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Certain q-Integrals Involving the Generalized Hypergeometric and Basic Hypergeometric Functions

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Abstract: The aim of this paper is to establish certain new q-integrals involving the generalized Heines hypergeometric function, generalized q-confluent hypergeometric function. Our main findings are capable of yielding a large number of new, interesting, and useful q-integrals, expansion formulas involving the hypergeometric and basic hypergeometric functions.

Keywords: Finite q-integrals, Generalized q-confluent hypergeometric function, Generalized Heine's hypergeometric function

1 Introduction

The q-calculus is the extension of the ordinary calculus. The subject deals with the investigations of q-integrals and q- derivatives of arbitrary order and has gained importance due to its various applications in the areas like ordinary calculus, solution of the q-differential and q-integral equations, q-transform analysis [3,6,8,9]. Motivated by these avenues of applications, a number of workers have made use of these operators to evaluate q-calculus, basic analogue of H-function, basic analogue of I-function, general class of q-polynomials etc.
Some fundamental properties and characteristics of the generalized Beta type function are as

$$B_p^{\alpha,\beta}(x,y)=\int_0^1 t^{x-1}(1-t)_1^{y-1}F_1(\alpha;\beta;\frac{-p}{t(1-t)})dt, \tag{1}$$

For $R(p)>0, minR(x),R(y),R(\alpha),R(\beta)>0$, and $B_0^{(\alpha,\beta)}(x,y)=B(x,y)$
were introduced and studied in [1], where

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

For $\Re(x)>0$ and $\Re(y)>0$ is the well-known Euler Beta function.
See also [2]and [1].
The q-gamma function was first introduced by Thomae and later by Jackson.
The q-analogue of gamma function which is defined by F.H. Jackson [7] is given by

$$\Gamma_q(x)=\frac{(q;q)_\infty(1-q)^{1-x}}{(q^x;q)_\infty}, 0<q<1$$

And the q-analogue of Beta function is defined as:

$$B_q(x,y)=\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)},$$

The q-analogue of the generalized Beta and Gamma function is defined as follows

$$B_p^{\alpha,\beta}(x,y;q)=\int_0^1 t^{x-1}(1-qt)_1^{y-1}\varphi_1(\alpha;\beta;\frac{-p}{t(1-qt)};q)d_qt, \tag{2}$$

$\Re(p)\geq 0$, $minR(x),R(y),R(\alpha),R(\beta)>0$, and $|q|<1$.

$$B_0^{\alpha,\beta}(x,y;q)=B(x,y;q)=\int_0^1 t^{x-1}(1-qt)^{y-1}d_qt,$$

We also know that

$$B_q(\lambda+n,\delta)=\frac{(\lambda,q)_n}{\lambda+\delta,q_n}B_q(\lambda,\delta), \tag{3}$$

$$\Gamma_p^{\alpha,\beta}(x;q)=\int_0^\infty t_1^{x-1}\varphi_1(\alpha;\beta;-t-\frac{p}{t};q)d_qt,$$

If $p=0, \alpha=\beta=1$, then

$$\Gamma_q(x;q)=\int_0^\infty t_1^{x-1}\varphi_1(\alpha;\beta;-t;q)d_qt,\ \Gamma(x;q)=\int_0^\infty t^{x-1}E_q^{-qt}d_qt$$

Heine's hypergeometric function is given by

$${}_2\varphi_1(a,b;c;z;q)=\sum_{n=0}^\infty\frac{(a;q)_n(b;q)_nz^n}{(c;q)_n(q;q)_n}$$

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Along with the generalized q-Beta function (2), the useful generalized Heine's hypergeometric functions

$$\varphi_p^{(\alpha,\beta)}(a,b;c;z;q) = \sum_{n=0}^{\infty} \frac{(a;q)_n B_p^{(\alpha,\beta)}(b+n,c-b;q)z^n}{B(b,c-b;q)(q;q)_n} \quad (4)$$

And the generalized q-confluent hypergeometric function

$${}_1\varphi_{1,p}^{(\alpha,\beta)}(b;c;z;q) = \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(b+n,c-b;q)z^n}{B(b,c-b;q)(q;q)_n}$$

for $|z| < 1, min\Re(\alpha), \Re(\beta) > 0, \Re(c) > 0, \Re(b) > 0$,and $\Re(p) \geq 0$, were also introduced and studied. See [2] and [1].

When $p = 0$, the functions $\varphi_p^{(\alpha,\beta)}(a,b;c;z;q)$ and ${}_1\varphi_{1,p}^{(\alpha,\beta)}(b;c;z;q)$ would reduce immediately to the extensively investigated Heine's hypergeometric functions ${}_2\varphi_1(.)$ and ${}_1\varphi_1(.)$ [4]. The functions ${}_2\varphi_1(.)$ and ${}_1\varphi_1(.)$ are the special cases of the well-known generalized basic hypergeometric series.

$${}_r\varphi_s\left[\begin{matrix}\alpha_1,\alpha_2,...,\alpha_r;z;q\\ \beta_1,\beta_2,...,\beta_s\end{matrix}\right] = {}_r\varphi_s(\alpha_1,\alpha_2,...,\alpha_r;\beta_1,\beta_2,...,\beta_s;z;q)$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1;q)_n(\alpha_2;q)_n...(\alpha_r;q)_n z^n}{(\beta_1)_n(\beta_2)_n...(\beta_s)_n(q;q)_n}$$

First we summarize some definitions and formulas; we need from the q-theory. For details the reader is referred to [4]

We always take $0 < |q| < 1$ in the sequel.

Where, $(a;q)_n$ symbol is defined for $a \in C$ by

$$(a;q)_n = \{ \begin{matrix} 1, n=0 \\ (1-a)(1-aq)...(1-aq^{n-1}), n=1,2,3,... \end{matrix}$$

is the q-shifted factorial.

Consider the following q-analogue of Hadamard's function

$${}_t\varphi_{s+r}^{(\alpha,\beta,\gamma,\rho)}[\begin{matrix}x_1,x_2,...,x_t;z;q\\ y_1,y_2,...,y_{s+r}\end{matrix}] = {}_1\varphi_r[\begin{matrix}1;z;q\\ y_1,y_2,...,y_r\end{matrix}]{}_t\varphi_s^{(\alpha,\beta,\gamma,\rho)}[\begin{matrix}x_1,x_2,...,x_t;z;q\\ y_{1+r},y_{2+r},...,y_{s+r}\end{matrix}] \quad (5)$$

2 Main Results

Theorem 1.

If $minR(\alpha), R(\beta) > 0, R(c) > R(b) > 0, R(p) > 0$, and $z, \lambda, \delta \in C$ such that $|z| < 1$, and $|q| < 1$, then

$$\int_0^{\infty} u^{\lambda-1}(1-qu)^{\delta-1}\varphi_p^{(\alpha,\beta)}(a,b;c;zu;q)d_q u \quad (6)$$

$$= B_q(\lambda,\delta){}_1\varphi_{p,1}^{(\alpha,\beta)}(a,b;\lambda;c,\lambda+\delta;z;q)$$

Proof. Using Eq.(4), we have

$$\int_0^{\infty} u^{\lambda-1}(1-qu)^{\delta-1}\varphi_{1,p}^{(\alpha,\beta)}(a,b;c;zu;q)d_q u$$

$$= \sum_{n=0}^{\infty} \frac{(a;q)_n B_p^{(\alpha,\beta)}(b+n,c-b;q)z^n}{B(b,c-b;q)(q;q)_n} \int_0^1 u^{\lambda+n-1}(1-qu)^{\delta-1}d_q u$$

$$= \sum_{n=0}^{\infty} \frac{(a;q)_n B_p^{(\alpha,\beta)}(b+n,c-b;q)z^n}{B(b,c-b;q)(q;q)_n} B_q(\lambda+n,\delta)$$

By applied Eq. (3), we get

$$= B_q(\lambda,\delta)\sum_{n=0}^{\infty} \frac{(a;q)_n B_p^{(\alpha,\beta)}(b+n,c-b;q)z^n(\lambda;q)_n}{B(b,c-b;q)(q;q)_n(\lambda+n;q)_n} \quad (7)$$

Interpreting the right-hand side of (7), in the view of the definition (4) and the concept of the Hadamard given by (5), we arrive at the required result

$$\int_0^{\infty} u^{\lambda-1}(1-qu)^{\delta-1}\varphi_p^{(\alpha,\beta)}(a,b;c;zu;q)d_q u$$

$$= B_q(\lambda,\delta){}_1\varphi_{p,1}^{(\alpha,\beta)}(a,b;\lambda;c,\lambda+\delta;z;q)$$

Theorem 2.

If $minR(\alpha), R(\beta) > 0, R(c) > R(b) > 0, R(p) > 0$, and $z, \lambda, \delta \in C$ such that $|z| < 1$, and $|q| < 1$, then

$$\int_0^{\infty} u^{\lambda-1}(1-qu)_1^{\delta-1}\varphi_1^{(\alpha,\beta;p)}(b;c;zu;q)d_q u \quad (8)$$

$$= B_q(\lambda,\delta){}_2\varphi_2^{(\alpha,\beta;p)}(b;\lambda;c,\lambda+\delta;z;q)$$

Proof. Using Eq.(4), we have

$$\int_0^{\infty} u^{\lambda-1}(1-qu)_1^{\delta-1}\varphi_1^{(\alpha,\beta;p)}(b;c;zu;q)d_q u$$

$$= \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(b+n,c-b;q)z^n}{B(b,c-b;q)(q;q)_n} \int_0^1 u^{\lambda+n-1}(1-qu)^{\delta-1}d_q u$$

$$= \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(b+n,c-b;q)z^n}{B(b,c-b;q)(q;q)_n} B_q(\lambda+n,\delta)$$

By applied Eq. (3), we get

$$= B_q(\lambda,\delta)\sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(b+n,c-b;q)z^n(\lambda;q)_n}{B(b,c-b;q)(q;q)_n(\lambda+n;q)_n} \quad (9)$$

Interpreting the right-hand side of (9) in the view of the definition (4) and the concept of the Hadamard given by (5), we arrive at the required result

$$\int_0^{\infty} u^{\lambda-1}(1-qu)_1^{\delta-1}\varphi_1^{(\alpha,\beta;p)}(b;c;zu;q)d_q u \quad (10)$$

$$= B_q(\lambda,\delta){}_2\varphi_2^{(\alpha,\beta;p)}(b;\lambda;c,\lambda+\delta;z;q)$$

Equation (10) is also known as q-fractional derivative formula.

Theorem 3.

If $minR(\alpha), R(\beta) > 0, R(c) > R(b) > 0, R(p) > 0$, and $z, \lambda, \delta \in C$ such that $|z| < 1$, and $|q| < 1$, then

$$\int_{qt}^{x} (x-s)^{\delta-1}(s-qt)^{\lambda-1}\varphi^{(\alpha,\beta;p)}(a,b;c;z(s-qt);q)d_q s$$

$$= B_q(\lambda,\delta)(x-t)_1^{\delta+\lambda-1}\varphi_1^{(\alpha,\beta;p)}(a,b,\lambda;c,\lambda+\delta;z(x-qt);q)$$

Proof. Change the variables to $u = \frac{(s-qt)}{(x-qt)}$, then the left-hand side of (11) becomes

$$\int_0^1 (x-t)^{\delta+\lambda-1}(1-qu)^{\delta-1}u^{\lambda-1}\varphi^{(\alpha,\beta;p)}(a,b;c;z[u(x-qt)];q)d_q u$$

$$= (x-t)^{\delta+\lambda-1}\sum_{n=0}^{\infty} \frac{(a;q)_n B_p^{(\alpha,\beta)}(b+n,c-b;q)[z(x-qt)]^n}{B(b,c-b;q)(q;q)_n} \int_0^1 u^{\lambda+n-1}(1-qu)^{\delta-1}d_q u$$

$$= (x-t)^{\delta+\lambda-1}\sum_{n=0}^{\infty} \frac{(a;q)_n B_p^{(\alpha,\beta)}(b+n,c-b;q)[z(x-qt)]^n}{B(b,c-b;q)(q;q)_n} B_q(\lambda+n,\delta)$$

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$$= B_q(\lambda,\delta)(x-t)^{\delta+\lambda-1}\sum_{n=0}^{\infty}\frac{(a;q)_n B_p^{(\alpha,\beta)}(b+n,c-b;q)[z(x-qt)]^n}{B(b,c-b;q)(q;q)_n}\frac{(\lambda;q)_n}{(\lambda+\delta;q)_n} \quad (11)$$

Interpreting the right-hand side of (12), in the view of the definition (4) and the concept of the Hadamard given by (5), we arrive at the following result as

$$\int_{qt}^{x}(x-s)^{\delta-1}(s-qt)^{\lambda-1}\varphi^{(\alpha,\beta;p)}(a,b;c;z(s-qt);q)d_qs$$
$$= B_q(\lambda,\delta)(x-t)_1^{\delta+\lambda-1}\varphi_1^{(\alpha,\beta;p)}(a,b,\lambda;c,\lambda+\delta;z(x-qt);q)$$

Theorem 4.
If $minR(\alpha),R(\beta)>0,R(c)>R(b)>0,R(p)>0$, and $z,\lambda,\delta\in C$ such that $|z|<1$, and $|q|<1$, then

$$\int_{qt}^{x}(x-s)^{\delta-1}(s-qt)_1^{\lambda-1}\varphi_1^{(\alpha,\beta;p)}(b;c;z(s-qt);q)d_qs$$
$$= B_q(\lambda,\delta)(x-t)_2^{\delta+\lambda-1}\varphi_2^{(\alpha,\beta;p)}(b,\lambda;c,\lambda+\delta;z(x-qt);q)$$

Proof. The proof follows the same lines as that employed in the proof of theorem 3.

Theorem 5.
If $z,\nu,\mu\in C, minR(\alpha),R(\beta)>0,R(c)>R(b)>0,R(p)>0,R(\nu)>0$ and $R(\mu)>0$, also $|z|<1$, and $|q|<1$, then

$$\int_{0}^{x}(t)^{\nu-1}(x-qt)^{\mu-1}\varphi_p^{(\alpha,\beta)}(a,b;c;z(x-qt);q)d_qt$$
$$= x^{\nu+\mu-1}B_q(\nu,\mu)_1\varphi_1^{(\alpha,\beta;p)}(a,b,\nu;\nu+\mu,c;zx;q) \quad (12)$$

Proof. Change the variables $t/x=u$, then

$$\int_{0}^{x}(t)^{\nu-1}(x-qt)^{\mu-1}\varphi_p^{(\alpha,\beta)}(a,b;c;z(x-qt);q)d_qt$$
$$= \int_{0}^{1}(ux)^{\nu-1}(x-qux)^{\mu-1}\varphi_p^{(\alpha,\beta)}(a,b;c;zx(1-qu);q)xd_qu$$
$$= x^{\nu+\mu-1}\int_{0}^{1}(u)^{\nu-1}(1-qu)^{\mu-1}\varphi_p^{(\alpha,\beta)}(a,b;c;zx(1-qu);q)d_qu$$

By using Eq. (4) to eq. (15) on RHS, we have

$$= x^{\nu+\mu-1}\sum_{n=0}^{\infty}\frac{(a;q)_n B_p^{(\alpha,\beta)}(b+n,c-b;q)(zx)^n}{B(b,c-b;q)(q;q)_n}\int_0^1(u)^{\nu+n-1}(1-qu)^{\mu-1}d_qu$$
$$= x^{\nu+\mu-1}\sum_{n=0}^{\infty}\frac{(a;q)_n B_p^{(\alpha,\beta)}(b+n,c-b;q)(zx)^n}{B(b,c-b;q)(q;q)_n}B_q(\nu+n,\mu)$$
$$= x^{\nu+\mu-1}\sum_{n=0}^{\infty}\frac{(a;q)_n B_p^{(\alpha,\beta)}(b+n,c-b;q)(zx)^n}{B(b,c-b;q)(q;q)_n}\frac{(\nu;q)_n}{(\nu+\mu)_n}B_q(\nu,\mu)$$
$$= x^{\nu+\mu-1}B_q(\nu,\mu)\sum_{n=0}^{\infty}\frac{(a;q)_n B_p^{(\alpha,\beta)}(b+n,c-b;q)(zx)^n}{B(b,c-b;q)(q;q)_n}\frac{(\nu;q)_n}{(\nu+\mu)_n}$$
$$= x^{\nu+\mu-1}B_q(\nu,\mu)\varphi_p^{(\alpha,\beta)}(a,b,c;zx;q)\times_1\varphi_1(\nu,\nu+\mu;zx;q) \quad (13)$$

Interpreting the right-hand side of (16) in the view of the definition (4) and the concept of the Hadamard given by (5), we arrive at the desired result (14).

$$\int_{0}^{x}(t)^{\nu-1}(x-qt)^{\mu-1}\varphi_p^{(\alpha,\beta)}(a,b;c;z(x-qt);q)d_qt$$
$$= x^{\nu+\mu-1}B_q(\nu,\mu)_1\varphi_1^{(\alpha,\beta;p)}(a,b,\nu;\nu+\mu,c;zx;q)$$

Theorem 6.
If $z,\nu,\mu\in C, minR(\alpha),R(\beta)>0,R(c)>R(b)>0,R(p)>0,R(\nu)>0$ and $R(\mu)>0$, also $|z|<1$, and $|q|<1$, then

$$\int_{0}^{x}(t)^{\nu-1}(x-qt)_1^{\mu-1}\varphi_1^{(\alpha,\beta;p)}(b;c;z(x-qt);q)d_qt$$
$$= x^{\nu+\mu-1}B_q(\nu,\mu)_2\varphi_2^{(\alpha,\beta;p)}(b,\nu;\nu+\mu,c;zx;q) \quad (14)$$

Proof: The proof follows the same lines as that employed in the proof of Theorem 5.

3 Special Cases

If we put $q=1$ then from above theorems we arrive at the results established by Praveen Agarwal et. al. [5].
For $\alpha=\beta$, then we obtain the following results from theorems 1 to 6 respectively.

Corollary.
If $\Re(c)>\Re(b)>0,\Re(p)>0,z,\lambda,\delta\in C$,and $|z|<1,|q|<1$, then

$$(i)\quad \int_0^1 u^{\lambda-1}(1-qu)^{\delta-1}\varphi_p(a,b;c;zu;q)d_qu$$
$$= B_q(\lambda,s)_1\varphi_{p,1}(a,b;\lambda;c,\lambda+\delta;z;q)$$

$$(ii)\quad \int_0^1 u^{\lambda-1}(1-qu)^{\delta-1}\varphi_p(b;c;zu;q)d_qu$$
$$= B_q(\lambda,s)_1\varphi_{p,1}(b;\lambda;c,\lambda+\delta;z;q)$$

$$(iii)\quad \int_{qt}^x(x-s)^{\delta-1}(s-qt)^{\lambda-1}\varphi_p(a,b;c;z(s-qt);q)d_qs$$
$$= B_q(\lambda,s)(x-t)_1^{\delta+\lambda-1}\varphi_{p,1}(a,b;\lambda;c,\lambda+\delta;z(x-qt);q)$$

$$(iv)\quad \int_{qt}^x(x-s)^{\delta-1}(s-qt)_1^{\lambda-1}\varphi_{p,1}(b;c;z(s-qt);q)d_qs$$
$$= B_q(\lambda,s)(x-t)_1^{\delta+\lambda-1}\varphi_{p,1}(b;\lambda;c,\lambda+\delta;z(x-qt);q).$$

If $z,\nu,\mu\in C,R(c)>R(b)>0,R(p)>0,R(\nu)>0$, and $R(\mu)>0$, also $|q|<1$, then

$$(v)\quad \int_0^x t^{\nu-1}(x-qt)^{\mu-1}\varphi_p(a,b;c;z(x-qt);q)d_qt$$
$$= x^{\nu+\mu-1}B_q(\nu,\mu)_1\varphi_{p,1}(a,b;\nu;\nu+\mu,c;zx;q)$$

$$(vi)\quad \int_0^x t^{\nu-1}(x-qt)^{\mu-1}\varphi_p(b;c;z(x-qt);q)d_qt$$
$$= x^{\nu+\mu-1}B_q(\nu,\mu)_1\varphi_{p,1}(b;\nu;\nu+\mu,c;zx;q)$$

4 Conclusion

The results proved in this paper give some contributions to the theory of the basic hypergeometric functions and are believed to be a new to the theory of q- calculus and are likely to find certain applications to the solution of the q-integral equations involving various basic hypergeometric functions.

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