Journal of Statistics Applications & Probability Letters
*An International Journal*

http://dx.doi.org/10.18576/jsapl/090103

# Improved Exponential Ratio Estimators in Adaptive Cluster Sampling

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Received: 15 Mar. 2021, Revised: 12 Sep. 2021, Accepted: 4 Nov. 2021
Published online: 1 Jan. 2022

**Abstract:** In this manuscript, we have proposed two improved exponential ratio estimators using auxiliary information for estimating the unknown population mean in Adaptive Cluster Sampling (ACS) and have found the optimum value of a (the base of the exponent) using a numerical study. The Mean Squared Error for the two estimators has been derived up to first order of approximations and studied along with the nature of their percentage relative efficiencies for each and every positive real value of a except for 1.

**Keywords:** Adaptive cluster sampling, Improved exponential ratio estimator, Auxiliary variable, Mean squared error, Percentage relative efficiency

## 1 Introduction

Adaptive cluster sampling (ACS), first proposed by Thompson [1] in 1990 is a type of sampling scheme which is used for sampling rare and hidden clustered population. In ACS, two things have to be defined before conducting the survey: the neighbourhood of a unit (or observation) and the condition of interest. Both of these definitions depends on the researcher but usually the condition of interest C is $y_i > 0$ where $y_i$ represents the $i^{th}$ unit or the $i^{th}$ observation on the variable of interest Y and the neighbourhood which is generally considered in ACS, is the 4-unit first order neighbourhood. In ACS an initial sample of size n is drawn using any probability sampling method (in this article the initial sample of size n is drawn using simple random sampling without replacement (SRSWOR)) from a population of size N. After the initial sample has been drawn, we draw the adaptive sample based on what has been drawn in the initial sample. For all the observations or units in the initial sample which satisfies the condition of interest C, we draw their 4 unit first order neighbourhood which includes that $i^{th}$ unit and its 4 adjacent units in the East, West, North and South directions. If any unit selected in the adaptive sample satisfies the condition of interest $y_i > 0$ then its 4-unit first order neighbourhood is selected as well. This process of drawing the adaptive samples is repeated till there is no unit left satisfying the conditions of interest $y_i > 0$.
There will be some observations in the neighbourhood satisfying the condition of interest C and there will be some observations in the neighbourhood, not satisfying the condition C. Observations satisfying the condition C are called network and those observations which do not satisfy the condition C are called edge units. The edge units are considered as networks of size 1. Selection of any observation or unit from a network leads to the selection on the entire network. These networks and edge units together form a cluster.
The clusters are not necessarily disjoint due to overlapping edge units but the networks are disjoint and thus the entire population can be partitioned into exhaustive sets of networks. So the final sample consists of the initial sample and the adaptive samples.
An auxiliary variable is a variable about which we have full information and is highly correlated (positively or negatively) with the variable of interest and it is well known that the variance of the estimator of population's parameter(s) of interest can be significantly reduced once such auxiliary variable is used in the study. The use of auxiliary variable was first suggested by Cochran (1940) [2] when he proposed the ratio estimator under SRSWOR. Since then, ratio estimators have been widely used and researched upon.

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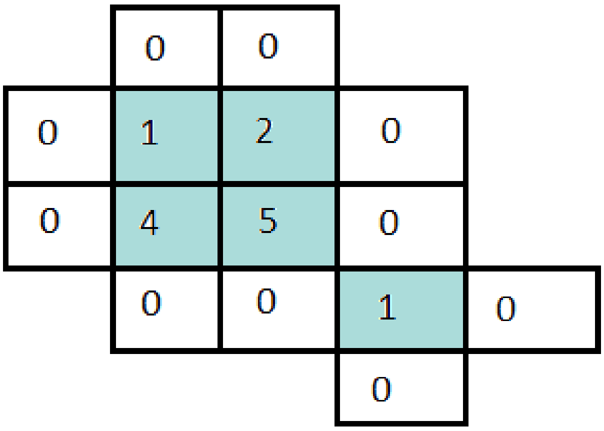


**Fig. 1:** This is an example of a hypothetical cluster. The condition of interest is $y_i > 0$. The units having y-values 1, 2, 4, 5 and 1 form a network of size 5. The edge units are the units with y values 0, adjacent to the y values 1, 2, 4, 5 and 1. Together they form a cluster.

Exponential ratio estimators have been used and studied a lot by researchers but generally with the base of the exponent as 2.718 approximately, which is, just a particular case of the exponential function, obviously if we vary this base of the exponent, we might find a better estimator to estimate the unknown parameter(s) of interest. This manuscript is driven by this idea.
In SRS (Simple random sampling) for estimating the unknown population mean using auxiliary information and known population parameters, generalized exponential ratio estimator have been studied by Singh et al. (2019) [3]. In ACS no such study has been done to find the optimum a for the exponential ratio estimator. Hanif (2016)[4] using known population median proposed some exponential ratio estimator with the base of the exponent as 2.718 (approximately) only, however, in this manuscript we propose, two generalised exponential ratio estimators for estimating the unknown population mean using the known population coefficient of variation of the auxiliary variable and the study variable and the correlation between the auxiliary and study variable of the transformed population. The expressions of the Mean squared errors (MSE) have been derived up to the first order of approximation for both the proposed estimators and the nature of their MSE and percentage relative efficiencies (PRE) have been studied for each and every value of $a > 0$ (except $a = 1$) to find the optimum a. The optimum a is that value of a which minimises the MSE. For any population the optimum value of a is $a \approx \exp(\frac{2\rho_{wxwy}C_{wy}}{C_{wx}})$ which we have confirmed using a numerical study in this manuscript. For the population studied in this manuscript, we calculated the optimum a using the formula presented above and then obtained the empirical value of a which gave the minimum MSE based on four different sample sizes. The value of empirical optimum a from the numerical illustration is very close to the value of optimum a obtained using the presented formula.

## 2 Estimators in Simple Random Sampling

Let Y denote the variable of our interest and X denote the auxiliary variable. The population size is N and $(x_i, y_i)$ where $i = 1, 2..., n$ be the bivariate observations on Y and X based on a sample of size n obtained using SRSWOR. Ratio estimators are used when the correlation between the auxiliary variable X and the study variable Y is positive and high (0.5 to 1) and the regression line between y and x passes through the origin.

Chochran (1940)[2] proposed the estimator $\bar{y}_R$ for estimating the unknown population mean $\bar{Y}$ as:

$$\bar{y}_R = \bar{y}\frac{\bar{X}}{\bar{x}} \tag{1}$$

where $\bar{y} = \sum_{i=1}^{n} y_i$, $\bar{x} = \sum_{i=1}^{n} x_i$ and $\bar{X}$ is the population mean of auxiliary variable. Mean Squared Error of ratio estimator up to the first order of approximation is

$$MSE(\bar{y}_R) = (\frac{1}{n} - \frac{1}{N})\bar{Y}^2[Cy^2 + Cx^2 - 2\rho_{XY}CyCx] \tag{2}$$

where $C_y = \frac{S_y}{\bar{Y}}$, $C_x = \frac{S_x}{\bar{X}}$, $S_y^2 = \frac{\sum_{i=1}^{N}(Y_i-\bar{Y})^2}{N-1}$

$S_x^2 = \frac{\sum_{i=1}^{N}(X_i-\bar{X})^2}{N-1}$ , $\rho_{XY} = \frac{S_{xy}}{S_x S_y}$ is the correlation between X and Y and

$S_{xy} = \frac{\sum_{i=1}^{N}(X_i-\bar{X})(Y_i-\bar{Y})}{N-1}$

The estimator proposed by Yadav and Kadilar (2013)[5] is

$$t_1 = k\bar{y}\exp(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}) \tag{3}$$

where k is a constant.
The MSE of the estimator $t_1$ up to the first order of approximation is given by:

$$MSE(t_1) = \bar{Y}^2(1 + k^2A - 2kB) \tag{4}$$

where $A = 1 + f(C_Y^2 + C_X^2 - 2\rho_{XY}C_YC_X)$

$B = 1 - f(\frac{\rho_{XY}C_XC_Y}{2} - \frac{3C_X^2}{8})$,

$k = \frac{B}{A}, f = \frac{1}{n} - \frac{1}{N}$.

Generalized exponential ratio estimator $t_2$ was proposed by Singh et al (2019)[3] as:

$$t_2 = \bar{y}a^{\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}} \tag{5}$$

The MSE of the estimator $t_2$ up to the first order of approximation is given by:

$$MSE(t_2) = \bar{Y}^2 f(C_Y^2 + \frac{1}{4}C_X^2(loga)^2 - \rho_{XY}C_XC_X loga) \tag{6}$$

where optimum $a = \exp(\frac{2\rho_{XY}C_Y}{C_X})$

## 3 Estimators in Adaptive Cluster Sampling

Let the population size be N and the sample size of initial sample selected using SRSWOR be n. Dryver and Chao (2007)[7] have stated that when we consider average of netwroks then ACS can be regarded as Simple Random Sampling. An unbiased estimator for population mean under ACS was proposed by Thompson (1990)[1].
This estimator was a modification of the Hansen-Hurvitz estimator(1943)[6]. Thompson's estimator is given by:

$$t_3 = (\frac{1}{n})\sum_{i=1}^{n} wy_i \tag{7}$$

where $w_{yi}$ denote the network means of a network containing the $i^{th}$ unit, so

$$w_{yi} = \sum_{j\varepsilon\Psi_i} (y_j)$$

where $\psi_i$ is the network containing unit i and $m_i$ be the number of units in the network $\psi_i$ The variance of Thompson (1990) estimator is given by:

$$V(t_3) = (\frac{1}{n} - \frac{1}{N})\bar{Y}^2 C_{wy}^2 = f\bar{Y}^2 C_{wy}^2 \tag{8}$$

where $f = \frac{1}{n} - \frac{1}{N}$, $C_{wy}^2 = \frac{S_{wy}^2}{\bar{Y}^2}$ and $S_{wy}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(w_{yi} - \bar{Y})^2$.

The usual regression estimator under ACS is:

$$\bar{y}_{Reg} = \bar{w_y} + \frac{S_{wxwy}}{S_{wx}^2}(\bar{X} - \bar{w_x}) \tag{9}$$

The MSE of the estimator $\bar{y}_{Reg}$ up to the first order of approximation is given by:

$$MSE(y\bar{_{Reg}}) = \frac{1-f}{n} S_{wy}^2 (1 - \rho_{wxwy}^2) \tag{10}$$

where $\rho_{wxwy}^2 = (\frac{S_{wxwy}}{S_{wx}S_{wy}})^2$, $S_{wx}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(w_{xi} - \bar{X})^2$

$S_{wy}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(w_{yi} - \bar{Y})^2$, $S_{wxwy} = \frac{1}{N-1}\sum_{i=1}^{n}(w_{x_i} - \bar{X})(w_{y_i} - \bar{Y})$

A modified ratio estimator was proposed by Dryver and Chao (2007)[7] as

$$t_4 = \frac{\sum_{i=1}^{n} w_{y_i}}{\sum_{i=1}^{n} w_{x_i}} \bar{X} \tag{11}$$

.
where $w_{yi}$ and $w_{xi}$ denote the network means of a network containing the $i^{th}$ unit, so $w_{yi} = \sum_{j\varepsilon\Psi_i}(y_j)$ and $w_{si} = \sum_{j\varepsilon\Psi_i}(x_j)$, where $\psi_i$ is the network containing unit i amd $m_i$ be the number of units in the network $\psi_i$.
The MSE of Dryver and Chao (2007)[7] estimator $t_4$ up to first order of approximation is

$$MSE(t_4) = (\frac{1}{n} - \frac{1}{N})\bar{w_y^2}[C_{wy}^2 + C_{wx}^2 - 2\rho_{wxwy}C_{wx}C_{wy}] \tag{12}$$

where $C_{wx}^2 = \frac{S_{wx}^2}{\bar{X}^2}$, $S_{wx}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(w_{xi} - \bar{X})^2$, $\rho_{wxwy} = \frac{S_{wxwy}}{S_{wx}S_{wy}}$,

$S_{wxwy} = \frac{1}{N-1}\sum_{i=1}^{n}(w_{x_i} - \bar{X})(w_{y_i} - \bar{Y})$

## 4 Proposed Estimators

Motivated by Yadav and Kadilar (2013)[5] and Singh et. al (2019)[3], we propose two generalized exponential ratio estimators $t_5$ and $t_6$ as:

$$t_5 = k_1 \bar{w}_y a_5^{\frac{\bar{X}-\bar{w}_x}{\bar{X}+\bar{w}_x}} \tag{13}$$

where $k_1$ is any constant and a is a positive real number.

$$t_6 = [k_2 \bar{w}_y + (1-k_2)(\bar{X}-\bar{w}_x)] a_6^{\frac{\bar{X}-\bar{w}_x}{\bar{X}+\bar{w}_x}} \tag{14}$$

The expression of MSE for the estimator $t_5$ is obtained as follows:

Using $w\bar{_{wy}} = \frac{\bar{w_y}}{\bar{Y}} - 1$, $w\bar{_{wx}} = \frac{\bar{w_x}}{\bar{X}} - 1$, $E(e\bar{_{wx}}^2) = fC_{wx}^2$, $E(e\bar{_{wy}}^2) = fC_{wy}^2$,

$E(e\bar{_{wx}} e\bar{_{wy}}) = f\rho_{wxwy} C_{wx} C_{wy}$.

Expanding the estimator $t_5$, we get

$k_1 \bar{Y}(1+e\bar{_{wy}}) a_5^{(\frac{e\bar{_{wx}}^2}{4} - \frac{e\bar{_{wx}}}{2})}$.

Expanding $a_5^{(\frac{e\bar{_{wx}}^2}{4} - \frac{e\bar{_{wx}}}{2})}$ and simplifying we get

$t_5 = k_1 \bar{Y}(1 + e\bar{_{wy}} - \frac{e\bar{_{wx}}}{2} log_e a_5 + \frac{e\bar{^2_{wx}}}{4} log_e a_5 - \frac{e\bar{_{wx}} e\bar{_{wy}}}{2} log_e a_5 + \frac{e\bar{_{wx}}^2}{8} (log_e a_5)^2)$

Subtracting $\bar{Y}$ from both sides, squaring and taking expectation, we get the Mean Square Error as:

$$MSE(t_5) = \bar{Y}^2 [1 + k_1^2 A^\gamma - 2k_1 B^\gamma] \tag{15}$$

where $A^\gamma = 1 + fC_{wy}^2 + \frac{1}{2} fC_{wx}^2 log_e a_5 + \frac{1}{2} fC_{wx}^2 (log_e a_5)^2 - 2\rho_{wxwy} fC_{wx} C_{wy} log_e a_5$

$B^\gamma = 1 + \frac{1}{4} fC_{wx}^2 log_e a_5 + \frac{1}{8} fC_{wx}^2 (log_e a_5)^2 - \frac{1}{2} \rho_{wxwy} fC_{wx} C_{wy} log_e a_5$

Differentiating MSE of the estimator $(t_5)$ with respect to $k_1$ we get

$$\frac{d}{dk_1} MSE(t_5) = 2k_1 \bar{Y}^2 A^\gamma - 2B^\gamma \bar{Y}^2 \tag{16}$$

equating equation (16) to zero we get the optimum value of $k_1$ as $k_{1(opt)} = \frac{B^\gamma}{A^\gamma} = k_0$(say).
Putting this optimum value of $k_0$ in equation (15) we get the minimum MSE of the estimator $t_5$:

$$minimumMSE(t_5) = \bar{Y}^2 [1 - \frac{(B^\gamma)^2}{A^\gamma}]$$

Expanding etimator $t_6$:

$$t_6 = [k_2 \bar{Y}(\bar{e}_{wy}+1) + (1-k_2)(\bar{X} - \bar{X}(\bar{e}_{wx}+1))] a_6^{(} - \frac{\bar{e}_{wx}}{2} + \frac{\bar{e}_{wx}^2}{4}) \tag{17}$$

expanding $a_6^{(} - \frac{e\bar{_{wx}}}{2} + \frac{\bar{e}_{wx}^2}{4})$ and simplifying equation (17) we get:

$$k_2 \bar{Y}[1 + \bar{e}_{wy} - \frac{\bar{e}_{wx}\bar{e}_{wy}}{2} log a_6 \frac{\bar{e}_{wx}^2}{4} log a_6 - \frac{\bar{e}_{wx}}{2} log a_6 + \frac{\bar{e}_{wx}^2}{8} (log a_6)^2] - \bar{X}\bar{e}_{wx} + \bar{X}\frac{\bar{e}_{wx}^2}{2} log a_6 + k_2 \bar{X}\bar{e}_{wx} - k_2 \bar{X}\frac{\bar{e}_{wx}^2}{2} log a_6 \tag{18}$$

Subtracting $\bar{Y}$ from both the sides of equation (18), squaring and taking expectation, we get the MSE as:

$$MSE(t_6) = \bar{Y}^2 + k_2^2 A^* - 2k_2 B^* + C^* \tag{19}$$

where

$$A^* = \bar{Y}^2[1+f(C_{wy}^2+\frac{1}{2}C_{wx}^2(log_e a_6)^2-2\rho_{wxwy}C_{wx}C_{wy}log_e a_6)]+\bar{X}^2 f C_{wx}^2+2\bar{X}\bar{Y}f(\rho_{wxwy}C_{wx}C_{wy}-C_{wx}^2 log_e a_6)$$

$$B^* = \bar{Y}^2[1+f(\frac{1}{4}C_{wx}^2 log_e a_6-\frac{1}{2}\rho_{wxwy}C_{wx}C_{wy}log_e a_6+\frac{1}{8}C_{wx}^2(log_e a_6)^2)]+\bar{X}^2 f C_{wx}^2-\bar{X}\bar{Y}f\frac{1}{2}C_{wx}^2 log_e a_6+\bar{X}\bar{Y}f(\rho_{wxwy}C_{wx}C_{wy}-C_{wx}^2 log_e a_6)$$

$$C^* = f C_{wx}^2 \bar{X}(\bar{X}-\bar{Y}log_e a_6)$$

Differentiating equation (19) with respect to $k_2$ we get:

$$\frac{d}{dk_2}MSE(t_6) = 2k_2A^* - 2B^* \quad (20)$$

Equating equation (20) to zero we get optimum value of $k_2$ as $k_{2(opt)} = \frac{B^*}{A^*}$ Differentiating $MSE(t_6)$ with respect to $k_2$ and equating to zero we get

$$k_2^* = \frac{B^*}{A^*} = k_o^*(say)$$

Putting this value of $k_0^*$ in equation (19) we get:

$$minimumMSE(t_6) = \bar{Y}^2 + (k_2^*)^2 - 2k_2^*B^* + C^*$$

## 5 Numerical Illustrations

In this section, we have studied the simulated x-values and y-values from Chutiman and Kumphon [8] and calculated the theoretical MSE of both the proposed estimators $t_5$ and $t_6$ for each and every value of $a > 0$, and obtained their PRE with respect to Thompson's [1] estimator $t_3$, Dryver and Chao's [7] estimator $t_4$ and the regression estimator under ACS $\bar{y}_{Reg}$, for 4 different sample sizes viz., 45, 50, 55 and 60. The optimum "a" is the value of "a" which gave minimum MSE. Results of the study are presented in table 2 to table 9. These tables contain just a sample of the output obtained for each iteration, here only those calculated values of the MSE and PRE are presented at which the value of a is reaching its optimum value. The data statistics of their population are taken from S. K. Yadav et al.[9] are given below in table 1.
The results obtained are presented in Table 2 to in Table 9.

**Table 1:** Data statistics

| | | | |
|---|---|---|---|
| N=400 | $\bar{Y}$=1.2225 | $\bar{X}$=0.5550 | $S_y$=5.050 |
| $\theta_1$=0.876 | $S_{wy}$=3.562 | $\theta_{w3}$=0.137 | $S_x$=2.400 |
| $\theta_2$=0.042 | $S_{wx}$=1.948 | $\theta_{w4}$=0.9357 | $S_{xy}$=11.037 |
| $\theta_3$=0.817 | $S_{wxwy}$=6.428 | $\theta_{w5}$=0.006 | $\rho_{XY}$=0.910 |
| $\theta_4$=0.064 | $\rho_{wxwy}$=0.926 | $\theta_{w6}$=0.375 | $C_y$=4.131 |
| $C_x$=4.325 | $C_{wy}$=2.914 | $\theta_{w8}$=0.864 | $C_{wx}$=3.510 |

**Table 2:** Mean Square Error and Percentage Relative Efficiency of $t_5$ for sample size 45

| $a_5$ | MSE $t_5$ | PRE w.r.t $t_3$ | PRE w.r.t $t_4$ | PRE w.r.t $\bar{y}_{Reg}$ |
|---|---|---|---|---|
| 4.85 | 0.032598 | 767.7989 | 168.9947 | 109.4073412 |
| 4.86 | 0.032595 | 767.8518 | 169.0064 | 109.4148861 |
| 4.87 | 0.032594 | 767.89 | 169.0148 | 109.4203318 |
| 4.88 | 0.032593 | 767.9136 | 169.02 | 109.4236932 |
| 4.89 | 0.032592 | 767.9227 | 169.022 | 109.4249851 |
| 4.9 | 0.032593 | 767.9173 | 169.0208 | 109.4242225 |
| 4.91 | 0.032593 | 767.8977 | 169.0165 | 109.4214206 |
| 4.92 | 0.032595 | 767.8638 | 169.009 | 109.4165946 |
| 4.93 | 0.032597 | 767.8158 | 168.9985 | 109.4097598 |
| 4.94 | 0.250284 | 767.7539 | 168.9848 | 109.4009318 |

**Table 3:** Mean Square Error and Percentage Relative Efficiency of $t_5$ for sample size 50

| $a_5$ | MSE $t_5$ | PRE w.r.t $t_3$ | PRE w.r.t $t_4$ | PRE w.r.t $\bar{y}_{Reg}$ |
|---|---|---|---|---|
| 4.81 | 0.029235 | 759.6411 | 167.1992 | 108.2449092 |
| 4.82 | 0.029233 | 759.7102 | 167.2144 | 108.2547565 |
| 4.83 | 0.02923 | 759.7643 | 167.2263 | 108.2624532 |
| 4.84 | 0.029229 | 759.8033 | 167.2349 | 108.2680138 |
| 4.85 | 0.029228 | 759.8274 | 167.2402 | 108.2714532 |
| 4.86 | 0.029228 | 759.8368 | 167.2423 | 108.2727863 |
| 4.87 | 0.029228 | 759.8315 | 167.2411 | 108.2720281 |
| 4.88 | 0.029229 | 759.8116 | 167.2367 | 108.2691938 |
| 4.89 | 0.02923 | 759.7772 | 167.2292 | 108.2642987 |
| 4.9 | 0.029232 | 759.7285 | 167.2184 | 108.2573583 |

**Table 4:** Mean Square Error and Percentage Relative Efficiency of $t_5$ for sample size 55

| $a_5$ | MSE $t_5$ | PRE w.r.t $t_3$ | PRE w.r.t $t_4$ | PRE w.r.t $\bar{y}_{Reg}$ |
|---|---|---|---|---|
| 4.8 | 0.02642 | 753.2443 | 165.7912 | 107.3333893 |
| 4.81 | 0.026419 | 753.2932 | 165.802 | 107.3403653 |
| 4.82 | 0.026417 | 753.327 | 165.8094 | 107.3451815 |
| 4.83 | 0.026417 | 753.3458 | 165.8136 | 107.3478527 |
| 4.84 | 0.026417 | 753.3496 | 165.8144 | 107.3483939 |
| 4.85 | 0.026417 | 753.3385 | 165.812 | 107.3468204 |
| 4.86 | 0.026418 | 753.3128 | 165.8063 | 107.3431472 |
| 4.87 | 0.026419 | 753.2724 | 165.7974 | 107.3373899 |
| 4.88 | 0.026421 | 753.2174 | 165.7853 | 107.329564 |

**Table 5:** Mean Square Error and Percentage Relative Efficiency of $t_5$ for sample size 60

| $a_5$ | MSE $t_5$ | PRE w.r.t $t_3$ | PRE w.r.t $t_4$ | PRE w.r.t $\bar{y}_{Reg}$ |
|---|---|---|---|---|
| 4.8 | 0.024035 | 748.0047 | 164.638 | 106.5867768 |
| 4.81 | 0.024034 | 748.0246 | 164.6424 | 106.5896053 |
| 4.82 | 0.024034 | 748.0293 | 164.6434 | 106.5902788 |
| 4.83 | 0.024034 | 748.019 | 164.6411 | 106.5888126 |
| 4.84 | 0.024035 | 747.9938 | 164.6356 | 106.5852219 |
| 4.85 | 0.024036 | 747.9538 | 164.6268 | 106.5795222 |
| 4.86 | 0.024038 | 747.8991 | 164.6147 | 106.571729 |
| 4.87 | 0.179781 | 747.8298 | 164.5995 | 106.561858 |
| 4.88 | 0.179781 | 747.7461 | 164.5811 | 106.5499251 |
| 4.89 | 0.179781 | 747.648 | 164.5595 | 106.535946 |

**Table 6:** Mean Square Error and Percentage Relative Efficiency of $t_6$ for sample size 45

| $a_6$ | MSE $t_6$ | PRE w.r.t $t_3$ | PRE w.r.t $t_4$ | PRE w.r.t $\bar{y}_{Reg}$ |
|---|---|---|---|---|
| 4.65 | 0.032412 | 772.1964 | 169.9627 | 110.0339681 |
| 4.66 | 0.03241 | 772.2516 | 169.9748 | 110.0418346 |
| 4.67 | 0.032408 | 772.2896 | 169.9832 | 110.04725 |
| 4.68 | 0.032407 | 772.3105 | 169.9878 | 110.05023 |
| 4.69 | 0.032407 | 772.3145 | 169.9886 | 110.0507902 |
| 4.7 | 0.032408 | 772.3015 | 169.9858 | 110.0489466 |
| 4.71 | 0.032409 | 772.2718 | 169.9793 | 110.0447155 |
| 4.72 | 0.032411 | 772.2255 | 169.9691 | 110.0381132 |
| 4.73 | 0.032413 | 772.1626 | 169.9552 | 110.0291563 |

**Table 7:** Mean Square Error and Percentage Relative Efficiency of $t_6$ for sample size 50

| $a_6$ | MSE $t_6$ | PRE w.r.t $t_3$ | PRE w.r.t $t_4$ | PRE w.r.t $\bar{y}_{Reg}$ |
|---|---|---|---|---|
| 4.64 | 0.029104 | 763.0649 | 167.9528 | 108.7327754 |
| 4.65 | 0.029102 | 763.1274 | 167.9665 | 108.7416864 |
| 4.66 | 0.0291 | 763.1728 | 167.9765 | 108.7481452 |
| 4.67 | 0.029099 | 763.201 | 167.9827 | 108.7521671 |
| 4.68 | 0.029098 | 763.2122 | 167.9852 | 108.7537678 |
| 4.69 | 0.029099 | 763.2066 | 167.984 | 108.7529629 |
| 4.7 | 0.0291 | 763.1841 | 167.979 | 108.7497684 |
| 4.71 | 0.029101 | 763.1451 | 167.9704 | 108.7442005 |
| 4.72 | 0.029103 | 763.0895 | 167.9582 | 108.7362756 |
| 4.73 | 0.029106 | 763.0174 | 167.9423 | 108.7260102 |

**Table 8:** Mean Square Error and Percentage Relative Efficiency of $t_6$ for sample size 55

| $a_6$ | MSE $t_6$ | PRE w.r.t $t_3$ | PRE w.r.t $t_4$ | PRE w.r.t $\bar{y}_{Reg}$ |
|---|---|---|---|---|
| 4.64 | 0.026328 | 755.8898 | 166.3735 | 107.7103555 |
| 4.65 | 0.026326 | 755.9449 | 166.3856 | 107.7182112 |
| 4.66 | 0.026325 | 755.9829 | 166.394 | 107.723626 |
| 4.67 | 0.026324 | 756.0039 | 166.3986 | 107.7266153 |
| 4.68 | 0.026324 | 756.0079 | 166.3995 | 107.7271946 |
| 4.69 | 0.026324 | 755.9952 | 166.3967 | 107.7253796 |
| 4.7 | 0.026325 | 755.9658 | 166.3902 | 107.7211863 |
| 4.71 | 0.026327 | 755.9198 | 166.3801 | 107.7146309 |
| 4.72 | 0.026329 | 755.8573 | 166.3664 | 107.7057297 |
| 4.73 | 0.026332 | 755.7785 | 166.349 | 107.6944992 |

**Table 9:** Mean Square Error and Percentage Relative Efficiency of $t_6$ for sample size 60

| $a_6$ | MSE $t_6$ | PRE w.r.t $t_3$ | PRE w.r.t $t_4$ | PRE w.r.t $\bar{y}_{Reg}$ |
|---|---|---|---|---|
| 4.63 | 0.023971 | 749.9996 | 165.0771 | 106.871035 |
| 4.64 | 0.023969 | 750.0662 | 165.0917 | 106.8805271 |
| 4.65 | 0.023967 | 750.1156 | 165.1026 | 106.8875724 |
| 4.66 | 0.023966 | 750.148 | 165.1097 | 106.892186 |
| 4.67 | 0.023966 | 750.1634 | 165.1131 | 106.8943833 |
| 4.68 | 0.023966 | 750.162 | 165.1128 | 106.8941798 |
| 4.69 | 0.023966 | 750.1438 | 165.1088 | 106.8915912 |
| 4.7 | 0.023967 | 750.109 | 165.1012 | 106.8866335 |
| 4.71 | 0.023969 | 750.0577 | 165.0899 | 106.8793227 |
| 4.72 | 0.023971 | 749.99 | 165.075 | 106.8696754 |

**Table 10:** MSE and PRE of all the estimators for sample size 45 (the MSE for $t_2$, $t_5$ and $t_6$ are at a=$a_{opt}$)

| estimators | MSE | PRE | optimum a |
|---|---|---|---|
| $t_1$ | 0.392597 | 63.75079 | not exponential |
| $t_2$ | 0.086465 | 289.4628 | 5.69 |
| $t_3$ | 0.250284 | 100 | not exponential |
| $t_4$ | 0.550088 | 45.49 | not exponential |
| $t_5$ | 0.032592 | 767.9227 | 4.89 |
| $t_6$ | 0.032407 | 772.3145 | 4.69 |

**Table 11:** MSE and PRE of all the estimators for sample size 50 (the MSE for $t_2$, $t_5$ and $t_6$ are at a=$a_{opt}$)

| estimators | MSE | PRE | optimum a |
|---|---|---|---|
| $t_1$ | 0.352588 | 62.98647 | not exponential |
| $t_2$ | 0.076722 | 289.4628 | 5.69 |
| $t_3$ | 0.222083 | 100 | not exponential |
| $t_4$ | 0.048881 | 454.3336 | not exponential |
| $t_5$ | 0.029228 | 759.8367 | 4.86 |
| $t_6$ | 0.029098 | 763.2123 | 4.68 |

**Table 12:** MSE and PRE of all the estimators for sample size 55 (the MSE for $t_2$, $t_5$ and $t_6$ are at a=$a_{opt}$)

| estimators | MSE | PRE | optimum a |
|---|---|---|---|
| $t_1$ | 0.319094 | 62.36703 | not exponential |
| $t_2$ | 0.068751 | 289.4627 | 5.69 |
| $t_3$ | 0.199009 | 100 | not exponential |
| $t_4$ | 0.043803 | 454.333 | not exponential |
| $t_5$ | 0.026417 | 753.3493 | 4.84 |
| $t_6$ | 0.026324 | 756.0077 | 4.68 |

**Table 13:** MSE and PRE of all the estimators for sample size 60 (the MSE for $t_2$, $t_5$ and $t_6$ are at a=$a_{opt}$)

| estimators | MSE | PRE | optimum a |
|---|---|---|---|
| $t_1$ | 0.29065 | 61.85489 | not exponential |
| $t_2$ | 0.062109 | 289.4628 | 5.69 |
| $t_3$ | 0.179781 | 100 | not exponential |
| $t_4$ | 0.03957 | 454.333 | not exponential |
| $t_5$ | 0.024034 | 748.0294 | 4.82 |
| $t_6$ | 0.023966 | 750.1634 | 4.67 |

## 6 Conclusion

In the present manuscript, we proposed two generalized exponential ratio estimators under ACS and obtained the optimum a values for both the proposed estimators using the numerical study. From Table 10 to Table 13, it is clear that the optimum a values obtained for both the estimators based on 4 different sample sizes, are too close to the value of optimum a obtained using the expression $\exp(\frac{2\rho_{wxwy}C_{wy}}{C_{wx}})$ which is $\exp(\frac{2\rho_{wxwy}C_{wy}}{C_{wx}}) = 4.65308$ for the population under study. Moreover from table 10 to table 13, it can be seen that when population is rare and hidden clustered, the proposed estimators have the maximum efficiency and the estimators for SRS do not perform adequately. Therefore the proposed estimators should be preferably adopted for estimating the unknown population mean in ACS.

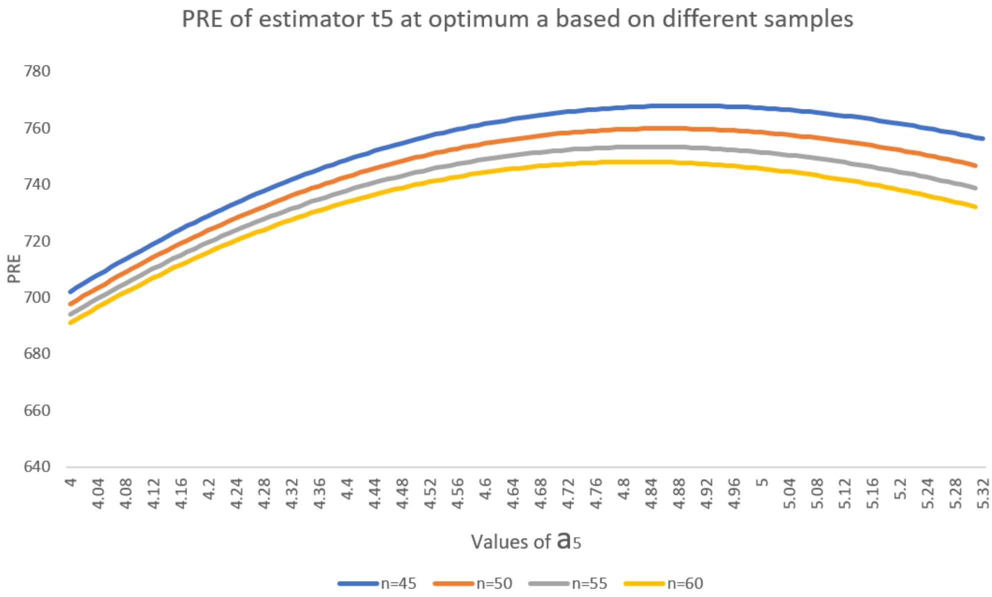


**Fig. 2:** The PRE of the estimator $t_5$ for the 4 sample sizes increases till it reaches the corresponding optimum value of $a_5$ and then starts declining.

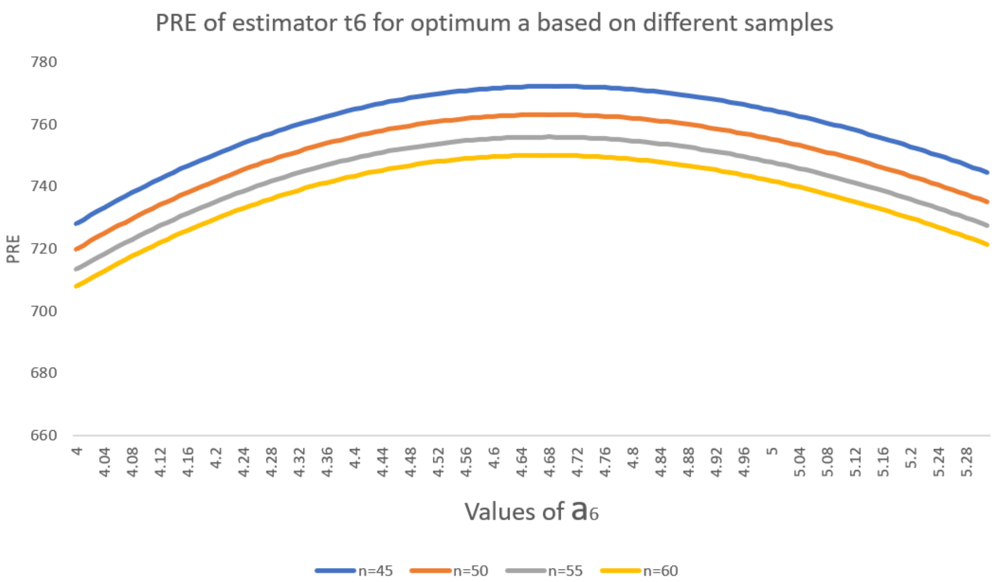


**Fig. 3:** The PRE of the estimator $t_6$ for the 4 sample sizes increases till it reaches the corresponding optimum value of $a_6$ and then starts declining.

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