\alpha,\lambda) = \int_0^{\infty}\int_0^{\infty} g_l(y_l|\alpha,\lambda)\pi(\alpha,\lambda|\underline{t})d\alpha d\lambda, \tag{34}$$

The analytical form of (33) is more difficult. Therefore, the consistent estimator for $g_l^*(y_l|\alpha,\lambda)$ under the MCMC sample is described. Suppose that $\{(\alpha^{(i)},\lambda^{(i)}),\ i = M^*+1, M^*+2, \ldots, N\}$ are the MCMC samples. Then, the consistent estimator of $g_l^*(y_l|\alpha,\lambda)$, is given by

$$g_l^*(y_l|\alpha,\lambda) = \frac{1}{N-M^*}\sum_{i=M^*+1}^{N} g_l(y_l|\alpha^{(i)},\lambda^{(i)})w_i, \tag{35}$$

where $w^{(i)}$ is given by (28), and M^* is burn-in.

Bayes point prediction

From (32), the Bayes point predictor under SEL of $l-$th order statistic y_s, $1 < l < s$ is given by

$$\hat{Y}_O = \int_0^{\infty} y_l\hat{g}_l^*(y_l|\alpha,\lambda)dy_l = \frac{1}{N-M^*}\sum_{i=M^*+1}^{N}\int_0^{\infty} y_l g_l(y_l|\alpha^{(i)},\lambda^{(i)})w_i dy_l. \tag{36}$$

Bayesian prediction intervals

A prediction interval (PI) is an interval that uses the results of a past sample and contains the results of a future sample from the same population with a specified probability. The distribution function of the density function, $g_l(y_l|\alpha,\lambda)$, is given by

$$G_l(y_l|\alpha,\lambda) = D\sum_{j=0}^{l-1} a_j\frac{1}{(s-l+j+1)}\left[1 - e^{(s-l+j+1)\left(1-(1+\lambda y_l)^{\alpha}\right)}\right], \tag{37}$$

If we want to estimate the predictive distribution of y_l, say $G_l^*(y_l|\alpha,\lambda)$, a simulation consistent estimator of $G_l^*(y_l|\alpha,\lambda)$ can be obtained as

$$\hat{G}_l^*(y_l|\alpha,\lambda) = \frac{1}{N-M^*}\sum_{i=M^*+1}^{N} G_l(y_l|\alpha^{(i)},\lambda^{(i)})w_i. \tag{38}$$

Then, the Bayesian predictive bounds of a two-sided equitailed $100(1-\gamma)\%$ interval for Y_l can be obtained by solving the following two equations for the lower bound L and upper bound U

$$P\left(Y_{(l)} > L|\underline{t}\right) = 1 - \hat{G}_l^*(L|\alpha,\lambda) = 1 - \frac{\gamma}{2},$$

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Hence

$$\hat{G}_l^*(L|\alpha,\lambda) = \frac{\gamma}{2}, \tag{39}$$

and

$$P\left(Y_{(l)} > U|\underline{t}\right) = 1 - \hat{G}_s^*(U|\alpha,\lambda) = \frac{\gamma}{2},$$

hence

$$\hat{G}_l^*(U|\alpha,\lambda) = 1 - \frac{\gamma}{2}. \tag{40}$$

Then, the two non-linear equations (39) and (40) cannot be solved with analytically methods. We can use numerical techniques for solving these non-linear equations, such as the Newton Rafson method.

4.2 Bayesian prediction for future record values

For given $T^{\mathbf{r}}_{1:m:n:k}$, $T^{\mathbf{r}}_{2:m:n:k}$, ..., $T^{\mathbf{r}}_{m:m:n:k}$ the progressive first-failure censored sample of size m can be drawn from the NH distribution with the parameters α and λ. Suppose that, X_1, X_2, ..., X_s is an independent future sample of the upper record values from the same population. The PDF of the X_l, $l = 1,...,s$ [21] is given by

$$h_l\left(x_l|\alpha,\lambda\right) = \frac{\left[-\log(1-F\left(x_l|\alpha,\lambda\right))\right]^{l-1}}{(l-1)!} f\left(x_l|\alpha,\lambda\right). \tag{41}$$

The Eqs (4), (5), and (41) are reduced

$$h_l\left(x_l|\alpha,\lambda\right) = \frac{\alpha\lambda}{(l-1)!}\sum_{j=0}^{l-1} b_j(1+\lambda x_l)^{\alpha(j+1)-1} e^{(1-(1+\lambda x_l)^{\alpha})}, \tag{42}$$

where $b_j = (-1)^{l+j-1}\binom{l-1}{j}$. Then, the predictive posterior density of the future record value under the progressively first-failure sample is given by

$$h_l^*(x_l|\alpha,\lambda) = \int_0^{\infty}\int_0^{\infty} h_l(x_l|\alpha,\lambda)\pi(\alpha,\lambda|\underline{t})d\alpha d\lambda, \tag{43}$$

The consistent estimator of $h_l^*(x_l|\alpha,\lambda)$ under the MCMC sample is given by

$$\hat{h}_l^*(x_l|\alpha,\lambda) = \frac{1}{N-M^*}\sum_{i=M^*+1}^{N} h_l\left(x_l|\alpha^{(i)},\lambda^{(i)}\right)w^{(i)}, \tag{44}$$

Bayesian point prediction

Under the SEL functions Bayes point predictors of X_l, is given by

$$\hat{X}_{\mathrm{U}} = \int_0^{\infty} x_l\hat{h}_l^*(x_l|\alpha,\lambda)dx_l = \frac{1}{N-M^*}\sum_{i=M^*+1}^{N}\int_0^{\infty} x_l h_l\left(x_l|\alpha^{(i)},\lambda^{(i)}\right)w_i dx_l. \tag{45}$$

Bayesian prediction intervals

A prediction interval (PI) is an interval that uses the results of a past sample and contains the results of a future sample from the same population with a specified probability. The distribution function of the density function, $h_l(x_l|\alpha,\lambda)$, is given

$$H_l(x_l|\alpha,\lambda) = \frac{e}{(l-1)!}\sum_{j=0}^{l-1} b_j\left[\Gamma(j+1,1) - \Gamma(j+1,(1+\lambda x_l)^{\alpha})\right], \tag{46}$$

If we want to estimate the predictive distribution of x_l, say $H_l^*(x_l|\alpha,\lambda)$, a simulation consistent estimator of $H_l^*(x_l|\alpha,\lambda)$ can be obtained as

$$H_l^*(x_l|\alpha,\lambda) = \frac{1}{N-M^*}\sum_{i=M^*+1}^{N} H_l(x_l|\alpha^{(i)},\lambda^{(i)})w_i. \tag{47}$$

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It should be noted that the MCMC samples $\{(\alpha^{(i)}, \lambda^{(i)}), i = M^*+1, M^*+2, ..., N\}$ can be used to compute $\hat{h}_l^*(x_l|\alpha,\lambda)$ or $\hat{H}_l^*(x_l|\alpha,\lambda)$ for all X_l. Moreover, a symmetric $100\gamma\%$ predictive interval for X_l can be obtained by solving the following two equations for the lower bound, L and the upper bound, U

$$P(X_l > L|\underline{t}) = 1 - \hat{H}_l^*(L|\alpha,\lambda) = 1 - \frac{\gamma}{2},$$

Hence

$$\hat{H}_l^*(L|\underline{t}) = \frac{\gamma}{2}, \tag{48}$$

and

$$P(x_l > U|\underline{t}) = 1 - \hat{H}_l^*(U|\alpha,\lambda) = \frac{\gamma}{2},$$

Hence

$$\hat{H}_l^*(U|\underline{t}) = 1 - \frac{\gamma}{2}. \tag{49}$$

The analytical solution, in this case, is not possible then. We need a numerical technique for solving these non-linear equations, such as the Newton Rafson method.

5 A Simulation Study

In this section, we analyze the effect of the estimators through a Monte Carlo simulation study. We study the effect of choosing the true value of the parameters and the combination of censoring parameters (k, m, n, r). The algorithm presented by [22], and the method of generating progressive first-failure samples mentioned in [23] are applied. All analysis is done through software R. the parameter with values such as α =1.0 and λ =1.5 and the corresponding hyper-parameters (a, b, c, d) of the prior distributions $\delta_1(\alpha)$ and $\delta_2(\lambda)$ as called, prior$_1$ = (a =0.5, b =1.0, c =10.0, d =5.0) are selected. Also, α =1.5 and λ =1.0 with corresponding hyper-parameters (a, b, c, d) of the prior distributions $\delta_1(\alpha)$ and $\delta_2(\lambda)$ as called,prior (2) = (a =3.0, b =2.0, c =0.5, d =1.0) are selected. The group size k is selected to be 1 and 3 and determines two sets of combination for n and m say $n = 30$, $m = 15, 25$; $n = 50$, $m = 25, 40$ with different r_i. In our study, we used four different censoring schemes (I, II, III,V) described in Table 1. In our simulation algorithm, we generated 11,000 MCMC samples, and discarded the first 1000 iterations. The average values (AV) of the MLE and Bayes estimate of α and λ along with their mean squared error (MSE) are computed and summarized in Tables 2 and 3, respectively. Also, we report the average length (AL) of 99% confidence interval and coverage percentages (CP) in Tables 4 and 5.

Table (1): Different censoring schemes that described simulation study.

scheme	(n,m)	Removals (Ri)	(n,m)	Removals (Ri)
I	$(30,15)$	$(15,0^{14})$	$(30,15)$	$(5,0^{24})$
II		$(0^{14},15)$		$(0^{24},5)$
III		$(0^7,15,0^7)$		$(0^{12},5,0^{12})$
V		(1^{15})		$(1,0^4)^5$
I	$(50,25)$	$(25,0^{24})$	$(50,40)$	$(10,0^{39})$
II		$(0^{24},25)$		$(0^{39},10)$
III		$(0^{12},25,0^{12})$		$(0^{19},10,0^{20})$
V		(1^{25})		$(1,0^3)^{10}$

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Table (2) The AV and MSEs of the parameters α and λ at $(\alpha = 1.0, \lambda = 1.5)$ with prior1.

k	(n,m)	CS	$(\cdot)_{\text{ML}}$		$(\cdot)_{\text{Bayes}}$	
			α	λ	α	λ
1	(30,15)	*I*	1.15(0.39)	1.47(0.65)	0.86(0.18)	2.02(0.52)
		II	1.11(0.44)	1.58(0.73)	0.79(0.24)	2.07(0.57)
		III	1.14(0.39)	1.51(0.66)	0.84(0.20)	2.03(0.53)
		IV	1.13(0.40)	1.50(0.65)	0.82(0.21)	2.05(0.55)
	(30,25)	*I*	1.12(0.37)	1.52(0.68)	0.87(0.15)	1.98(0.50)
		II	1.10(0.36)	1.54(0.65)	0.85(0.18)	2.02(0.53)
		III	1.13(0.36)	1.52(0.63)	0.88(0.15)	1.99(0.50)
		IV	1.11(0.36)	1.52(0.66)	0.86(0.16)	1.99(0.50)
	(50,25)	*I*	1.14(0.36)	1.46(0.62)	0.88(0.14)	1.97(0.49)
		II	1.11(0.42)	1.60(0.68)	0.82(0.20)	2.07(0.57)
		III	1.08(0.34)	1.55(0.63)	0.85(0.17)	1.99(0.50)
		IV	1.09(0.37)	1.60(0.67)	0.84(0.18)	2.03(0.54)
	(50,40)	*I*	1.12(0.34)	1.48(0.60)	0.90(0.13)	1.93(0.46)
		II	1.12(0.36)	1.51(0.64)	0.86(0.15)	1.99(0.50)
		III	1.14(0.36)	1.46(0.63)	0.89(0.13)	1.91(0.45)
		IV	1.11(0.34)	1.53(0.62)	0.89(0.13)	1.95(0.47)
3	(30,15)	*I*	1.06(0.37)	1.63(0.66)	0.81(0.23)	2.07(0.58)
		II	1.05(0.45)	1.70(0.69)	0.80(0.27)	2.09(0.61)
		III	1.03(0.37)	1.68(0.69)	0.79(0.25)	2.08(0.58)
		IV	1.10(0.46)	1.60(0.73)	0.77(0.26)	2.08(0.58)
	(30,25)	*I*	1.08(0.37)	1.62(0.67)	0.83(0.20)	2.05(0.56)
		II	1.14(0.48)	1.56(0.72)	0.80(0.22)	2.07(0.57)
		III	1.08(0.38)	1.60(0.66)	0.82(0.21)	2.06(0.56)
		IV	1.06(0.37)	1.64(0.65)	0.82(0.21)	2.06(0.57)
	(50,25)	*I*	1.11(0.40)	1.56(0.66)	0.82(0.20)	2.05(0.55)
		II	1.07(0.46)	1.66(0.70)	0.80(0.24)	2.11(0.64)
		III	1.04(0.36)	1.65(0.67)	0.80(0.22)	2.06(0.57)
		IV	1.10(0.45)	1.61(0.71)	0.83(0.22)	2.07(0.61)
	(50,40)	*I*	1.10(0.37)	1.57(0.63)	0.84(0.18)	2.02(0.53)
		II	1.13(0.46)	1.56(0.70)	0.80(0.20)	2.06(0.56)
		III	1.09(0.39)	1.58(0.65)	0.83(0.19)	2.03(0.54)
		IV	1.09(0.39)	1.58(0.65)	0.94(0.17)	2.03(0.64)

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Table (3). The AV and MSE of the parameters α and λ at ($\alpha = 1.5$, $\lambda = 1.0$) with prior2.

k	(n,m)	CS	$(\cdot)_{ML}$		$(\cdot)_{Bayes}$	
			α	λ	α	λ
1	(30,15)	*I*	1.56(0.54)	1.22(0.62)	1.68(0.23)	1.11(0.19)
		II	1.83(0.81)	1.06(0.61)	1.70(0.22)	1.09(0.19)
		III	1.62(0.62)	1.21(0.69)	1.67(0.22)	1.12(0.23)
		IV	1.53(0.62)	1.26(0.72)	1.67(0.20)	1.10(0.19)
	(30,25)	*I*	1.53(0.52)	1.25(0.66)	1.66(0.24)	1.13(0.25)
		II	1.50(0.58)	1.29(0.61)	1.67(0.20)	1.12(0.19)
		III	1.61(0.62)	1.21(0.69)	1.66(0.23)	1.11(0.25)
		IV	1.57(0.56)	1.20(0.64)	1.66(0.22)	1.10(0.21)
	(50,25)	*I*	1.59(0.61)	1.23(0.69)	1.67(0.23)	1.12(0.25)
		II	1.77(0.71)	1.12(0.65)	1.71(0.23)	1.11(0.18)
		III	1.63(0.64)	1.20(0.68)	1.66(0.24)	1.11(0.25)
		IV	1.52(0.51)	1.27(0.71)	1.66(0.21)	1.13(0.21)
	(50,40)	*I*	1.52(0.51)	1.22(0.62)	1.82(0.47)	0.84(0.31)
		II	1.41(0.57)	1.31(0.72)	1.84(0.45)	0.81(0.27)
		III	1.59(0.58)	1.21(0.65)	1.81(0.60)	0.70(0.38)
		IV	1.51(0.54)	1.26(0.66)	1.84(0.57)	0.74(0.36)
3	(30,15)	*I*	1.62(0.73)	1.25(0.74)	1.69(0.22)	1.11(0.21)
		II	1.99(0.86)	0.902(0.55)	1.70(0.26)	1.06(0.22)
		III	1.54(0.66)	1.23(0.68)	1.67(0.19)	1.08(0.17)
		IV	1.83(0.82)	1.09(0.70)	1.71(0.22)	1.08(0.11)
	(30,25)	*I*	1.55(0.69)	1.29(0.71)	1.68(0.21)	1.13(0.19)
		II	1.66(0.76)	1.18(0.66)	1.61(0.23)	1.11(0.20)
		III	1.48(0.71)	1.33(0.76)	1.65(0.18)	1.11(0.17)
		IV	1.53(0.72)	1.33(0.77)	1.68(0.20)	1.13(0.11)
	(50,25)	*I*	1.51(0.69)	1.36(0.79)	1.68(0.21)	1.13(0.21)
		II	1.90(0.72)	0.92(0.47)	1.74(0.41)	1.12(0.34)
		III	1.42(0.64)	1.31(0.79)	1.66(0.19)	1.13(0.21)
		IV	1.69(0.81)	1.22(0.76)	1.70(0.23)	1.10(0.19)
	(50,40)	*I*	1.51(0.65)	1.32(0.75)	1.66(0.20)	1.14(0.21)
		II	1.74(0.76)	1.09(0.57)	1.71(0.23)	1.10(0.18)
		III	1.41(0.67)	1.30(0.74)	1.63(0.11)	1.12(0.11)
		IV	1.55(0.67)	1.28(0.72)	1.67(0.22)	1.14(0.11)

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Table (4) The AL and CP of α and λ at ($\alpha = 1.0, \lambda = 1.5$) with prior1.

k	(n,m)	CS	$(\cdot)_{\text{ML}}$		$(\cdot)_{\text{Bayes}}$	
			α	λ	α	λ
1	(30,15)	*I*	6.51(0.99)	6.14(1.00)	3.10(1.00)	1.83(1.00)
		II	18.6(1.00)	11.0(1.00)	3.30(1.00)	2.14(1.00)
		III	7.61(0.99)	5.90(1.00)	2.94(1.00)	2.59(1.00)
		IV	9.98(0.99)	8.07(1.00)	3.20(1.00)	2.41(1.00)
	(30,25)	*I*	4.82(0.97)	4.70(1.00)	3.18(1.00)	1.42(1.00)
		II	6.94(0.98)	6.14(1.00)	3.38(1.00)	1.54(1.00)
		III	5.55(0.96)	4.48(1.00)	3.11(1.00)	1.60(1.00)
		IV	5.51(0.97)	4.79(1.00)	3.14(1.00)	1.56(1.00)
	(50,25)	*I*	5.49(0.99)	4.61(1.00)	3.16(1.00)	1.47(1.00)
		II	13.7(1.00)	8.61(1.00)	3.38(1.00)	1.82(1.00)
		III	5.83(0.97)	4.38(1.00)	2.77(1.00)	2.05(1.00)
		IV	7.43(0.98)	6.08(1.00)	3.13(1.00)	1.78(1.00)
	(50,40)	*I*	3.69(0.97)	3.53(1.00)	3.15(1.00)	0.92(1.00)
		II	5.79(0.94)	5.16(1.00)	3.35(1.00)	0.91(0.92)
		III	4.13(0.93)	3.52(1.00)	2.98(1.00)	1.04(0.96)
		IV	4.09(0.92)	3.82(1.00)	3.04(1.00)	0.97(1.00)
3	(30,15)	*I*	14.0(0.99)	9.69(1.00)	3.09(1.00)	2.28(1.00)
		II	30.2(1.00)	16.9(1.00)	3.45(1.00)	3.04(0.96)
		III	14.1(0.99)	10.1(1.00)	3.00(1.00)	3.01(1.00)
		IV	28.4(1.00)	16.9(1.00)	3.25(1.00)	2.90(1.00)
	(30,25)	*I*	10.28(0.99)	7.66(1.00)	3.17(1.00)	2.27(1.00)
		II	13.22(1.00)	9.12(1.00)	3.38(1.00)	2.48(1.00)
		III	9.47(0.97)	7.36(1.00)	3.10(1.00)	2.49(1.00)
		IV	9.99(1.00)	7.73(1.00)	3.17(1.00)	2.49(1.00)
	(50,25)	*I*	9.21(0.99)	7.51(1.00)	3.09(1.00)	2.07(1.00)
		II	21.3(1.00)	12.0(1.00)	3.49(1.00)	2.83(0.96)
		III	9.35(0.99)	7.88(1.00)	2.90(1.00)	3.08(1.00)
		IV	13.7(0.99)	9.39(1.00)	3.23(1.00)	2.74(1.00)
	(50,40)	*I*	7.53(0.97)	5.71(1.00)	3.12(1.00)	1.78(1.00)
		II	11.7(1.00)	7.39(1.00)	3.39(1.00)	1.87(1.00)
		III	7.52(0.95)	5.68(1.00)	3.01(1.00)	2.09(1.00)
		IV	8.38(0.97)	6.22(1.00)	3.11(1.00)	2.03(1.00)

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Table (5) The AL and CP of α and λ at ($\alpha=1.5$, $\lambda=1.0$) with prior2.						
k	(n,m)	CS	$(\cdot)_{\text{ML}}$		$(\cdot)_{\text{Bayes}}$	
			α	λ	α	λ
1	(30,15)	I	3.76(0.99)	6.44(0.99)	1.01(0.99)	2.21(1.00)
		II	9.45(1.00)	13.3(1.00)	2.01(0.97)	2.26(1.00)
		III	3.80(0.99)	6.23(1.00)	1.20(1.00)	2.34(1.00)
		IV	5.52(1.00)	8.34(1.00)	1.48(1.00)	2.28(1.00)
	(30,25)	I	2.67(0.99)	4.97(0.99)	0.95(1.00)	2.02(1.00)
		II	3.81(0.99)	6.48(1.00)	1.26(1.00)	2.01(1.00)
		III	2.73(0.99)	4.93(1.00)	1.01(1.00)	2.08(1.00)
		IV	2.77(1.00)	5.09(0.99)	1.01(1.00)	2.05(1.00)
	(50,25)	I	2.75(0.99)	4.79(0.99)	0.96(1.00)	2.04(1.00)
		II	7.15(1.00)	10.3(1.00)	1.48(1.00)	2.11(1.00)
		III	2.55(0.99)	4.68(0.99)	1.07(1.00)	2.26(1.00)
		IV	3.84(0.99)	6.50(1.00)	1.41(1.00)	2.16(1.00)
	(50,40)	I	2.04(0.99)	3.79(0.98)	0.97(1.00)	1.80(1.00)
		II	3.19(0.99)	5.17(1.00)	1.39(1.00)	1.78(1.00)
		III	2.12(0.99)	3.63(0.98)	0.98(1.00)	1.89(1.00)
		IV	2.14(0.99)	4.03(0.99)	1.05(1.00)	1.86(1.00)
3	(30,15)	I	6.60(1.00)	10.5(1.00)	1.12(0.99)	2.34(1.00)
		II	14.7(1.00)	23.5(1.00)	2.49(0.82)	2.39(1.00)
		III	6.82(1.00)	10.9(1.00)	1.35(1.00)	2.47(1.00)
		IV	11.3(1.00)	15.5(1.00)	1.73(0.99)	2.41(1.00)
	(30,25)	I	5.06(0.99)	7.95(1.00)	1.04(1.00)	2.24(1.00)
		II	8.68(1.00)	11.0(1.00)	1.77(0.98)	2.23(1.00)
		III	5.31(0.99)	7.90(1.00)	1.08(1.00)	2.29(1.00)
		IV	5.27(1.00)	8.34(1.00)	1.41(1.00)	2.28(1.00)
	(50,25)	I	5.35(0.99)	7.69(1.00)	1.03(0.99)	2.25(1.00)
		II	11.0(1.00)	16.5(1.00)	2.03(0.93)	2.30(1.00)
		III	5.13(1.00)	7.90(1.00)	1.20(1.00)	2.45(1.00)
		IV	8.73(1.00)	11.7(1.00)	2.03(0.89)	2.36(1.00)
	(50,40)	I	3.99.(1.00)	5.97(1.00)	1.00(1.00)	2.11(1.00)
		II	6.85(1.00)	9.08(1.00)	1.50(1.00)	2.09(1.00)
		III	4.13(0.99)	6.05(1.00)	1.04(1.00)	2.22(1.00)
		IV	4.47(0.99)	6.43(1.00)	1.92(1.00)	2.19(1.00)

6 Simulate Data Analysis

In this section, we analyze a simulate data set generated from the NH distribution with parameters values α=1.0 and λ=0.5 using the algorithm described in [22]. The data is generated under k =2, n =30, m =25 and censoring scheme $\mathbf{r}=\{0^{(10)},1^{(5)},0^{(10)}\}$. The prior information are selected to be a=0.5, b=1.0, c=4.0, d=8.0. Then the generated progressive first failure data are given by {**0.0079, 0.0356, 0.0474, 0.0725, 0.0842, 0.1421, 0.2191, 0.2956, 0.3277, 0.3719, 0.3801, 0.3978, 0.5088, 0.5965, 0.6124, 0.6769, 0.6977, 0.7682, 0.9223, 1.1889, 1.2158, 1.4409, 1.9434, 2.2252, 3.1607**}.Under the above data, the point and 95% interval estimate of the ML Bayes methods are given in Table (6). The quality of the convergence of the MCMC method is measured by the plot of the simulation number and corresponding histogram generated by the MCMC method which are reported in Figures (1-4). For the prediction results, the point and interval prediction of the future set of order statistic and record values are presented in Table (7).

Table (6): Two-sided 95% confidence and HP intervals of α and λ.						
	MLE			Bayes(MCMC)		
Method	$(.)_{\text{ML}}$	95% C.I	Length	$(.)_{\text{B}}$	95% C.I of λ	Length h
α	1.185	$(-0.920, 3.2914)$	4.2114	1.123	$(0.8217, 3.3540)$	2.5322
λ	0.481	$(-0.6265, 1.5890)$	2.2155	0.576	$(0.2326, 1.07170)$	0.8391

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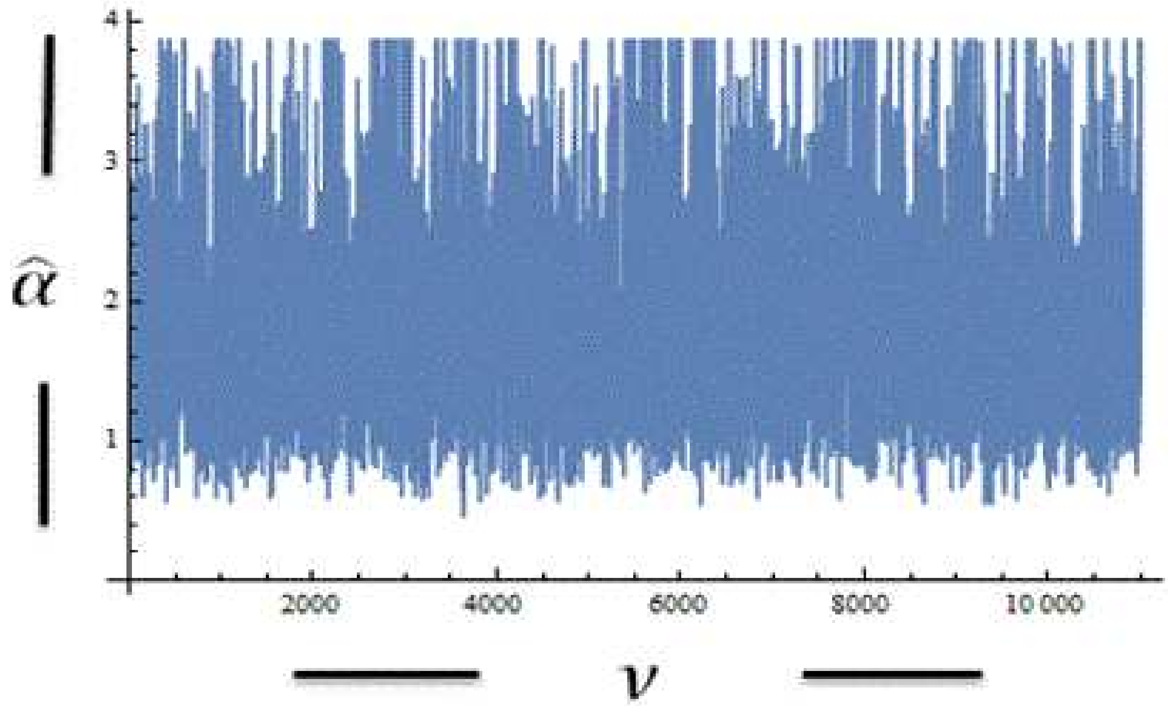
Table (7): The point and 95% Pls predictions of a future order statistics and record values.

Variables	Order statistics				Record values		
$Y_{(s)}$	SEL	L	U	$X_{(s)}$	SEL	L	U
$Y_{(1)}$	0.0595	0.0012	0.1819	$X_{(1)}$	1.7558	0.0288	3.3680
$Y_{(2)}$	0.1205	0.0096	0.2816	$X_{(2)}$	3.5400	0.2587	5.0648
$Y_{(3)}$	0.1834	0.0286	0.3750	$X_{(3)}$	5.3709	0.6062	6.5806
$Y_{(4)}$	0.2500	0.0511	0.4670	$X_{(4)}$	6.8182	0.9774	8.2299
$Y_{(5)}$	0.3170	0.0759	0.5552	$X_{(5)}$	9.4679	1.3236	9.9927

7 Conclusion and Brief Comments

The NH distribution presents a suitable model for fitting data, especially the one that contains zero values than weibull, exponential ,and gamma distributions. This model can be modeling different lifetime products. In this paper, we developed and discussed two methods of estimation for unknown NH parameters. Also, we discussed problem of the prediction of a future sample in two case order statistics and record values based on general censoring schemes whose several censoring scheme is considered as a special case. The results are assessed and compared through the Monto Carlo study and numerical example. From the numerical results, we observe

1.Tables 2 to 5 show that the proposed methods are more acceptable.
2.The results in tables 2 to 5 show that the Bayes estimators perform better than the MLEs in the terms of MSEs, especially in the informative prior.
3.The interval estimation Bayes estimators perform better than the MLEs in terms of the mean length and probability coverage.
4.For the cases that increasing proportion $\frac{m}{n}$, MSEs and withdraw a length of all estimators have been decreasing.
5.The prediction of the future order statistic and record values the proposed methods performs better.



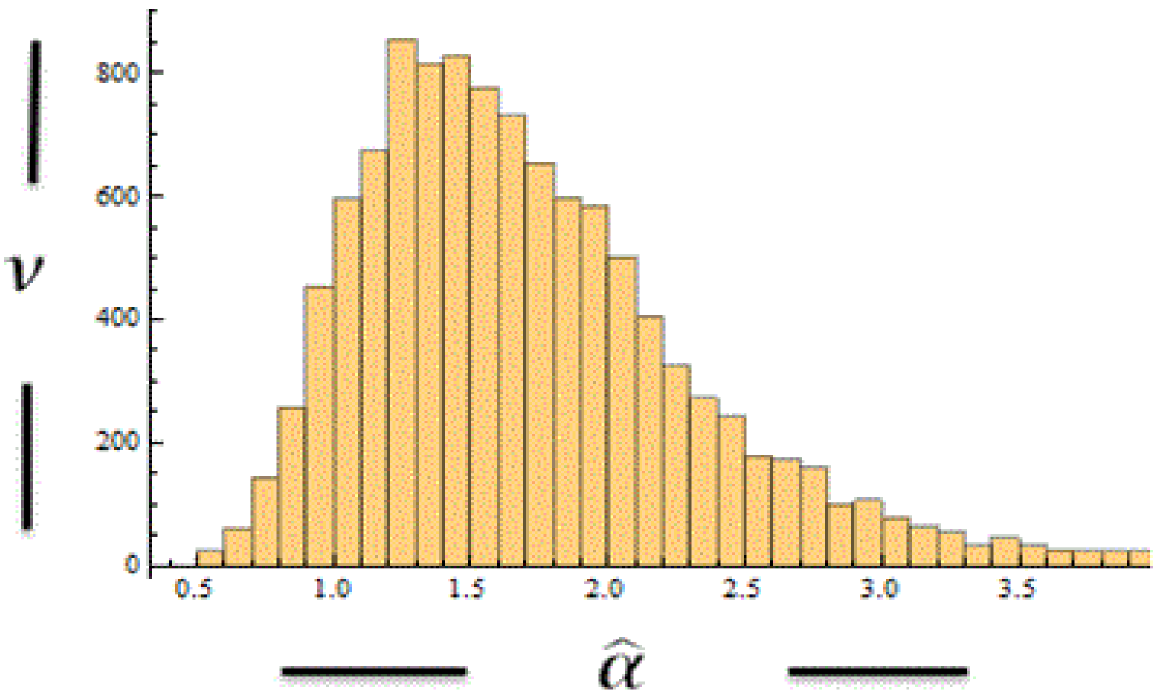
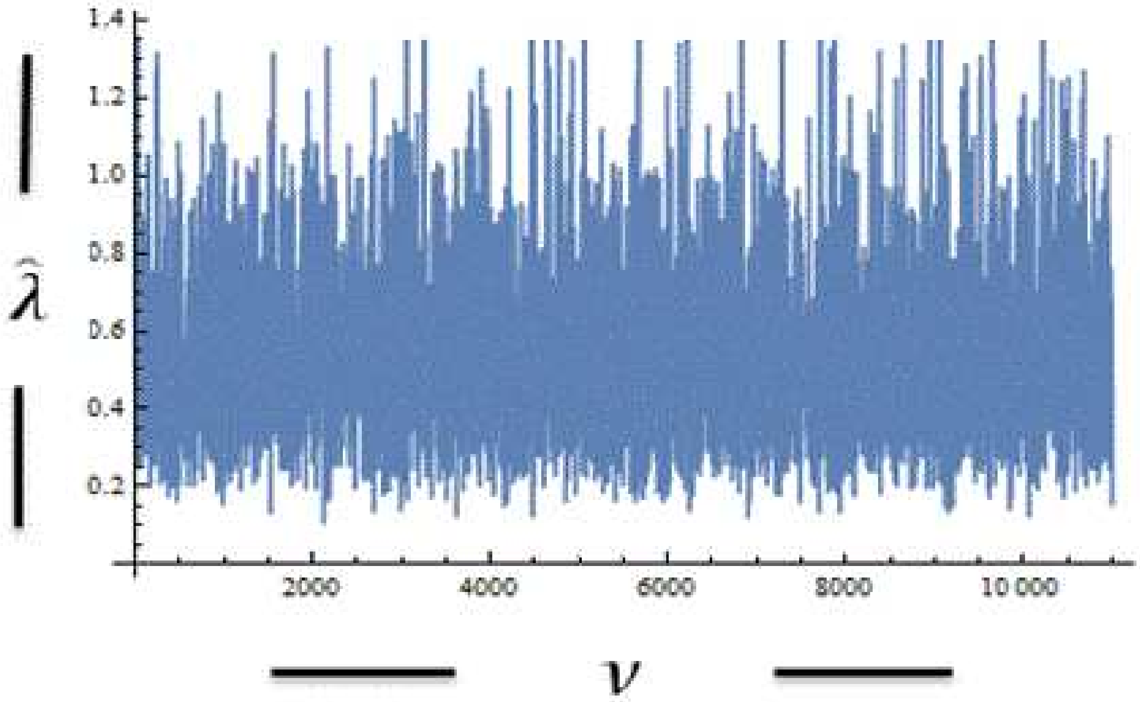


Fig. 1: Simulation number of α generated by MCMC method and Histogram of α generated by MCMC method.



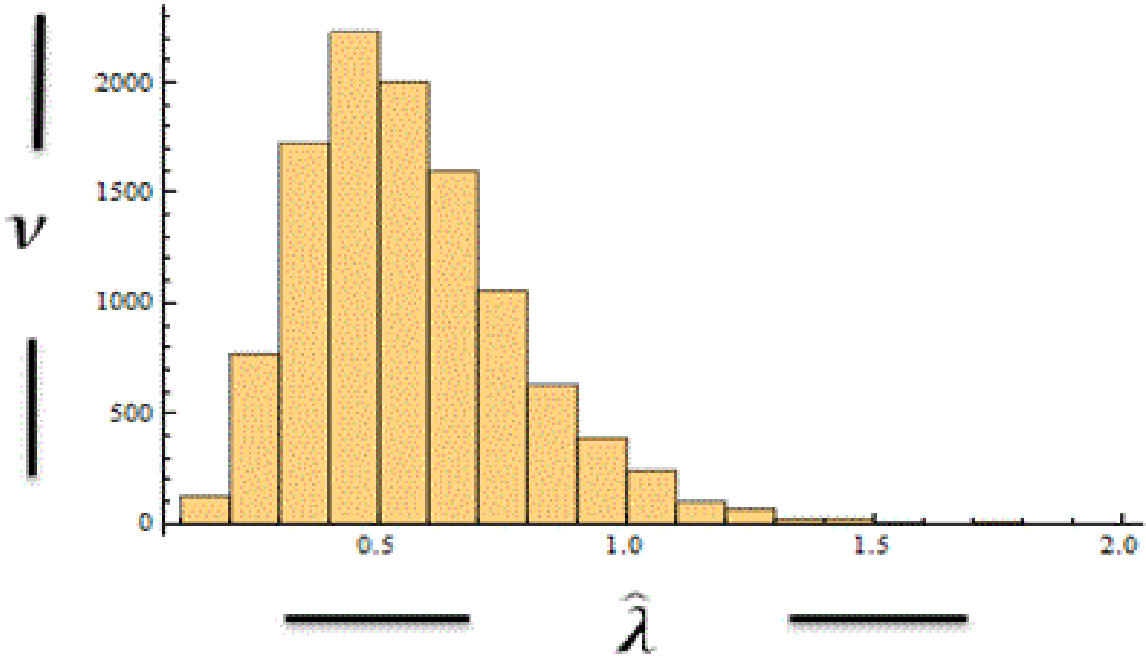


Fig. 2: Simulation number of λ generated by MCMC method and Histogram of λ generated by MCMC method.

Conflict of interest

The authors declare that they have no conflict of interest.

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