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pa-BCL$^+$ Algebras and Groups

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Abstract: A BCL^+ algebras can be considered as a fragment of propositional logic containing only a logical connective implication a binary operation $\rightarrow$ and 1 which is interpreted as the value true. This paper offers a new abstract structure, which is called the pseudo-association BCL^+ algebras (simply pa-BCL^+ algebras). Because of its origin in BCL^+ algebras, we increase the pseudo-association law and ordered structure, but the connection is not a simple derivation, it involved groups, abelian and semigroups. We will give some concepts and characterizations in the pa-BCL^+ algebras. As a result, the pa-BCL^+ algebras have more complex algebraic structures.

Keywords: groups, semigroups, abelian groups, BCL-algebras, BCL^+ algebras, pa-BCL^+ algebras

1 Introduction

BCL-algebras and BCL^+ algebras are introduced by author in [1] and [2], which is a relative new-comer in the history of algebra. Al-Kadi and Hosny in [3] introduced the concepts of deformation of such algebra in BCL-algebras and illustrate the connection between divisible algebra and deformation function. Soft BCL-algebras are studied by Al-Kadi in [4]. The BCL^+ algebras developed recently by author in [5,6,7,8,9,10,11,12], some concepts involved partial order, lattice, topology, pseudocomplement, filtrations, deductive systems, funnels and standard ideals in BCL^+ algebras.

We know that the algebraic theory of semigroups occurs naturally in many areas of mathematics, such as combinatorics, automata theory, operator algebras, probability theory and BCL^+ algebras [13]. We do also believe that semigroups, historically, to large extent, has based its prosperity by Clifford in [14,15,16], Hall in [17,18,19] and Nambooripad in [20] with their good works.

We actually wants to know the relations between the BCL^+ algebras and the pseudo-association, how to choose a suitable group and semigroup is becoming very important, and the structure provides theory for *cryptography*. In this paper, we introduce the pseudo-association BCL^+ algebras (simply pa-BCL^+ algebras) is based on the BCL^+ algebras and is increase the pseudo-association law and ordered structures (the ordered pa-BCL^+ algebras, simply opa-BCL^+ algebras). The new algebras or pa-BCL^+ algebras linked up with several groups, e.g., *involution groups*, *abelian groups*, *adjoint groups* and *semigroups*. We research that the properties of pa-BCL^+ algebras and the results are interesting. We will also illustrate it with three examples. In this paper, a binary operation, denoted by $\rightarrow$ (the original denoted by $*$ (cf. Ref. [2,5,6,7,8,9,10,11,12,13])), is that makes obvious the connection with logic.

2 Preliminaries

In this section, let's first review some relevant concepts and state some results, as follows.

Definition 2.1. ([2], Definition 2.1.) A BCL^+ algebra is a triple $(A); \rightarrow, 1)$, where A is a nonempty set, $\rightarrow$ is a binary operation on A, and $1 \in A$ is an element such that the following three axioms hold for any $x, y, z \in A$.
(BCL^+1). $x \rightarrow x = 1$.
(BCL^+2). $x \rightarrow y = 1$ and $y \rightarrow x = 1$ imply $x = y$.
(BCL^+3). $((x \rightarrow y) \rightarrow z) \rightarrow ((x \rightarrow z) \rightarrow y) = (z \rightarrow y) \rightarrow x$.

Theorem 2.1. ([8], Theorem 3.1.) Let Y be a BCL^+algebra. Let $x \leq y$ iff $x \rightarrow y = 1$. Then $\leq$ is a partially ordering on Y; that is $(Y; \leq)$ is a partially ordered set with greatest element 1 (and unit element) of Y.

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3 Main results

Definition 3.1. A BCL^+ algebra $\mathscr{A} = (A; \rightarrow, 1)$ is called pseudo-association if it satisfies the axiom
(BCL^+4). $(x \rightarrow y) \rightarrow z = x \rightarrow (z \rightarrow y)$ for all $x, y, z \in A$.
Then we say that A is a pseudo-association BCL^+ algebra (*pa-*BCL^+ algebra).

Intuitively, the basic case is simply stated though (or some object we plan to construct directly from A). The following examples illustrate the concept of a pseudo-association.

Example 3.1. Let $A = \{1, a, b, c\}$. We define an operation $\rightarrow$ on A by the following Cayley Table 3.1 and Figure 3.1 display Hasse diagram.

Table 3.1. *pa-*BCL^+ operation.

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	b	c	1	c
c	a	b	c	1

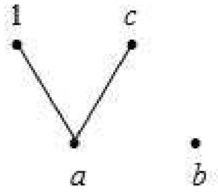

Fig. 3.1. The Hasse Diagram of $(\{1, a, b, c\})$.

Then $\mathscr{A} = (A; \rightarrow, 1)$ is a pseudo-association BCL^+ algebra.
Solution. Since

$$(a \rightarrow b) \rightarrow c = 1 \text{ and } a \rightarrow (c \rightarrow b) = 1,$$

for instance.

The next example point out that pseudo-association axiom is independent. In algebra structure, the concept of a pseudo-association also makes a great deal of sense.

Example 3.2. Let $A = \{0, a, b, c, 1\}$. We define an operation $\rightarrow$ on A by the following Cayley Table 3.2 and Figure 3.2 display Hasse diagram.

Table 3.2. BCL^+ operation.

$\rightarrow$	0	a	b	c	1
0	1	0	0	0	0
a	a	1	1	c	1
b	b	a	1	c	1
c	c	b	c	1	1
1	0	1	1	1	1

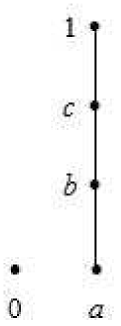

Fig. 3.2. The Hasse Diagram of $(\{0, a, b, c, 1\})$.

Then $\mathscr{A} = (A; \rightarrow, 1)$ is a BCL^+ algebra.
Solution. It is easy to check that

$$(a \rightarrow b) \rightarrow c = 1 \text{ and } a \rightarrow (c \rightarrow b) = c,$$

but $1 \neq c$ is not a pseudo-association of $\mathscr{A}$.

Theorem 3.1. Let $\mathscr{A} = (A; \rightarrow, 1)$ be a pseudo-association BCL^+ algebra and assume that for all $x, y, z \in A$. The following are then equivalent:
(PAB 1). A is pseudo-association.
(PAB 2). $1 \rightarrow x = x$.
(PAB 3). $x \rightarrow y = y \rightarrow x$.
(PAB 4). $1 \rightarrow (x \rightarrow y) = y \rightarrow x$.
(PAB 5). $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

Proof. First assume (PAB 1) and prove (PAB 2). Then

$$(1 \rightarrow x) \rightarrow x = 1 \rightarrow (x \rightarrow x) = 1,$$

and

$$x \rightarrow (1 \rightarrow x) = (x \rightarrow x) \rightarrow 1 = 1,$$

so that $1 \rightarrow x = x$, and (PAB 2) is proved.

We assume (PAB 2) and prove (PAB 3). Then
$x \rightarrow y$
$= (1 \rightarrow x) \rightarrow (1 \rightarrow y)$
$= 1 \rightarrow ((1 \rightarrow y) \rightarrow x)$
$= (1 \rightarrow y) \rightarrow x$
$= y \rightarrow x$,
and (PAB 3) is proved.

To show that (PAB 3) implies (PAB 4). Then
$1 \rightarrow (x \rightarrow y)$
$= (x \rightarrow y) \rightarrow 1$
$= x \rightarrow (1 \rightarrow y)$
$= x \rightarrow y$
$= y \rightarrow x$,
and (PAB 4) is proved.

We assume (PAB) 4) and prove (PAB 5). Then
$y \rightarrow (x \rightarrow z)$
$= 1 \rightarrow ((x \rightarrow z) \rightarrow y)$
$= (x \rightarrow z) \rightarrow y$
$= x \rightarrow (y \rightarrow z)$,
and (PAB 5) is proved.

Finally, assume (PAB 5) and prove (PAB) 1). Then
$x \rightarrow (z \rightarrow y)$
$= z \rightarrow (x \rightarrow y)$

$= (x \to y) \to z.$
This proof is now complete. □

The following theorem is a useful characterization of pseudo-association BCL^+ algebra and clarifies, to some extent, the connection between pseudo-association BCL^+ algebra and involution group.

Theorem 3.2. For any involution group G, let

$$a \to b = a \cdot b.$$

Then $(G; \to, 1)$ is a pseudo-association BCL^+ algebra with at identity element 1.

Proof. First, the fact that $x \to x = 1$ for $x \in G$ is obvious.

Now we prove

$$x \to y = 1 = y \to x \text{ implies } x = y.$$

From this, we get

$$(x \to y) \to y = 1 \to y = y.$$

By Definition 3.1, however, we have

$$\begin{aligned} &(x \to y) \to y \\ &= x \to (y \to y) \\ &= x \to 1 \\ &= 1 \to x \\ &= x. \end{aligned}$$

Thus $x = y$.

Finally, we prove (BCL^+ 3), suppose that $x, y, z \in G$. By Definition 3.1, we can write

$$\begin{aligned} &(((x \to y) \to z) \to ((x \to z) \to y)) \to ((z \to y) \to x) \\ &= ((x \to y) \to z) \to (((z \to y) \to x) \to ((x \to z) \to y)) \\ &= ((x \to y) \to z) \to (((z \to y) \to x) \to (x \to (y \to z))) \\ &= ((x \to y) \to z) \to 1 \end{aligned}$$

and

$$\begin{aligned} &((z \to y) \to x) \to (((x \to y) \to z) \to ((x \to z) \to y)) \\ &= (((z \to y) \to x) \to ((x \to z) \to y)) \to ((x \to y) \to z) \\ &= (((z \to y) \to x) \to (x \to (y \to z))) \to ((x \to y) \to z) \\ &= 1 \to ((x \to y) \to z). \end{aligned}$$

Then

$$\begin{aligned} &((x \to y) \to z) \to 1 \\ &= 1 \to ((x \to y) \to z) \\ &= ((x \to y) \to z). \end{aligned}$$

This proof is now complete. □

Theorem 3.3. Let A be a pseudo-association BCL^+ algebra. Then A is involution group with at identity element 1.

Proof. Let A is pseudo-association and let $\to$ is a binary operation. Then we see that

$$1 \to x = x \to 1 = x \text{ and } x \to x = 1,$$

for each $x \in A$. The proof is complete. □

Example 3.3. Let $A = \{1, a, b, c\}$. We define an operation $\to$ on A by the following Cayley Table 3.3 and Figure 3.3 display Hasse diagram.

Table 3.3. *pa-BCL*$^+$ operation.

$\to$	1	a	b	c
1	1	a	b	c
a	a	1	a	1
b	b	c	1	c
c	c	b	c	1

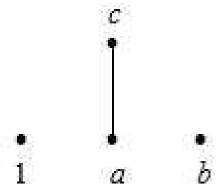

Fig. 3.3. The Hasse Diagram of $(\{1, a, b, c\})$.

Then $\mathscr{A} = (A; \to, 1)$ is a pseudo-association BCL^+ algebra.
Solution. It is easy to check that

$$(a \to b) \to c = 1 \text{ and } a \to (c \to b) = 1,$$

On the other hand, $(A, \to)$ is involution group with at identity element 1.

Definition 3.2. Let A be a BCL^+ algebra. We say that minimal positive integer n is degree if $1 \to x^n = 1$ for all $x \in A$, denoted by $|A|$.

Definition 3.3. Let A be a BCL^+ algebra and let $x, y \in A$ such that

$$x \to y^0 = x,$$

and

$$x \to y^n = (\cdots((x \to \underbrace{y) \to y) \to \cdots) \to y}_{n}.$$

The following result shows that $|A|$.

Theorem 3.4. Let $\mathscr{A} = (A; \to, 1)$ be a pseudo-association BCL^+ algebra and $\mathscr{G} = (G, \to)$ be an involution group, are same-degree.

Proof. Since we can take $1 \to x = x$, and we choose $x \in A$ and $x \in G$, we have

$$\begin{aligned} &1 \to x^n \\ &= (\cdots((1 \to x) \to x) \to \cdots) \to x \\ &= \underbrace{x \to x \to \cdots \to x}_{n} \\ &= x^n, \end{aligned}$$

and n exists and such that

$$1 \to x^n = 1 \text{ iff } x^n = 1,$$

and we deduce that

$$\deg(x) = |A| = |G|.$$

This proof is now complete. □

Definition 3.4. Let $(G, +)$ be an abelian, $(G; -, 1)$ be an adjoint BCL^+ algebra and $(G; \to, 1)$ be a pseudo-association BCL^+ algebra such that
$(ADB\ 1)$. $x - (1 - y) = x + y$ and
$(ADB\ 2)$. $x - y = x \to y$.
Then adjoint group of $(G; -, 1)$ is abelian $(G, +)$, and adjoint BCL^+ algebra of abelian $(G, +)$ is $(G; \to, 1)$.

Theorem 3.5. $\mathscr{A} = (A; \to, 1)$ be a pseudo-association BCL^+ algebra and $\mathscr{P} = (P, +)$ be an adjoint group, are same-degree.

Proof. If $x \in A$ and $x \in P$, we write $1 \to x = x$, then

$$\begin{aligned} & nx \\ =\ & \underbrace{x+x+\cdots+x}_{n} \\ =\ & (\cdots((x \to (1 \to x) \to (1 \to x) \to \cdots) \to (1 \to x) \\ =\ & (1 \to (1 \to x)) \to (1 \to x)^{n-1} \\ =\ & 1 \to (1 \to x)^{n} \\ =\ & 1 \to (1 \to x^{n}), \end{aligned}$$

and n exists. If $1 \to x^n = 1$. Then $nx = 1 \to 1 = 1$.
Conversely, if $nx = 1$. Then

$$\begin{aligned} & 1 \to x^{n} \\ =\ & 1 \to (1 \to (1 \to x^{n})) \\ =\ & 1. \end{aligned}$$

Finally, $1 \to x^n = 1\ iff\ nx = 1$, and we deduce that

$$\deg(x) = |A| = |P|.$$

This proof is now complete. □

Theorem 3.6. In the situation of Definition 3.1, we have $\deg(\beta) = |A| > 1$.

Proof. Let $\beta \in A$. We show next that A is a pseudo-association. Then

$$\begin{aligned} & ((\beta \to 1) \to 1) \to (\beta \to (1 \to 1)) \\ =\ & (\beta \to (1 \to 1)) \to (\beta \to 1) \\ =\ & (\beta \to 1) \to (\beta \to 1) \\ =\ & 1, \end{aligned}$$

and

$$\begin{aligned} & (\beta \to (1 \to 1)) \to ((\beta \to 1) \to 1) \\ =\ & (\beta \to 1) \to ((\beta \to 1) \to 1) \\ =\ & (\beta \to 1) \to (\beta \to 1) \\ =\ & 1. \end{aligned}$$

Therefore

$$\begin{aligned} & ((\beta \to 1) \to 1) \\ =\ & \beta \to (1 \to 1) \\ =\ & \beta \to 1. \end{aligned}$$

Now, we have

$$\begin{aligned} & 1 \to ((\beta \to (1 \to 1)) \to ((\beta \to 1) \to 1)) \\ =\ & (1 \to (\beta \to 1) \to 1) \to (\beta \to 1) \\ =\ & ((\beta \to 1) \to 1) \to (\beta \to 1) \\ =\ & (\beta \to (1 \to 1)) \to (\beta \to 1) \\ =\ & (\beta \to 1) \to (\beta \to 1) \\ =\ & 1. \end{aligned}$$

Then

$$|(\beta \to 1) \to ((\beta \to 1) \to 1)| \leq 1.$$

Conversely, let A is a pseudo-association and $|A| = 1$, we have $\beta = 1$. If $\beta \neq 1$.Then $\deg(\beta) = |A| > 1$.
This proof is now complete. □

The following result is closely linked with abelian.

Theorem 3.7. Let $\mathscr{A} = (A; \to, 1)$ be a pseudo-association BCL^+ algebra. For all $x, y \in A$, we define the following for abelian:

$$x + y = y \to (x \to 1).$$

Then $(A, +)$ is abelian, and $y \to 1$ is negative element of y and is adjoint group of A.

Proof. Let $x, y, z \in A$, by Theorem 3.1, we get

$$\begin{aligned} & x + y \\ =\ & y \to (x \to 1) \\ =\ & (y \to 1) \to x \\ =\ & x \to (y \to 1) \\ =\ & y + x. \end{aligned}$$

Similarly, $z + y = y + z$. We have

$$\begin{aligned} & (x + y) + z \\ =\ & (x \to (y \to 1)) \to (z \to 1) \\ =\ & x \to ((z \to 1) \to (y \to 1)) \\ =\ & x + (z + y) \\ =\ & x + (y + z). \end{aligned}$$

Clearly,

$$\begin{aligned} & 1 + y \\ =\ & 1 \to (y \to 1) \\ =\ & y \to 1 \\ =\ & y, \end{aligned}$$

and

$$\begin{aligned} & (y \to 1) + y \\ =\ & (y \to 1) \to (y \to 1) \\ =\ & 1. \end{aligned}$$

This proof is now complete. □

Theorem 3.8. Let M is an adjoint group of pseudo-association BCL^+ algebra, $N \subseteq M$. Then N is a subalgebra of M iff N is a subgroup of M.
Proof. Let $x, y \in N$ with $1 = x \to x \in N$. Then

$$-y = 1 \to y \in N,$$

and we have

$$x + y = y \to (x \to 1) \in N.$$

Conversely, let N be a subgroup of M. In fact $1 \in N$ and $x, y \in N$. Then

$$\begin{aligned} & x \to y \\ =\ & x \to (1 \to (1 \to y)) \\ =\ & x + (1 \to y) \\ =\ & x + (-y) \end{aligned}$$

$= x - y \in N$.
We deduce that N is a subalgebra of M. The proof is now complete. ☐

Theorem 3.9. Let H is an abelian. For all $x, y \in H$, define: $x \to y = x + y^{-1}$. Then adjoint BCL^+ algebra $(H; \to, 1)$ is pseudo-association.
Proof. If $x \in H$, we write

$$\begin{aligned}&1 \to x\\ &= x \to 1\\ &= x,\end{aligned}$$

then

$$\begin{aligned}&1 \to (1 \to x)\\ &= 1 + (1 + x^{-1})^{-1}\\ &= x,\end{aligned}$$

and

$$\begin{aligned}&(1 \to x) \to 1\\ &= (1 + x^{-1}) + 1^{-1}\\ &= x + 1^{-1}\\ &= x.\end{aligned}$$

We have

$$1 \to (1 \to x) = (1 \to x) \to 1.$$

The proof is complete. ☐

Corollary 3.1. The pseudo-association BCL^+ algebra and abelian, are equivalent.
Proof. This is immediate since, by Theorem 3.1, Theorem 3.7 and Theorem 3.9, this yields the desired equivalent. The proof is complete. ☐

Definition 3.5. A BCL^+ algebra $\mathscr{A} = (A; \to, 1)$ is called ordered pseudo-association if it satisfies the axiom

$$(x \to y) \to z \leq x \to (z \to y)$$

for all $x, y, z \in A$. We say that A be ordered pseudo-association BCL^+ algebras (simply opa-BCL^+ algebras).

Theorem 3.10. Let A be a pseudo-association BCL^+ algebra and suppose that D is a subalgebra of A. Then there exists $x \in D$. Also, D is a filtration in A iff $1 \to x \in D$.
Proof. Since D is a filtration. Then $1 \to x \in D$ implies $x \in D$.

Conversely, let $x \in D$ and choose $x \to y \in D$. Since D is a subalgebra, we have

$$((x \to y) \to x) \to ((x \to x) \to y) \leq (x \to y) \to x.$$

It follows that

$$((x \to y) \to x) \to (1 \to y) \leq (x \to y) \to x.$$

We also find that

$$\begin{aligned}&1 \to y\\ &\leq (1 \to x) \to (1 \to y)\\ &\leq (x \to y) \to x\\ &\leq x \to (x \to y) \in D.\end{aligned}$$

We see that $1 \to y \in D$, then $y \in D$ and so D is a filtration. The proof is complete. ☐

Theorem 3.11. Let A be an ordered pseudo-association iff deg $(x) = |A| \leq 2$.
Proof. Let $x \in A$. We have

$$\begin{aligned}&(1 \to x) \to x\\ &= 1 \to (x \to x)\\ &= 1 \to 1\\ &= 1.\end{aligned}$$

Then deg $(x) = |A| \leq 2$.

Conversely, Let $x, y, z \in A$. Since deg $(y) = |A| \leq 2$. Then $(1 \to y) \to y = 1$, by Definition 3.1, we see that

$$\begin{aligned}&((y \to z) \to y) \to z\\ &= (y \to (y \to z)) \to z\\ &= y \to (z \to (y \to z))\\ &= y \to ((z \to z) \to y)\\ &= y \to (1 \to y)\\ &= (1 \to y) \to y\\ &= 1 \to y^2\\ &= 1.\end{aligned}$$

Since deg $(z) = |A| \leq 2$. Then $(1 \to z) \to z = 1$. But in another way, by Definition 3.1 and Theorem 3.1 (PAB 3), we have

$$\begin{aligned}&((y \to z) \to y) \to z\\ &= (y \to z) \to (z \to y)\\ &= 1,\end{aligned}$$

and let $y = 1$, we write $1 \to z = z \to 1 = z$. Then

$$\begin{aligned}&(1 \to z) \to (z \to 1)\\ &= 1 \to ((z \to 1) \to z)\\ &= (1 \to z) \to z\\ &= 1 \to z^2\\ &= 1.\end{aligned}$$

As required, now

$$\begin{aligned}&(x \to y) \to z\\ &= (((y \to z) \to y) \to z) \to ((x \to y) \to z)\\ &\leq ((y \to z) \to y) \to (x \to y)\\ &\leq (y \to z) \to x\\ &= y \to (x \to z).\end{aligned}$$

and let $x = z$, $y = x$ and $z = y$, we have

$$y \to (x \to z) = x \to (z \to y),$$

so that $(x \to y) \to z \leq x \to (z \to y)$. The proof is now complete. ☐

Theorem 3.12. Let G be an opa-BCL^+ algebra. Assume that for all $x, y \in G$, we define the following for semigroup: $x + y = x \to y$. Then $(G, +)$ is ordered commutative semigroup.
Proof. Let $x, y, z \in G$. We have

$$\begin{aligned}&x + y\\ &= x \to y\end{aligned}$$

$$\begin{aligned}
&= 1 \to (x \to y)\\
&= (1 \to y) \to x\\
&= y \to x\\
&= y + x,
\end{aligned}$$

since $y + z = z + y$, we see that $y \to z = z \to y$, therefore, it suffices to show that

$$\begin{aligned}
&\quad (x + y) + z\\
&= (x \to y) \to z\\
&= x \to (z \to y)\\
&= x \to (y \to z)\\
&= x + (y + z).
\end{aligned}$$

Then $(G, +)$ is commutative semigroup.

As required, now let $x \leq y$. Then $z \to y \leq z \to x$. We have

$$1 \to (z \to x) \leq 1 \to (z \to y),$$

we see that

$$z + x \leq z + y \text{ and } x + z \leq y + z.$$

Also, $(G, +)$ is ordered commutative semigroup and is a law of order-preserving. The proof is now complete. □

Lemma 3.1. Let $\mathscr{A} = (A; \to, 1)$ be an *opa-BCL*$^+$ algebra and assume that for all $x, y, z \in A$. The following are then equivalent:

$(OPA\ 1)$. $1 \to (1 \to x) = x$.

$(OPA\ 2)$. An ordered commutative semigroup $(A, +)$ is identity semigroup with at identity element 1.

$(OPA\ 3)$. An ordered commutative semigroup $(A, +)$ is group with at identity element 1.

Proof. Assume $(OPA\ 1)$ and prove $(OPA)\ 2)$. Let $x \in A$, and by Theorem 3.12. Then

$$\begin{aligned}
&\quad 1 + x\\
&= 1 \to (1 \to x)\\
&= 1 \to x\\
&= x,
\end{aligned}$$

and $(OPA)\ 2)$ is proved.

Assume $(OPA\ 2)$ and prove $(OPA\ 3)$. By Theorem 3.12, we have

$$\begin{aligned}
&\quad (1 \to x) + x\\
&= 1 \to ((1 \to x) \to x)\\
&= (1 \to x) \to (1 \to x)\\
&= 1.
\end{aligned}$$

Then $-x = 1 \to x$, and $(OPA\ 3)$ is proved.

Assume $(OPA\ 3)$ and prove $(OPA\ 1)$. Then

$$1 \to (1 \to x) = 1 + x = x.$$

This proof is now complete. □

Theorem 3.13. Let $\mathscr{A} = (A; \to, 1)$ be a *BCL*$^+$ algebra and assume that for all $x, y, z \in A$. we define the following for addition: $x + y = x \to y$. Then $(A, +)$ is pseudo commutative semigroup.

Proof. Let $x, y, z \in A$. We have

$$\begin{aligned}
&\quad (x + y) + z\\
&= (x \to y) \to z\\
&= x \to (z \to y)\\
&= x + (z + y)\\
&= (z + y) + x.
\end{aligned}$$

Then

$$\begin{aligned}
&\quad (x + y) + z\\
&= x + (z + y)\\
&= (z + y) + x.
\end{aligned}$$

(These additions are laws of associativity of pseudo.) This proof is now complete. □

Lemma 3.2. Let $\mathscr{A} = (A; \to, 1)$ be a *BCL*$^+$ algebra and assume that for all $x, y, z \in A$. Then

$(ADD\ 1)$. $(x + y) + 1 = x + y$.

$(ADD\ 2)$. $x + (1 \to x) = 1$.

$(ADD\ 3)$. $x + ((y + z) \to 1) = x + (y + z)$.

Proof. We have

$$\begin{aligned}
&\quad (x + y) + 1\\
&= (x \to y) \to 1\\
&= x \to (1 \to y)\\
&= x \to y\\
&= x + y.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\quad x + (1 \to x)\\
&= x \to (1 \to x)\\
&= x \to x\\
&= x + x\\
&= 1.
\end{aligned}$$

For $(ADD\ 3)$ we write $x \to 1 = x = 1 \to x$. Then

$$\begin{aligned}
&\quad x + ((y + z) \to 1)\\
&= x \to ((y \to z) \to 1)\\
&= (x \to 1) \to (y \to z)\\
&= x \to (y \to z)\\
&= x + (y + z).
\end{aligned}$$

This proof is now complete. □

4 Conclusion

It turns out that *BCL*$^+$ algebras are a very interesting area of research in the theory of algebraic systems in mathematics. In the present paper, we introduce the pseudo-association *BCL*$^+$ algebras, and the group is actually helping one important aspect of *BCL*$^+$ algebras, so we put some useful definitions and properties into Section 3, Main results.

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