J. Ana. Num. Theor. **5**, No. 1, 13-17 (2017)

Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/050103

Hermite-Hadamard Type Inequalities for Operator α-Preinvex Functions

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Received: 27 Jun. 2016, Revised: 2 Aug. 2016, Accepted: 4 Aug. 2016
Published online: 1 Jan. 2017

Abstract: In the paper, we introduce the concept of operator α-preinvex function, establish some new Hermite-Hadamard type inequalities for operator α-preinvex functions, and provide the estimates of both sides of Hermite-Hadamard type inequality in which some operator α-preinvex functions of positive selfadjoint operators in Hilbert spaces are involved.

Keywords: Hermite-Hadamard type inequality, operator α-convex function, operator preinvex function, operator α-preinvex function

1 Introduction

Throughout this paper, let $\mathbb{R} = (-\infty,\infty)$ and $\mathbb{R}_0 = [0,\infty)$.

First we review the operator order in $B(H)$ which is the set of all bounded linear operators on a Hilbert space $(H;\langle .,.\rangle)$, and the continuous functional calculus for a bounded self-adjoint operator. For self-adjoint operators $A,B \in B(H)$, we write $A \le B$ if $\langle Ax,x\rangle \le \langle Bx,x\rangle$ for every vector $x \in H$, we call it the operator order.

Let A be a bounded self-adjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous complex-valued functions defined on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [2], p.3). For any $f,g \in C(Sp(A))$ and any $\alpha,\beta \in \mathbb{C}$, we have

(i) $\Phi(\alpha f+\beta g) = \alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f^*) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t\in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A)$.

With this notation, we define

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A)) \tag{1}$$

and we call it the continuous functional calculus for a bounded self-adjoint operator A.

If A is a bounded self-adjoint operator and f is a real-valued continuous function on $Sp(A)$, then $f(t) \ge 0$ for any $t \in Sp(A)$ implies that $f(A) \ge 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real-valued functions on $Sp(A)$ such that $f(t) \le g(t)$ for any $t \in Sp(A)$, then $f(A) \le g(A)$ in the operator order in $B(H)$.

A real valued continuous function f on an interval $I \subseteq \mathbb{R}$ is said to be operator convex (operator concave) if

$$f((1-\lambda)A+\lambda B) \le (\ge)(1-\lambda)f(A)+\lambda f(B)$$

in the operator order in $B(H)$, for all $\lambda \in [0,1]$ and for every bounded self-adjoint operators A and B in $B(H)$ whose spectra are contained in I.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [2], [5], [6] and the references therein.

In [3], Ghazanfari et al. gave the concept of operator preinvex function and obtained Hermite-Hadamard type inequality for operator preinvex function.

Definition 1.1.[[3]] Let X be a real vector space, a set $S \subseteq X$ is said to be invex with respect to the map $\eta : S\times S \to X$, if for every $x,y \in S$ and $t \in [0,1]$,

$$x+t\eta(x,y) \in S. \tag{2}$$

It is obvious that every convex set is invex with respect to the map $\eta(x,y) = x-y$, but there exist invex sets which are not convex (see [1]).

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Let $S \subseteq X$ be an invex set with respect to $\eta : S \times S \to X$. For every $x, y \in S$, the η-path P_{xv} joining the points x and $v := x + \eta(y,x)$ is defined as follows

$$P_{xv} := \{z : z = x + t\eta(y,x), t \in [0,1]\}.$$

The mapping η is said to be satisfies the condition (C) if for every $x, y \in S$ and $t \in [0,1]$,

$$\begin{aligned} &\eta(y, y + t\eta(x,y)) = -t\eta(x,y), \\ &\eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y). \end{aligned} \tag{C}$$

Note that for every $x, y \in S$ and every $t_1, t_2 \in [0,1]$ from condition (C) we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y), \tag{3}$$

see [4], [7] for details.

Let A be a C^*-algebra, denote by A_{sa} the set of all self-adjoint elements in A.

Definition 1.2.[[3]] Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)_{sa}$. Then, the continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be operator preinvex with respect to η on S, if for every $A, B \in S$ and $t \in [0,1]$,

$$f(A + t\eta(B,A)) \le (1-t)f(A) + tf(B) \tag{4}$$

in the operator order in $B(H)$.

Every operator convex function is operator preinvex with respect to the map $\eta(A,B) = A - B$, but the converse does not holds (see [3]).

Theorem 1.1.[[3]] Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)_{sa}$ and η satisfy condition (C). If for every $A, B \in S$ and $V = A + \eta(B,A)$ the function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is operator preinvex with respect to η on η-path P_{AV} with spectra of A and spectra of V in the interval I. Then we have the inequality

$$f\left(\frac{A+V}{2}\right) \le \int_0^1 f((A + t\eta(B,A))\mathrm{d}t \le \frac{f(A)+f(B)}{2}. \tag{5}$$

Motivated by the above results we investigate in this paper the operator version of the Hermite-Hadamard inequality for operator α-preinvex functions.

2 Operator α-preinvex functions

In order to verify our main results, the following definition and lemmas are necessary.

Definition 2.1. Let I be an interval in $\mathbb{R}_0$ and $S \subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)^+_{sa}$. Then, the continuous function $f : I \to \mathbb{R}$ is said to be operator α-preinvex with respect to η on I for operators in S, if

$$f(A + t\eta(B,A)) \le (1-t^\alpha)f(A) + t^\alpha f(B) \tag{6}$$

in the operator order in $B(H)$, for all $t \in [0,1]$ and every positive operators A and B in S whose spectra are contained in I and for some fixed $\alpha \in [0,1]$.

It is obvious that every operator 1-preinvex function is operator preinvex, and every operator α-preinvex with respect to the map $\eta(A,B) = A - B$ is operator α-convex function, that is,

Definition 2.2. Let I be an interval in $\mathbb{R}_0$. Then, the continuous function $f : I \to \mathbb{R}$ is said to be operator α-convex on I for operators in $B(H)^+_{sa}$, if

$$f(tA + (1-t)B) \le t^\alpha f(A) + (1-t^\alpha)f(B) \tag{7}$$

in the operator order in $B(H)$, for all $t \in [0,1]$ and every positive operators A and B in $B(H)^+_{sa}$ whose spectra are contained in I and for some fixed $\alpha \in [0,1]$.

Lemma 2.1. Let $S \subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)^+_{sa}$ and $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a continuous function on the interval I. Suppose that η satisfies condition (C) on S. Then for every $A, B \in S$ and $V = A + \eta(B,A)$ and for some fixed $\alpha \in [0,1]$, the function f is operator α-preinvex with respect to η on η-path P_{AV} with spectra of A and V in the interval I if and only if the function $\varphi_{x,A,B} : [0,1] \to \mathbb{R}$ defined by

$$\varphi_{x,A,B}(t) := \langle f(A + t\eta(B,A))x, x\rangle \tag{8}$$

is α-convex on $[0,1]$ for every $x \in H$.

Proof. Suppose that $x \in H$ and $\varphi_{x,A,B} : [0,1] \to \mathbb{R}$ is α-convex on $[0,1]$ for some fixed $\alpha \in [0,1]$. For every $C_1 := A + t_1\eta(B,A) \in P_{AV}$, $C_2 := A + t_2\eta(B,A) \in P_{AV}$, fix $\lambda \in [0,1]$, by (8) we have

$$\begin{aligned} &\langle f(C_1 + \lambda\eta(C_2, C_1))x, x\rangle \\ &= \langle f(A + ((1-\lambda)t_1 + \lambda t_2)\eta(B,A))x, x\rangle \\ &= \varphi_{x,A,B}((1-\lambda)t_1 + \lambda t_2) \\ &\le (1-\lambda^\alpha)\varphi_{x,A,B}(t_1) + \lambda^\alpha\varphi_{x,A,B}(t_2) \\ &= (1-\lambda^\alpha)\langle f(C_1)x, x\rangle + \lambda^\alpha\langle f(C_2)x, x\rangle. \end{aligned} \tag{9}$$

Hence, f is operator α-preinvex with respect to η on η-path P_{AV}.

Conversely, let $A, B \in S$ and f be operator α-preinvex with respect to η on η-path P_{AV} for some fixed $\alpha \in [0,1]$. Suppose that $t_1, t_2 \in [0,1]$. Then for every $\lambda \in [0,1]$ and $x \in H$, we have

$$\begin{aligned} &\varphi_{x,A,B}((1-\lambda)t_1 + \lambda t_2) \\ &= \langle f(A + ((1-\lambda)t_1 + \lambda t_2)\eta(B,A))x, x\rangle \\ &= \langle f(A + t_1\eta(B,A) + \lambda\eta(A + t_2\eta(B,A), \\ &\quad A + t_1\eta(B,A)))x, x\rangle \\ &\le (1-\lambda^\alpha)\langle f(A + t_1\eta(B,A))x, x\rangle \\ &\quad + \lambda^\alpha\langle f(A + t_2\eta(B,A))x, x\rangle \\ &= (1-\lambda^\alpha)\varphi_{x,A,B}(t_1) + \lambda^\alpha\varphi_{x,A,B}(t_2). \end{aligned} \tag{10}$$

Therefore, $\varphi_{x,A,B}$ is α-convex on $[0,1]$. The proof of Lemma 2 is complete.

3 Hermite-Hadamard type inequalities for the operator α-preinvex functions

The following theorem is the generalization of Hermite-Hadamard's inequality for operator α-preinvex functions.

Theorem 3.1. Let $S \subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta : S\times S \to B(H)^+_{sa}$ and η satisfy condition (C) on S. If for every $A,B \in S$ and $V = A+\eta(B,A)$ and for some fixed $\alpha \in [0,1]$, the continuous function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is operator α-preinvex with respect to η on η-path P_{AV} with spectra of A and V in the interval I. Then we have the inequality

$$f\left(\frac{A+V}{2}\right) \le \int_0^1 f(A+t\eta(B,A))\mathrm{d}t \le \frac{\alpha f(A)+f(B)}{\alpha+1}. \tag{11}$$

Proof. For $x \in H$ and $t \in [0,1]$, we have

$$\langle (A+t\eta(B,A))x,x\rangle = \langle Ax,x\rangle + t\langle \eta(B,A)x,x\rangle \in I, \tag{12}$$

since $\langle Ax,x\rangle \in Sp(A) \subseteq I$ and $\langle Vx,x\rangle \in Sp(V) \subseteq I$.

Continuity of f and (12) imply that the operator valued integral $\int_0^1 f(A+t\eta(B,A))\mathrm{d}t$ exists.

Since η satisfies condition (C) and f is α-preinvex with respect to η, for every $t\in[0,1]$, we have

$$\begin{aligned}
&f\left(A+\frac{1}{2}\eta(B,A)\right)\\
=&f\left(A+t\eta(B,A)+\frac{1}{2}\eta(A+(1-t)\eta(B,A),A+t\eta(B,A))\right)\\
\le&\left(1-\frac{1}{2^\alpha}\right)f(A+t\eta(B,A))+\frac{1}{2^\alpha}f(A+(1-t)\eta(B,A))\\
\le&\left\{1-t^\alpha+\frac{1}{2^\alpha}\left[t^\alpha-(1-t)^\alpha\right]\right\}f(A)\\
&+\left\{t^\alpha-\frac{1}{2^\alpha}\left[t^\alpha-(1-t)^\alpha\right]\right\}f(B).
\end{aligned} \tag{13}$$

Integrating the inequality (13) over $t \in [0,1]$ and taking into account that

$$\int_0^1 f(A+t\eta(B,A))\mathrm{d}t = \int_0^1 f(A+(1-t)\eta(B,A))\mathrm{d}t, \tag{14}$$

we obtain the inequality (11), which complete the proof of Theorem 3.

Remark 3.1.1. Choosing $\alpha = 1$, we obtain Theorem 1.

For some fixed $\alpha_1,\alpha_2 \in [0,1]$, let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be an operator α_1-preinvex function and $g : I \to \mathbb{R}$ be an operator α_2-preinvex function on the interval I. Then for all positive operators A and B on a Hilbert space H with spectra in I and for any $x\in H$, we define real functions $M(A,B)$ and $N(A,B)$ on H by

$$\begin{aligned}
M(A,B)(x) &= \langle f(A)x,x\rangle\langle g(A)x,x\rangle + \langle f(B)x,x\rangle\langle g(B)x,x\rangle,\\
N(A,B)(x) &= \langle f(A)x,x\rangle\langle g(B)x,x\rangle + \langle f(B)x,x\rangle\langle g(A)x,x\rangle.
\end{aligned} \tag{15}$$

Theorem 3.2. Let $S \subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta : S\times S \to B(H)^+_{sa}$ and η satisfy condition (C) on S. If for every $A,B\in S$ and $V = A+\eta(B,A)$ and for some fixed $\alpha_1,\alpha_2 \in [0,1]$, the continuous function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is an operator α_1-preinvex function and $g : I \to \mathbb{R}$ is an operator α_2-preinvex function on the interval I with respect to η on η-path P_{AV} with spectra of A and V in the interval I. Then we have the inequality

$$\begin{aligned}
&\int_0^1 \langle f(A+t\eta(B,A))x,x\rangle\langle g(A+t\eta(B,A))x,x\rangle\mathrm{d}t\\
&\le \frac{\alpha_1\alpha_2-1}{(\alpha_1+1)(\alpha_2+1)}\langle f(A)x,x\rangle\langle g(A)x,x\rangle\\
&\quad+\frac{1}{\alpha_2+1}\langle f(A)x,x\rangle\langle g(B)x,x\rangle\\
&\quad+\frac{1}{\alpha_1+1}\langle f(B)x,x\rangle\langle g(A)x,x\rangle\\
&\quad+\frac{1}{\alpha_1+\alpha_2+1}\left[M(A,B)(x)-N(A,B)(x)\right]
\end{aligned} \tag{16}$$

holds for any $x\in H$, where $M(A,B)$ and $N(A,B)$ are defined in (15).

Proof. For $x\in H$ and $t\in[0,1]$, we have

$$\langle (A+t\eta(B,A))x,x\rangle = \langle Ax,x\rangle + t\langle\eta(B,A)x,x\rangle \in I,$$

since $\langle Ax,x\rangle \in Sp(A)\subseteq I$ and $\langle Vx,x\rangle \in Sp(V) \subseteq I$.

From the continuity of f, g, it shows that the operator valued integral $\int_0^1 f(A+t\eta(B,A))\mathrm{d}t$, $\int_0^1 g(A+t\eta(B,A))\mathrm{d}t$, and $\int_0^1 (fg)(A+t\eta(B,A))\mathrm{d}t$ exist.

Since $f : I \to \mathbb{R}$ is operator α_1-preinvex and $g : I \to \mathbb{R}$ is operator α_2-preinvex for some fixed $\alpha_1,\alpha_2\in[0,1]$, therefore for every $t\in[0,1]$ we drive

$$\begin{aligned}
&\langle f(A+t\eta(B,A))x,x\rangle\langle g(A+t\eta(B,A))x,x\rangle\\
&\le (1-t^{\alpha_1})(1-t^{\alpha_2})\langle f(A)x,x\rangle\langle g(A)x,x\rangle\\
&\quad+(1-t^{\alpha_1})t^{\alpha_2}\langle f(A)x,x\rangle\langle g(B))x,x\rangle\\
&\quad+t^{\alpha_1}(1-t^{\alpha_2})\langle f(B)x,x\rangle\langle g(A)x,x\rangle\\
&\quad+t^{\alpha_1+\alpha_2}\langle f(B)x,x\rangle\langle g(B))x,x\rangle.
\end{aligned} \tag{17}$$

Integrating both sides of (17) over $t\in[0,1]$, we obtain the required inequality (16). The proof of Theorem 3 is complete.

Corollary 3.2.1. Under the assumptions of Theorem 3, if $\alpha_1 = \alpha_2 = \alpha$, then

$$\begin{aligned}
&\int_0^1 \langle f(A+t\eta(B,A))x,x\rangle\langle g(A+t\eta(B,A))x,x\rangle\mathrm{d}t\\
&\le \frac{\alpha-1}{\alpha+1}\langle f(A)x,x\rangle\langle g(A)x,x\rangle + \frac{1}{2\alpha+1}M(A,B)(x)\\
&\quad+\frac{\alpha}{(\alpha+1)(2\alpha+1)}N(A,B)(x).
\end{aligned} \tag{18}$$

Specially, if $\alpha_1 = \alpha_2 = 1$, then

$$\int_0^1 \langle f(A+t\eta(B,A))x,x\rangle\langle g(A+t\eta(B,A))x,x\rangle \mathrm{d}t \leq \frac{2M(A,B)(x)+N(A,B)(x)}{6}. \tag{19}$$

Corollary 3.2.2. With the conditions of Theorem 3, if $\eta(B,A) = B-A$, then

$$\begin{aligned}
&\int_0^1 \langle f(tB+(1-t)A)x,x\rangle\langle g(tB+(1-t)A)x,x\rangle \mathrm{d}t \\
&\leq \frac{\alpha_1\alpha_2-1}{(\alpha_1+1)(\alpha_2+1)}\langle f(A)x,x\rangle\langle g(A)x,x\rangle \\
&\quad + \frac{1}{\alpha_2+1}\langle f(A)x,x\rangle\langle g(B)x,x\rangle \\
&\quad + \frac{1}{\alpha_1+1}\langle f(B)x,x\rangle\langle g(A)x,x\rangle \\
&\quad + \frac{1}{\alpha_1+\alpha_2+1}\left[M(A,B)(x)-N(A,B)(x)\right].
\end{aligned} \tag{20}$$

Theorem 3.3. Let $S \subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta : S\times S \to B(H)^+_{sa}$ and η satisfy condition (C) on S. If for every $A,B \in S$ and $V = A+\eta(B,A)$ and for some fixed $\alpha_1,\alpha_2 \in [0,1]$, the continuous function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is an operator α_1-preinvex function and $g : I \to \mathbb{R}$ is an operator α_2-preinvex function on the interval I with respect to η on η-path P_{AV} with spectra of A and V in the interval I. Then we have the inequality

$$\begin{aligned}
&\frac{2^{\alpha_1+\alpha_2}}{(2^{\alpha_1}-1)(2^{\alpha_2}-1)+1} \\
&\quad\times \left\langle f\left(\frac{A+V}{2}\right)x,x\right\rangle\left\langle g\left(\frac{A+V}{2}\right)x,x\right\rangle \\
&\leq \int_0^1 \langle f(A+t\eta(B,A))x,x\rangle\langle g(A+t\eta(B,A))x,x\rangle \mathrm{d}t \\
&\quad + \frac{\alpha_1-1}{(2^{\alpha_1}-1)(2^{\alpha_2}-1)+1}\langle f(A)x,x\rangle\langle g(B)x,x\rangle \\
&\quad + \frac{\alpha_2-1}{(2^{\alpha_1}-1)(2^{\alpha_2}-1)+1}\langle f(B)x,x\rangle\langle g(A)x,x\rangle
\end{aligned} \tag{21}$$

holds for any $x \in H$.

Proof. Since $f : I \to \mathbb{R}$ is operator α_1-preinvex and $g : I \to \mathbb{R}$ be operator α_2-preinvex for some fixed $\alpha_1,\alpha_2 \in [0,1]$, therefore for every $t \in [0,1]$ we have

$$\begin{aligned}
&\left\langle f\left(\frac{A+V}{2}\right)x,x\right\rangle\left\langle g\left(\frac{A+V}{2}\right)x,x\right\rangle \\
&= \left\langle f\left(A+t\eta(B,A)+\frac{1}{2}\eta(A+(1-t)\eta(B,A),\right.\right. \\
&\qquad \left.\left. A+t\eta(B,A))\right)x,x\right\rangle \\
&\quad \times \left\langle g\left(A+t\eta(B,A)+\frac{1}{2}\eta(A+(1-t)\eta(B,A),\right.\right. \\
&\qquad \left.\left. A+t\eta(B,A))\right)x,x\right\rangle \\
&\leq \left\langle \left[\left(1-\frac{1}{2^{\alpha_1}}\right)f(A+t\eta(B,A))\right.\right. \\
&\qquad \left.\left. +\frac{1}{2^{\alpha_1}}f(A+(1-t)\eta(B,A))\right]x,x\right\rangle \\
&\quad \times \left\langle \left[\left(1-\frac{1}{2^{\alpha_2}}\right)g(A+t\eta(B,A))\right.\right. \\
&\qquad \left.\left. +\frac{1}{2^{\alpha_2}}g(A+(1-t)\eta(B,A))\right]x,x\right\rangle \\
&\leq \left(1-\frac{1}{2^{\alpha_1}}\right)\left(1-\frac{1}{2^{\alpha_2}}\right)\langle f(A+t\eta(B,A))x,x\rangle \\
&\quad \times \langle g(A+t\eta(B,A))x,x\rangle \\
&\quad + \frac{1}{2^{\alpha_1+\alpha_2}}\langle f(A+(1-t)\eta(B,A))x,x\rangle \\
&\quad \times \langle g(A+(1-t)\eta(B,A))x,x\rangle \\
&\quad + \left(1-\frac{1}{2^{\alpha_1}}\right)\frac{1}{2^{\alpha_2}}\langle f(A)x,x\rangle\langle g(B)x,x\rangle \\
&\quad + \left(1-\frac{1}{2^{\alpha_2}}\right)\frac{1}{2^{\alpha_1}}\langle f(B)x,x\rangle\langle g(A)x,x\rangle].
\end{aligned} \tag{22}$$

By integrating over $t \in [0,1]$ and taking into account that

$$\begin{aligned}
&\int_0^1 \langle f(A+t\eta(B,A))x,x\rangle\langle g(A+t\eta(B,A))x,x\rangle \mathrm{d}t \\
&= \int_0^1 \langle f(A+(1-t)\eta(B,A))x,x\rangle \\
&\quad \times \langle g(A+(1-t)\eta(B,A))x,x\rangle \mathrm{d}t,
\end{aligned}$$

we obtain the required inequality (21). Thus Theorem 3 is thus proved.

Corollary 3.3.1. Under the assumptions of Theorem 3, if $\alpha_1 = \alpha_2 = \alpha$, then

$$\begin{aligned}
&\frac{4^\alpha}{(2^\alpha-1)^2+1}\left\langle f\left(\frac{A+V}{2}\right)x,x\right\rangle\left\langle g\left(\frac{A+V}{2}\right)x,x\right\rangle \\
&\leq \int_0^1 \langle f(A+t\eta(B,A))x,x\rangle\langle g(A+t\eta(B,A))x,x\rangle \mathrm{d}t \\
&\quad + \frac{\alpha-1}{(2^\alpha-1)^2+1}N(A,B)(x).
\end{aligned} \tag{23}$$

In particular, if $\alpha_1 = \alpha_2 = 1$, then

$$2\left\langle f\left(\frac{A+V}{2}\right)x,x\right\rangle\left\langle g\left(\frac{A+V}{2}\right)x,x\right\rangle \leq \int_0^1 \langle f(A+t\eta(B,A))x,x\rangle\langle g(A+t\eta(B,A))x,x\rangle \mathrm{d}t. \quad (24)$$

where $N(A,B)$ is defined in (15).

Corollary 3.3.2. With the conditions of Theorem 3, if $\eta(B,A) = B - A$, then

$$\frac{2^{\alpha_1+\alpha_2}}{(2^{\alpha_1}-1)(2^{\alpha_2}-1)+1} \times \left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle\left\langle g\left(\frac{A+B}{2}\right)x,x\right\rangle \leq \int_0^1 \langle f(tB+(1-t)A)x,x\rangle\langle g(tB+(1-t)A)x,x\rangle \mathrm{d}t + \frac{\alpha_1-1}{(2^{\alpha_1}-1)(2^{\alpha_2}-1)+1}\langle f(A)x,x\rangle\langle g(B)x,x\rangle + \frac{\alpha_2-1}{(2^{\alpha_1}-1)(2^{\alpha_2}-1)+1}\langle f(B)x,x\rangle\langle g(A)x,x\rangle. \quad (25)$$

Corollary 3.3.3. With the assumptions of Theorem 3 and Theorem 3, we obtain

$$\frac{1}{(2^{\alpha_1}-1)(2^{\alpha_2}-1)+1}\Bigg[2^{\alpha_1+\alpha_2}\left\langle f\left(\frac{A+V}{2}\right)x,x\right\rangle \times \left\langle g\left(\frac{A+V}{2}\right)x,x\right\rangle - (\alpha_1-1)\langle f(A)x,x\rangle\langle g(B)x,x\rangle - (\alpha_2-1)\langle f(B)x,x\rangle\langle g(A)x,x\rangle\Bigg] \leq \int_0^1 \langle f((1-t)A+tB)x,x\rangle\langle g((1-t)A+tB)x,x\rangle \mathrm{d}t \leq \frac{\alpha_1\alpha_2-1}{(\alpha_1+1)(\alpha_2+1)}\langle f(A)x,x\rangle\langle g(A)x,x\rangle + \frac{1}{\alpha_2+1}\langle f(A)x,x\rangle\langle g(B)x,x\rangle + \frac{1}{\alpha_1+1}\langle f(B)x,x\rangle\langle g(A)x,x\rangle + \frac{1}{\alpha_1+\alpha_2+1}[M(A,B)(x)-N(A,B)(x)]. \quad (26)$$

where $M(A,B)$ and $N(A,B)$ are defined in (15).

Acknowledgement

This work was supported by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant No. NJZZ16175.
The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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