On Homotopy of Volterrian Quadratic Stochastic Operators

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In the present paper we introduce a notion of homotopy of two Volterra operators which is related to fixed points of such operators. We establish a criterion for determining when two Volterra operators are homotopic, and as a consequence we obtain that the corresponding tournaments of that operators are the same. This gives us a possibility to know some information about the trajectory of homotopic Volterra operators. Moreover, it is shown that any Volterra q.s.o. given on a face has at least two homotopic extensions to the whole simplex.

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1 Introduction

Since Lotka and Volterra’s seminal and pioneering works [29, 30] many decades ago, modeling of interacting, competing species has received considerable attention in the fields of biology, ecology, mathematics [7, 9, 17, 25, 26] and, more recently, in the physics literature as well [2, 12–14, 21, 23]. In their remarkably simple deterministic model, Lotka and Volterra considered two coupled nonlinear differential equations that mimic the temporal evolution of a two-species system of competing predator and prey populations. They demonstrated that coexistence of both species was not only possible but inevitable in their model. Moreover, similar to observations in real populations, both predator and prey densities in this deterministic system display regular oscillations in time, with both the amplitude and the period determined by the prescribed initial conditions.
To investigate the computational aspects of such dynamical systems, we need to consider discretization of such systems. This leads to the study of the trajectory of discrete time Volterra operators. Therefore, in [3–5, 18, 19] discrete time Volterra operators were considered and investigated. (Note that the more general quadratic operators were studied by many authors, see for example, [1, 10, 16]). A connection between such dynamical systems and the theory of tournaments were established. This allows us to get some information about the trajectory of Volterra operators by looking at the corresponding tournaments, which are related to fixed points of Volterra operators. Moreover, some ergodic properties of such operators, in small dimensions, were studied in [6, 28, 31]. However, still much information is unknown about behavior of Volterra operators.

In the present paper we introduce a notion of homotopy of two Volterra operators which is related to fixed points of such operators. Further, we will establish a criterion for determining when two Volterra operators are homotopic, and as a consequence we obtain that the corresponding tournaments of that operators are the same. This, due to [3], gives us a possibility to know some information about the trajectory of homotopic Volterra operators. Moreover, it is shown that any Volterra q.s.o. given on a face has at least two homotopic extensions to the whole simplex.

2 Preliminaries

We denote by

\[ S^{m-1} = \{ x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : \sum_{k=1}^m x_k = 1, \ x_k \geq 0 \} \]

the \((m - 1)\)-dimensional simplex. The vertices of the simplex \(S^{m-1}\) are described by the elements \(e_k = (\delta_{1k}, \delta_{2k}, \ldots, \delta_{mk})\), where \(\delta_{ik}\) is the Kronecker’s symbol. Let \(I = \{1, 2, \ldots, m\}\) and \(\alpha \subset I\) be an arbitrary subset. By \(\Gamma_\alpha\) we denote the convex hull of the vertices \(\{e_i\}_{i \in \alpha}\). The set \(\Gamma_\alpha\) is usually called \((|\alpha| - 1)\)-dimensional face of the simplex, where \(|\alpha|\) stands for the cardinality of \(\alpha\). An interior of \(\Gamma_\alpha\) in the induced topology of \(\mathbb{R}^m\) to affine hull \(\Gamma_\alpha\) is called relative interior and is denoted by \(rI_\Gamma_\alpha\). One can see that

\[ rI_\Gamma_\alpha = \{ x \in S^{m-1} : x_k > 0 \ \forall k \in \alpha; \ x_k = 0 \ \forall k \notin \alpha \} \]

Similarly, one can define relative boundary \(\partial I_\Gamma_\alpha\) of the face \(\Gamma_\alpha\). In particular, we have

\[ rI S^{m-1} = \{ x \in S^{m-1} : x_k > 0 \ \forall k \in I \} \]

\[ \partial S^{m-1} = \{ x \in S^{m-1} : \exists k \in I; x_k = 0 \} \]

A Volterra quadratic stochastic operator (q.s.o.) \(V : S^{m-1} \rightarrow S^{m-1}\) is defined by

\[ (V(x))_k = x_k \left( 1 + \sum_{i=1}^m a_{ki}x_i \right), \ k = 1, m, \]  

(2.1)
where \( a_{ki} = -a_{ik}, |a_{ki}| \leq 1 \), i.e. \( A_m = (a_{ki})_{k,i=1}^m \) is a skew-symmetrical matrix.

Note that in what follows sometimes for the sake for shortness we will use a terminology Volterra operator instead of Volterra q.s.o.

Let
\[
Fix(V) = \{x \in S^{m-1} : Vx = x\}
\]

be the set of all fixed points of the Volterra operator \( V \).

One can see that for any Volterra operator \( V \) the set \( Fix(V) \) contains all the vertices of the simplex \( S^{m-1} \). Therefore, it is nonempty.

For given \( x \in S^{m-1} \) we consider a sequence \( \{x, Vx, \ldots, V^n x, \ldots\} \) called the trajectory of a Volterra operator \( V \). Limit points of such a sequence is denoted by \( \omega_V(x) \).

For an arbitrary \( x \in \mathbb{R}^m \),
\[
supp(x) = \{i \in I : x_i \neq 0\}
\]
is the support of \( x \). The following statements can be easily proved.

**Theorem 2.1 ([3]).** Let \( V : S^{m-1} \rightarrow S^{m-1} \) be a Volterra operator. Let \( V_\alpha \) be the restriction of \( V \) to the face \( \Gamma_\alpha \). Then the following assertions are true:

(i) For any \( \alpha \subset I \) one has \( V(\Gamma_\alpha) \subset \Gamma_\alpha \).

(ii) For any \( \alpha \subset I \) one has \( V(ri\Gamma_\alpha) \subset ri\Gamma_\alpha \) and \( V(\partial\Gamma_\alpha) \subset \partial\Gamma_\alpha \).

(iii) The restriction \( V_\alpha : \Gamma_\alpha \rightarrow \Gamma_\alpha \) is also a Volterra q.s.o.

(iv) If \( x \in Fix(V) \), then \( Supp(x) \cap Supp(A_m x) = \emptyset \). In particular, if \( x \in Fix(V) \cap riS^{m-1} \), then \( x \in \text{Ker}A_m \), where \( \text{Ker}A_m \) is the kernel of the matrix \( A_m \).

(v) For any \( x \in S^{m-1} \), the set \( \omega_V(x) \) either consists of a single point or is infinite.

(vi) The set of all volterra q.s.o.s geometrically can be considered as a \((m(m-1)/2)\)-dimensional cub on \( \mathbb{R}^{m(m-1)/2} \).

Let \( A_m \) be a skew-symmetrical matrix corresponding to a Volterra operator given by (2.1). It is known [24] that if the order of a skew-symmetrical matrix is odd, then the determinant of this matrix is 0, otherwise the determinant is the square of some polynomial of its entries, which situated above the main diagonal. Such a polynomial is called pfaffian and can by calculated by the following rule:

**Lemma 2.1 ([24]).** Let \( p_m \) be a pfaffian of an even order skew-symmetrical matrix \( A_m \), \( m > 2 \). By \( p_{im} \), we denote a pfaffian of the skew-symmetrical matrix \( A_{im} \), which is obtained from \( A_m \) by deleting of the \( m \)-th and \( i \)-th rows and columns, where \( i = \overline{1, m-1} \). Then one has
\[
p_m = \sum_{i=1}^{m-1} (-1)^{i-1} p_{im} a_{im} \quad p_2 = a_{12},
\]
where \( p_{im} \) is obtained via \( p_{m-2} \) by adding 1 to all indexes greater or equal to \( i \).
A pfaffian of the main minor of an even order skew-symmetric matrix $A_m$ with rows and columns $\{i_1, i_2, \ldots, i_{2k}\}$ is called main subpfaffian of order $2k$, and is denoted by $gp_{i_1i_2 \ldots i_{2k}}$.

For example,

$$gp_{i_1i_2} = a_{i_1i_2}; \quad gp_{i_1i_2i_3i_4} = a_{i_1i_2}a_{i_3i_4} + a_{i_1i_4}a_{i_2i_3} - a_{i_1i_3}a_{i_2i_4}.$$

**Definition 2.1.** A skew-symmetric matrix $A_m$ is called transversal if all even order main minors are nonzero.

It is clear that if a skew-symmetric matrix $A_m$ is transversal, then all even order main subpfaffians are nonzero, that is

$$gp_{i_1i_2 \ldots i_{2k}} \neq 0, \quad \forall i_1, i_2, \ldots, i_{2k} \in I,$$

in particular, $a_{ki} \neq 0$ for $k \neq i$.

**Definition 2.2.** We say that a Volterra q.s.o. $V$ is transversal if the corresponding skew-symmetric matrix $A_m$ is transversal.

We denote $V^m_{t-1}$ the set of all transversal Volterra q.s.o.s defined in the simplex $S^{m-1}$.

Henceforth, we will consider only transversal Volterra q.s.o. without using the word "transversal".

**Theorem 2.2 ([3]).** Let $V \in V^m_{t-1}$, then

(i) $Fix(V)$ is a finite set;

(ii) if $x \in Fix(V)$, then the cardinality of $Supp(x)$ is odd;

(iii) for any face $\Gamma_\alpha$ of the simplex $S^{m-1}$ one has $|Fix(V) \cap ri\Gamma_\alpha| \leq 1$.

**Remark 2.1.** Note that there is no fixed points of any (transversal) Volterra q.s.o. in the interior of odd-dimensional faces.

## 3 Homotopy of Volterra Operators

In this section we are going to define a notion of homotopy for Volterra operator. Further, we will show that two homotopic Volterra operators have 'similar' trajectories under some conditions.

**Definition 3.1.** Two Volterra operators $V_0, V_1 \in V^m_{t-1}$ are called homotopic, if there exists a family of Volterra operators $\{V_\lambda\}_{\lambda \in [0,1]} \subset V^m_{t-1}$ such that it is continuous with respect to $\lambda$ with $V_\lambda |_{\lambda=0} = V_0$, $V_\lambda |_{\lambda=1} = V_1$ and one has $|Fix(V_\lambda)| = |Fix(V_0)| = |Fix(V_1)|$ for any $\lambda \in [0,1]$. 


Remark 3.1. Note that if a family \( \{ V_\lambda \}_{\lambda \in [0,1]} \subset \mathcal{V}_t^{m-1} \) is continuous then one can see that the main subpfaffians \( gp_{i_1 i_2 \ldots i_{2k}}^{(\lambda)} \) of the corresponding skew-symmetric matrices \( A_m^{(\lambda)} \) are also continuous with respect to \( \lambda \).

One can see that the introduced homotopy defines an equivalency relation in the set \( \mathcal{V}_t^{m-1} \). Therefore, two operators \( V_0, V_1 \in \mathcal{V}_t^{m-1} \) are called equivalent and denoted by \( V_0 \sim V_1 \), if they are homotopic. Hence, one can consider a factor set \( \mathcal{V}_t^{m-1}/\sim \).

Example 3.1. Let \( m = 2 \). Then Volterra operators corresponding to the matrices

\[
A_a = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad 0 < a \leq 1
\]

are always homopotic.

Let \( m = 3 \). Then Volterra operators corresponding to the matrices

\[
A_{abc} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \quad 0 < a, b, c \leq 1
\]

are always homopotic.

Now we are interested in determining when two Volterra q.s.o. are equivalent.

Theorem 3.1. Let \( V_0, V_1 \in \mathcal{V}_t^{m-1} \) with \( V_0 \sim V_1 \) and \( A_m^{(0)}, A_m^{(1)} \) be their corresponding skew-symmetric matrices. Then all corresponding even order main subpfaffians of the matrices \( A_m^{(0)} \) and \( A_m^{(1)} \) have the same sign i.e.

\[
\text{Sign} \left( gp_{i_1 i_2 \ldots i_{2k}}^{(0)} \right) = \text{Sign} \left( gp_{i_1 i_2 \ldots i_{2k}}^{(1)} \right) \quad \forall i_1, i_2, \ldots, i_{2k} \in I.
\]

Proof. Due to \( V_0 \sim V_1 \) there exists a continuous family \( \{ V_\lambda \}_{\lambda \in [0,1]} \subset \mathcal{V}_t^{m-1} \) such that \( V_\lambda |_{\lambda = 0} = V_0 \) and \( V_\lambda |_{\lambda = 1} = V_1 \). Let us consider a skew-symmetrical matrix \( A_m^{(\lambda)} \) corresponding to \( V_\lambda \). Then \( A_m^{(\lambda)} |_{\lambda = 0} = A_m^{(0)} \) and \( A_m^{(\lambda)} |_{\lambda = 1} = A_m^{(1)} \).

Assume that the assertion of the theorem is not true, that is there are \( 2k_0 \) order main subpfaffians of the matrices \( A_m^{(0)} \) and \( A_m^{(1)} \) such that

\[
\text{Sign} \left( gp_{i_1 i_2 \ldots i_{2k_0}}^{(0)} \right) \neq \text{Sign} \left( gp_{i_1 i_2 \ldots i_{2k_0}}^{(1)} \right),
\]

which implies

\[
gp_{i_1 i_2 \ldots i_{2k_0}}^{(0)} \cdot gp_{i_1 i_2 \ldots i_{2k_0}}^{(1)} < 0.
\]

Continuity of \( gp_{i_1 i_2 \ldots i_{2k_0}}^{(\lambda)} \) with respect to \( \lambda \) (see Remark 3.1) yields the existence of \( \lambda_0 \in [0,1] \) such that \( gp_{i_1 i_2 \ldots i_{2k_0}}^{(\lambda_0)} = 0 \). But the last contradicts to \( V_{\lambda_0} \in \mathcal{V}_t^{m-1} \). \( \square \)
Let us recall some definitions relating to tournaments associated with a skew-symmetrical matrix $A_m = (a_{ki})_{k,i=1}^m$. Put

$$\text{Sign}(A_m) = (\text{Sign } a_{ki})_{k,i=1}^m.$$ 

Define a tournament $T_m$, as a complete (full) graph consisting of $m$ vertices labeled with $\{1, 2, \ldots, m\}$, corresponding to a skew-symmetrical matrix $A_m$ by the following rule: there is an arrow from $i$ to $k$ if $a_{ki} < 0$, a reverse arrow otherwise. Note that if signs of two skew-symmetric matrices are the same, then the corresponding tournaments are the same as well.

Recall that a tournament is said to be strong if it is possible to go from any vertex to any other vertex with directions taken into account. A strong component of a tournament is a maximal strong subtournament of the tournament. The tournament with the strong components of $T_m$ as vertices and with the edge directions induced from $T_m$ is called the factor tournament of the tournament $T_m$ and denoted by $\tilde{T}_m$. Transitivity of the tournament means that there is no strong subtournament consisting of three vertices of the given tournament. A tournament containing fewer than three vertices is regarded as transitive by definition.

As is known [8], the factor tournament $\tilde{T}_m$ of any tournament $T_m$ is transitive. Further, after a suitable renumbering of the vertices of $T_m$ we can assume that the subtournament $T_r$ contains the vertices of $T_m$ as its vertices, i.e., $\{1\}, \{2\}, \ldots, \{r\}$. Obviously, $r \geq m$, and $r = m$ if and only if $T_m$ is a strong tournament.

**Corollary 3.1.** If $V_0 \sim V_1$, then the corresponding tournaments $T_m^{(0)}$ and $T_m^{(1)}$ are the same.

**Proof.** Since $g_{ki} = a_{ki}$, Theorem 3.1 implies that

$$\text{Sign}(A_m^{(0)}) = \text{Sign}(A_m^{(1)}).$$

Hence, the corresponding tournaments $T_m^{(0)}$ and $T_m^{(1)}$ are the same. \qed

This corollary gives some information about the trajectory of equivalent Volterra operators. Namely, due to results of [3] and Corollary 3.1 we get the following:

**Corollary 3.2.** Let $V_0 \sim V_1$. The following assertions hold true:

(i) Assume that the tournament $T_m^{(0)}$ corresponding to $V_0$ is not strong. Then for any $x^0 \in riS^{m-1}$ one has $\omega_{V_0}(x^0) \subset \Gamma_\alpha$, $\omega_{V_1}(x^0) \subset \Gamma_\alpha$, here $\alpha = \{1, 2, \ldots, r\}$. 

(ii) Assume that $T_m^{(0)}$ is transitive, then for any $x^0 \in riS^{m-1}$ one has $\omega_{V_0}(x^0) = \omega_{V_1}(x^0) = (1, 0, \ldots, 0)$.

In Theorem 3.1 we have formulated a necessary condition of equivalence of two Volterra operators. Now in small dimensions, we are going to provide certain criterions for the equivalence.
Theorem 3.2. Let $m \leq 3$. Then $V_0 \sim V_1$ if and only if

$$\text{Sign} \left( A_m^{(0)} \right) = \text{Sign} \left( A_m^{(1)} \right).$$

Proof. Necessity immediately follows from Theorem 3.1. Therefore, we shall prove sufficiency. Let us separately consider two distinct cases with respect to $m$.

Let $m = 2$. Then

$$A_2^{(0)} = \begin{pmatrix} 0 & a_{12}^{(0)} & a_{13}^{(0)} \\ -a_{12}^{(0)} & 0 & a_{23}^{(0)} \\ -a_{13}^{(0)} & a_{23}^{(0)} & 0 \end{pmatrix} \quad \text{and} \quad A_2^{(1)} = \begin{pmatrix} 0 & a_{12}^{(1)} & a_{13}^{(1)} \\ -a_{12}^{(1)} & 0 & a_{23}^{(1)} \\ -a_{13}^{(1)} & -a_{23}^{(1)} & 0 \end{pmatrix},$$

where \(\text{Sign} a_{ij}^{(0)} = \text{Sign} a_{ij}^{(1)}\) for $i < j$.

One can check that the skew-symmetrical matrix \(A_3^{(\lambda)} = (1 - \lambda)A_3^{(0)} + \lambda A_3^{(1)}\) is transversal for any \(\lambda \in [0, 1]\). Let $V_\lambda$ be the corresponding Volterra q.s.o. (see (2.1)). Then one has $V_\lambda = (1 - \lambda)V_0 + \lambda V_1$ and \(\{V_\lambda\}_{\lambda \in [0,1]} \subset V^1_t\). According to Theorem 2.2 and Remark 2.1 the set of all fixed points of $V^{\lambda} \in V^1_t$ consists of the vertices of $S^3$. Therefore, \(|Fix(V_\lambda)| = |Fix(V_0)| = |Fix(V_1)| = 2\) for any $\lambda \in [0, 1]$, which means $V_0 \sim V_1$.

Let $m = 3$. Then

$$A_3^{(0)} = \begin{pmatrix} 0 & a_{12}^{(0)} & a_{13}^{(0)} \\ -a_{12}^{(0)} & 0 & a_{23}^{(0)} \\ -a_{13}^{(0)} & a_{23}^{(0)} & 0 \end{pmatrix} \quad \text{and} \quad A_3^{(1)} = \begin{pmatrix} 0 & a_{12}^{(1)} & a_{13}^{(1)} \\ -a_{12}^{(1)} & 0 & a_{23}^{(1)} \\ -a_{13}^{(1)} & -a_{23}^{(1)} & 0 \end{pmatrix},$$

where \(\text{Sign} a_{ij}^{(0)} = \text{Sign} a_{ij}^{(1)}\) for $i < j$.

One can check that the skew-symmetrical matrix \(A_3^{(\lambda)} = (1 - \lambda)A_3^{(0)} + \lambda A_3^{(1)}\) is transversal for any $\lambda \in [0, 1]$. Therefore, the corresponding Volterra q.s.o. $V_\lambda$ belongs to $V^2_t$ and one has $V_\lambda = (1 - \lambda)V_0 + \lambda V_1$ for any $\lambda \in [0, 1]$.

Assume that $V \in V^2_t$, and its corresponding matrix be

$$A_3 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & a_{23} & 0 \end{pmatrix}.$$

Then one can find that if

$$\text{Sign}(a_{12}) = \text{Sign}(a_{13}) = \text{Sign}(a_{23}),$$

then \(|Fix(V)| = 4\), otherwise \(|Fix(V)| = 3\).

Hence, if the condition of the theorem is satisfied, i.e.,

$$\text{Sign} \left( A_3^{(0)} \right) = \text{Sign} \left( A_3^{(1)} \right),$$

then we conclude that either \(|Fix(V_0)| = |Fix(V_1)| = 4\) or \(|Fix(V_0)| = |Fix(V_1)| = 3\).
Due to
\[
\text{Sign} \left( A_3^{(\lambda)} \right) = \text{Sign} \left( A_3^{(0)} \right) = \text{Sign} \left( A_3^{(1)} \right),
\]
one gets
\[
|\text{Fix}(V_\lambda)| = |\text{Fix}(V_0)| = |\text{Fix}(V_1)|
\]
for any \( \lambda \in [0, 1] \). These imply that \( V_0 \sim V_1 \). 

Corollary 3.3. If \( m = 2 \), then \( |V_i^2 \sim | = 2 \). If \( m = 3 \), then \( |V_i^3 \sim | = 8 \).

Theorem 3.3. Let \( m = 4 \). Then \( V_0 \sim V_1 \) if and only if
\[
\text{Sign} \left( A_4^{(0)} \right) = \text{Sign}(A_4^{(1)}), \quad \text{Sign} \left( gp_{1234}^{(0)} \right) = \text{Sign}(gp_{1234}^{(1)}),
\]
where
\[
gp_{1234}^{(i)} = a_{12}^{(i)} a_{34}^{(i)} + a_{14}^{(i)} a_{23}^{(i)} - a_{13}^{(i)} a_{24}^{(i)}, \quad i = 0, 1.
\]

Proof. As before, the necessity immediately follows from Theorem 3.1. Let us prove the sufficiency.

Let
\[
A_4^{(0)} = \begin{pmatrix}
0 & a_{12}^{(0)} & a_{13}^{(0)} & a_{14}^{(0)} \\
-a_{12}^{(0)} & 0 & a_{23}^{(0)} & a_{24}^{(0)} \\
-a_{13}^{(0)} & -a_{23}^{(0)} & 0 & a_{34}^{(0)} \\
-a_{14}^{(0)} & -a_{24}^{(0)} & -a_{34}^{(0)} & 0
\end{pmatrix}, \quad A_4^{(1)} = \begin{pmatrix}
0 & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} \\
-a_{12}^{(1)} & 0 & a_{23}^{(1)} & a_{24}^{(1)} \\
-a_{13}^{(1)} & -a_{23}^{(1)} & 0 & a_{34}^{(1)} \\
-a_{14}^{(1)} & -a_{24}^{(1)} & -a_{34}^{(1)} & 0
\end{pmatrix},
\]

where \( \text{Sign} a_{ij}^{(0)} = \text{Sign} a_{ij}^{(1)} \) for \( i < j \) and
\[
\text{Sign} \left( a_{12}^{(0)} a_{34}^{(0)} + a_{14}^{(0)} a_{23}^{(0)} - a_{13}^{(0)} a_{24}^{(0)} \right) = \text{Sign} \left( a_{12}^{(1)} a_{34}^{(1)} + a_{14}^{(1)} a_{23}^{(1)} - a_{13}^{(1)} a_{24}^{(1)} \right).
\]

Let us consider the following skew-symmetric matrix \( A_4^{(A)} \) defined by
\[
\begin{pmatrix}
0 & A_{12} & A_{13} & A_{14} \\
-A_{12} & 0 & A_{23} & A_{24} \\
-A_{13} & -A_{23} & 0 & A_{34} \\
-A_{14} & -A_{24} & -A_{34} & 0
\end{pmatrix},
\]

where
\[
A_{12} = (1 - \lambda)a_{12}^{(0)} + \lambda a_{12}^{(1)}, \quad A_{13} = (1 - \lambda)a_{13}^{(0)} + \lambda a_{13}^{(1)},
\]
\[
A_{14} = (1 - \lambda)a_{14}^{(0)} + \lambda a_{14}^{(1)}, \quad A_{23} = \frac{(1 - \lambda)a_{14}^{(0)} a_{23}^{(0)} + \lambda a_{14}^{(1)} a_{23}^{(1)}}{(1 - \lambda)a_{14}^{(0)} + \lambda a_{14}^{(1)}},
\]
\[
A_{24} = \frac{(1 - \lambda)a_{13}^{(0)} a_{24}^{(0)} + \lambda a_{13}^{(1)} a_{24}^{(1)}}{(1 - \lambda)a_{13}^{(0)} + \lambda a_{13}^{(1)}}, \quad A_{34} = \frac{(1 - \lambda)a_{12}^{(0)} a_{34}^{(0)} + \lambda a_{12}^{(1)} a_{34}^{(1)}}{(1 - \lambda)a_{12}^{(0)} + \lambda a_{12}^{(1)}}.
\]
It is then clear that

\[
\text{Sign}\left(a^{(0)}_{ij}\right) = \text{Sign}\left(a^{(1)}_{ij}\right) = \text{Sign}\left(a^{(\lambda)}_{ij}\right)
\]

for \(i < j\) and

\[
\text{Sign}\left(a^{(\lambda)}_{12} a^{(\lambda)}_{34} + a^{(\lambda)}_{14} a^{(\lambda)}_{23}\right)
\]

\[
= \text{Sign}\left((1 - \lambda)a^{(0)}_{12} a^{(0)}_{34} + \lambda a^{(1)}_{12} a^{(1)}_{34} + (1 - \lambda)a^{(0)}_{14} a^{(0)}_{23} + \lambda a^{(1)}_{14} a^{(1)}_{23}\right)
\]

\[
- (1 - \lambda)a^{(0)}_{13} a^{(0)}_{24} - \lambda a^{(1)}_{13} a^{(1)}_{24}
\]

\[
= \text{Sign}\left(a^{(0)}_{12} a^{(0)}_{34} + a^{(0)}_{14} a^{(0)}_{23} - a^{(0)}_{13} a^{(0)}_{24}\right)
\]

\[
= \text{Sign}\left(a^{(1)}_{12} a^{(1)}_{34} + a^{(1)}_{14} a^{(1)}_{23} - a^{(1)}_{13} a^{(1)}_{24}\right)
\]

for any \(\lambda \in [0, 1]\). This implies that the corresponding Volterrian operator \(V_{\lambda}\) belongs to \(V_4\) for any \(\lambda \in [0, 1]\).

Since \(m = 4\) is even, according to Remark 2.1 there is no fixed point in the interior of the simplex \(S^3\). Thanks to Theorem 3.2 one gets

\[
|\text{Fix}(V_0) \cap \partial S^3| = |\text{Fix}(V_1) \cap \partial S^3| = |\text{Fix}(V_1)|
\]

for any \(\lambda \in [0, 1]\). Therefore, \(|\text{Fix}(V_\lambda)| = |\text{Fix}(V_0)| = |\text{Fix}(V_1)|\) for any \(\lambda \in [0, 1]\).

**Corollary 3.4.** If \(m = 4\), then \(|V_4^\sim| = 112\).

The following theorem can be considered a reverse to Theorem 3.1.

**Theorem 3.4.** Let \(V_0, V_1 \in V_3^{m-1}\) and \(A^{(0)}_m, A^{(1)}_m\) be their the corresponding skew-symmetric matrices. If all corresponding even order main pfaffians of the matrices \(A^{(0)}_m\) and \(A^{(1)}_m\) have the same sign, that is

\[
\text{Sign}(\text{gp}_{i_1, \ldots, i_{2k}}) = \text{Sign}(\text{gp}_{i_1, i_2, \ldots, i_{2k}}), \quad \forall i_1, i_2, \ldots, i_{2k} \in I,
\]

then \(|\text{Fix}(V_0)| = |\text{Fix}(V_1)|\).

**Proof.** We will prove this theorem by the induction with respect to the dimension \(m\) of the simplex \(S^{m-1}\). For small dimensions our assumption is true (see Theorems 3.2 and 3.3). Let us assume that the statement of the theorem is true for dimension \(m - 1\). Now we prove it for dimension \(m\).

Since the restriction of any transversal Volterrian operator to any face \(\Gamma_\alpha\) of the simplex is also transversal Volterra operator (see Theorem 2.1), by the assumption of the induction we get that operators \(V_0\) and \(V_1\) have the same number of fixed points in \(\partial S^{m-1}\), i.e.

\[
\text{Fix}(V_0) \cap \partial S^{m-1} = \text{Fix}(V_1) \cap \partial S^{m-1}.
\]
Let us show that the operators $V_0$ and $V_1$ have the same number of fixed points in $r_S^{m-1}$.

If $m$ is even, then due to Remark 2.1 there is no fixed point of Volterra q.s.o. in the interior of the simplex. Therefore, we have to prove the theorem only when $m$ is odd. In this case, Theorem 2.2 implies that $|\text{Fix}(V) \cap r_S^{m-1}| \leq 1$ for any $V \in \mathcal{V}_1^{m-1}$. According to Theorem 2.1 $x \in \text{Fix}(V) \cap r_S^{m-1}$ if and only if $x \in \ker A_m \cap r_S^{m-1}$.

Due to oddness of $m$ the determinant of $A_m$ equals to 0, but the transversality of the operator $V_0$ yields that the minor of order $m-1$ is not zero, which means the dimension of image $\text{Im}(A_m)$ is $m-1$. Hence, the equality $\dim(\ker A_m) + \dim(\text{Im}A_m) = m$ implies that $\ker A_m$ is a one dimensional space.

Now we are going to describe $\ker A_m$. Keeping in mind that $\det A_m$ is zero, one finds

$$\sum_{k=1}^{m} a_{ki}A_{ki} = \det A_m = 0, \quad \forall k = \overline{1, m},$$

where $A_{ki}$ is an algebraic completion (i.e. algebraic minor) of entry $a_{ki}$. It is known [24] that

$$A_{ki} = (-1)^{k+i} gp_{I_k},$$

where as before $gp_{I_k}$ and $gp_{I_i}$ are pfaffians of the minors $M_{kk}$ and $M_{ii}$, respectively, $I_k = I \setminus \{k\}$ and $I_i = I \setminus \{i\}$. It then follows from (3.1) and (3.2) that

$$(-1)^k gp_{I_k} \sum_{i=1}^{m} a_{ki}(-1)^i gp_{I_i} = 0, \quad \forall k = \overline{1, m}.$$ 

Thanks to $gp_{I_k} \neq 0$, one finds

$$\sum_{i=1}^{m} a_{ki}(-1)^i gp_{I_i} = 0, \quad \forall k = \overline{1, m}.$$  

This means that for an element defined by

$$x_0 = (-gp_{I_1}, gp_{I_2}, \ldots, (-1)^i gp_{I_i}, \ldots (-1)^m gp_{I_m})$$

one has $A_m(x_0) = 0$, i.e. $x_0 \in \ker A_m$. The one-dimensionality of $A_m$ implies that $\ker A_m = \{\lambda x_0 : \lambda \in \mathbb{R}\}$. This means that there is an interior fixed point for the Volterra operator $V_0$ if and only if

$$\text{Sign}(-1)^i gp_{I_i} = \text{Sign}(-1)^2 gp_{I_2} = \cdots = \text{Sign}(-1)^i p_{I_i} = \cdots = \text{Sign}(-1)^m gp_{I_m}$$

and that fixed point is given by

$$x = \frac{1}{G_0} x_0.$$  

(3.4)
where
\[ GP = \sum_{i=1}^{m} (-1)^k g_{pi}. \]

Now if the condition of the theorem is satisfied, then from (3.4) one concludes that
\[ |\text{Fix}(V_0) \cap r_i S^{m-1}| = |\text{Fix}(V_1) \cap r_i S^{m-1}|. \]
Consequently, one gets \(|\text{Fix}(V_0)| = |\text{Fix}(V_1)|\). This completes the proof.

According to Theorem 2.1 the set of all Volterra q.s.o. geometrically forms a \((m(m-1)/2)\)-dimensional cube \(V^{m-1}_m\) in \(R^{m(m-1)/2}\). Now let us consider the following manifolds
\[ \left\{ V \in V^{m-1}: g_{pi_1i_2...i_{2k}}(V) = 0, \exists i_1, i_2, \ldots, i_{2k} \in I \right\}. \]
These manifolds divide the cube into several connected components.

From Theorems 3.1 and 3.4 one can prove the following:

**Theorem 3.5.** Two Volterra q.s.o. \(V_0\) and \(V_1\) \((V_0, V_1 \in V^{m-1}_m)\) are homotopic if and only if the operators \(V_0\) and \(V_1\) belong to only one connected component of the cube.

**Proof.** ‘If’ part of the proof immediately follows from Theorem 3.1. Therefore, let us prove ‘only if’ part.

Let us assume that \(V_0\) and \(V_1\) belong to the same connected component. Then from the definition of component one can conclude that such operators can be connected by a continuous path \(\{V_\lambda\} \in V^{m-1}_m\) located in that component. On the other hand, we see that the corresponding all main subpafffians of all operators \(V_\lambda\) have the same signs. So, thanks Theorem 3.4 one finds that \(|\text{Fix}(V_\lambda)| = |\text{Fix}(V_0)| = |\text{Fix}(V_1)|\) which implies that \(V_0 \sim V_1\).

**Remark 3.2.** Note that in small dimensions \((m \leq 4)\) the necessity condition, for homotopy of Volterra operators, is sufficient as well.

**Corollary 3.5.** If \(V_0 \sim V_1\), then for any face \(\Gamma_\alpha\) of the simplex \(S^{m-1}\) one has \(V_0 \mid_{\Gamma_\alpha} \sim V_1 \mid_{\Gamma_\alpha}\).

**Proof.** Let \(I \setminus \Gamma_\alpha = \{i_1, i_2, \ldots, i_k\}\). According to Theorem 2.1 the restriction of Vorterra q.s.o. \(V\) to \(\Gamma_\alpha\), i.e. \(V_\alpha: \Gamma_\alpha \to \Gamma_\alpha\) is also Volterra q.s.o. Therefore, the corresponding the skew-symmetrical matrix \(A_\alpha\) to \(V_\mid_{\Gamma_\alpha}\) is a matrix which can be obtained from the matrix \(A_m\) by eliminating the rows \(\{i_1, i_2, \ldots, i_m\}\) and the columns \(\{i_1, i_2, \ldots, i_m\}\). Now if \(V_0 \sim V_1\), then from Theorem 3.5 it follows that the operators \(V_0\) and \(V_1\) lie on the same connected component. From the definition of the subpafffinas one concludes that the operators \(V_0 \mid_{\Gamma_\alpha}, V_1 \mid_{\Gamma_\alpha}\) also belong to the same connected component, hence again Theorem 3.5 implies that \(V_0 \mid_{\Gamma_\alpha}\) and \(V_1 \mid_{\Gamma_\alpha}\) are homotopic.
Corollary 3.6. Let $V_0 \sim V_1$. Then for any face $\Gamma_\alpha$ of the simplex $S^{m-1}$ one has 

(i) $|\text{Fix}(V_0) \cap \Gamma_\alpha| = |\text{Fix}(V_1) \cap \Gamma_\alpha|.$
(ii) $|\text{Fix}(V_0) \cap r_i \Gamma_\alpha| = |\text{Fix}(V_1) \cap r_i \Gamma_\alpha|.$

Proof. (i). Corollary 3.5 yields that $V_0 |_{\Gamma_\alpha} \sim V_1 |_{\Gamma_\alpha}$ for any $\alpha \subset I$. Therefore, $|\text{Fix}(V_0) \cap \Gamma_\alpha| = |\text{Fix}(V_1) \cap \Gamma_\alpha|.$

(ii) Now suppose that $\alpha = \{i_1, i_2, \ldots, i_k\}$. One can see that $\partial \Gamma_\alpha = \bigcup_{n=1}^{k} \Gamma_{\alpha n},$ (3.5)

where $\alpha_i = \alpha \setminus \{i_n\}$. From (i) one finds that $|\text{Fix}(V_0) \cap \Gamma_{\alpha n}| = |\text{Fix}(V_0) \cap \Gamma_{\alpha n}|$ for any $n = 1, k$. Hence from (3.5) we get $|\text{Fix}(V_0) \cap \partial \Gamma_\alpha| = |\text{Fix}(V_0) \cap r_i \Gamma_\alpha|$, which implies $|\text{Fix}(V_0) \cap r_i \Gamma_\alpha| = |\text{Fix}(V_0) \cap \partial \Gamma_\alpha|$. 

Remark 3.3. The proved corollaries imply that equivalent Volterra operators have the same number of fixed points on every face and its interior as well. Due to these facts one can ask: are there homotopic extensions of a given Volterra operator on a face to whole simplex? Next we are going to study this question.

Let $\alpha \subset I$ and $V_0 : \Gamma_\alpha \rightarrow \Gamma_\alpha$ be a transitive Volterra q.s.o. on a face $\Gamma_\alpha$. Denote

$$F_{V_0}(\alpha) = \{V \in \mathcal{V}_m^{m-1} : V |_{\Gamma_\alpha} \sim V_0\}. $$ (3.6)

Remark 3.4. From the definition of $F_{V_0}$ it follows for any $V_1, V_2 \in F_{V_0}(\alpha)$ one has $V_1 |_{\Gamma_\alpha} \sim V_2 |_{\Gamma_\alpha}$. Since $\Gamma_\alpha$ is a $(|\alpha| - 1)$-dimensional simplex, so Corollary 3.5 implies that

Lemma 3.1. For any $\alpha, \beta \subset I$ one has

$$\beta \subset \alpha \Rightarrow F_{V_0}(\alpha) \subset F_{V_0}(\beta).$$

In particular, $\forall \alpha \subset I$ one gets

$$F_{V_0}(I) \subset F_{V_0}(\alpha).$$

Let

$$\mathcal{V}_m^{m-1}/\sim = \{\mathcal{V}_1, \ldots, \mathcal{V}_r\}. $$ (3.7)

Then from (3.6) and (3.7) one can see that for any $V_0 \in \mathcal{V}_m^{m-1}$ there exists $i \in \{1, 2, \ldots, r\}$ such that

$$F_{V_0}(I) = \mathcal{V}_i.$$ 

Therefore, we are interested in $|\alpha| \leq n - 1$. In this case, it is clear that any Volterra operator given on $\Gamma_\alpha$ can be extended to a transversal Volterra operator defined on the simplex $S^{m-1}$. Note that such an extension is not unique.

In what follows we shall assume that a Volterra operator $V_0$ is defined on the whole simplex $S^{m-1}$, i.e. $V_0 \in \mathcal{V}_m^{m-1}$. 


Theorem 3.6. Let $|\alpha| \leq n - 1$. Then there are $i, j \in \{1, 2, \ldots, r\}$, $i \neq j$ such that

$$F_{V_0} \cap V_i \neq \emptyset \quad \text{and} \quad F_{V_0} \cap V_j \neq \emptyset.$$ 

Proof. Due to $V_0 \in \mathcal{V}_i^{m-1}/\sim$ there is $i \in \{1, 2, \ldots, r\}$ such that $V_0 \in \mathcal{V}_i$. As before, by $A^0_m$ we denote the corresponding skew-symmetric matrix. From $|\alpha| \leq n - 1$ we have $I \setminus \alpha \neq \emptyset$. Let $p_0 \in I \setminus \alpha, q_0 \in I$. Then $a^0_{p_0q_0}$ is not an element of $A^0_m = A^0_m |_{\Gamma_n}$. Without loss of generality we may assume that $a^0_{p_0q_0} > 0$. Now we are going to construct a skew-symmetric matrix $A^1_m = (a^1_{ij})_{i,j=1}^m$ as follows: if $i, j \notin \{p_0, q_0\}$, then we put $a^1_{ij} = a^0_{ij}$. We choose $a^1_{p_0q_0}$ from the segment $[-1, 0)$ (Note that $a^0_{p_0q_0} \in (0, 1)$) such that all pfaffians of the matrix $A^1_m$ are not zero. The existence of such a number comes from that fact that each pfaffian is a polynomial with respect to $a^1_{p_0q_0}$ (since all the rest elements are defined), therefore, its zeros are finite, and such pfaffians are finite as well. So, all pfaffians are not zero except for finite numbers of $[-1, 0)$. According to the construction $A^1_m$ is a skew-symmetric, hence the corresponding Volterra q.s.o. $V_1$ is transversal. Moreover, $A^1_m |_{\Gamma_n} = A^0_m |_{\Gamma_n}$, i.e., $V_1 |_{\Gamma_n} \sim V_0 |_{\Gamma_n}$. But $V_1$ and $V_0$ are not homotopic, since $a^1_{p_0q_0}$ and $a^0_{p_0q_0}$ have different signs. This means that the second order pfaffians have different signs too (see Theorem 3.1). Let $\mathcal{V}_j$ be a set of Volterra operators which are equivalent to $V_1$. Then the construction shows that $i \neq j$. \qed

Corollary 3.7. Note also that if $|\alpha| \leq n - 1$, then

(i) $F_{V_0}(\alpha)$ is not a subset of any $\mathcal{V}_i$ (here $i \in \{1, 2, \ldots, r\}$);

(ii) $F_{V_0}(\alpha)$ is not a linearly connected set.

Remark 3.5. From Theorem 3.6 we conclude that any transversal Volterra operator given on a face has no unique homotopic extension.

It is clear that

$$F_{V_0}(\alpha) = \bigcup_{i=1}^{r} \left( F_{V_0}(\alpha) \cap \mathcal{V}_i \right).$$

Theorem 3.7. If there is some $i \in \{1, 2, \ldots, r\}$ such that $F_{V_0}(\alpha) \cap \mathcal{V}_i$ is not empty, then it is linearly connected.

Proof. Let us assume that $F_{V_0}(\alpha) \cap \mathcal{V}_i$ is not empty for some $i \in \{1, 2, \ldots, r\}$. Then we take two elements $V_1, V_2 \in F_{V_0}(\alpha) \cap \mathcal{V}_i$. Now we are going to show such element can be connected with a path lying in $F_{V_0}(\alpha) \cap \mathcal{V}_i$. Taking into account that $V_1, V_2 \in \mathcal{V}_i$ and Theorem 3.5 we find that there is a path $\{V_\lambda\}_{\lambda \in [1,2]} \subset \mathcal{V}_i$ connecting them. For any $\lambda \in [1, 2]$ one has $V_\lambda \sim V_1$, hence $V_\lambda |_{\Gamma_n} \sim V_1 |_{\Gamma_n}$. From $V_1 |_{\Gamma_n} \sim V_0 |_{\Gamma_n}$ we obtain $V_\lambda |_{\Gamma_n} \sim V_0 |_{\Gamma_n}$, this means that $\{V_\lambda\}_{\lambda \in [1,2]} \subset F_{V_0}(\alpha)$. Therefore, $F_{V_0}(\alpha) \cap \mathcal{V}_i$ is linearly connected. \qed
Corollary 3.8. For any $\alpha \subset I$ one has

$$F_{V_0}(\alpha)/\sim = \left\{ F_{V_0}(\alpha) \cap V_i \right\}_{i=1}^r,$$

where for some $i$ the set $F_{V_0}(\alpha) \cap V_i$ can be empty. Therefore,

$$|F_{V_0}(\alpha)/\sim| \leq r.$$

In particular, the equality occurs when $|\alpha| = 1$, i.e.,

$$F_{V_0}(\alpha)/\sim = \{ V_i \}_{i=1}^r = V_{i=1}^{m-1}/\sim.$$

From Theorem 3.6 we conclude that if two Volterra operators are homotopic on a face, then they need not be homotopic on the simplex $S^{m-1}$. But there arises the following problem:

Problem 3.1. How many faces, on which two Volterra operators are homotopic, need to be homotopic of such operators on the simplex $S^{m-1}$?

Now we are going to show that the formulated problem has negative solution when $m$ is even.

Example 3.2. Consider a case when $m = 4$, then according to Theorem 3.3 we know that two Volterra q.s.o. are homotopic iff the signature of corresponding matrices are the same, and moreover, Pfaffians of their determinants have the same sign. Let us consider two transversal Volterra q.s.o. corresponding to the following matrices

$$A_4^{(1)} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}, \quad A_4^{(2)} = \begin{pmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ -1 & 0 & \frac{1}{2} & 1 \\ -1 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -1 & -\frac{1}{2} & 0 \end{pmatrix}. $$

Then one can check that the operators $V_1$ and $V_2$ are homotopic on any proper face of the simplex $S^3$(see Theorem 3.2). Since the pfaffians corresponding to determinants of the matrices $A_4^{(1)}$ and $A_4^{(2)}$ are 1 and $-1/2$, respectively, from Theorem 3.3 we conclude that they are not homotopic on the whole simplex $S^3$.

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References


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