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p-Schatten Norm Generalized Fractional Ostrowski and Grüss Type Inequalities Involving Several Functions

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Abstract: Employing generalized Caputo fractional left and right vectorial Taylor formulae we establish generalized fractional Ostrowski and Grüss type inequalities involving several functions that take values in the von Neumann-Schatten class $\mathscr{B}_p(H)$, $1 \leq p < \infty$. The estimates are with respect to all p-Schatten norms, $1 \leq p < \infty$. We finish with applications.

Keywords: p-Schatten norms, von Neumann-Schatten class, Ostrowski and Grüss inequalities, generalized fractional derivative, generalized fractional inequalities.

1 Introduction

The following results motivate our work.

Theorem 1.
(1938, Ostrowski [1]) Let $f:[a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \tag{1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Ostrowski type inequalities have great applications to integral approximations in Numerical Analysis.

Theorem 2.
(1882, Čebyšev [2]) Let $f,g:[a,b] \to \mathbb{R}$ be absolutely continuous functions with $f', g' \in L_\infty([a,b])$. Then

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x)\,dx - \left(\frac{1}{b-a} \int_a^b f(x)\,dx \right) \left(\frac{1}{b-a} \int_a^b g(x)\,dx \right) \right|$$

$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty. \tag{2}$$

The above integrals are assumed to exist.

The related Grüss type inequalities have many applications to Probability Theory. We presented also ([3], Ch. 8,9) mixed fractional Ostrowski and Grüss-Cebysev type inequalities for several functions, acting to all possible directions. The estimates involve the left and right Caputo fractional derivatives. See also the monographs written by the author [4], Chapters 24-26 and [5], Chapters 2-6.

We are motivated also by S. Dragomir [6] recent work:

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An operator $A \in \mathscr{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathscr{B}_p(H)$, $1 \leq p < \infty$ if the p-Schatten norm is finite

$$\|A\|_p := [tr(|A|^p)]^{\frac{1}{p}} < \infty.$$

Assume that $A : [a,b] \to \mathscr{B}_p(H)$, $B : [a,b] \to \mathscr{B}_q(H)$, $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, are continuous and B is strongly differentiable on (a,b), then

$$\left\| \int_a^b A(t) B(t) dt - \left(\int_a^b A(s) ds \right) B(u) \right\|_1 \leq$$

$$\sup_{t \in [a,b]} \|B'(t)\|_q \times \begin{cases} \left[\frac{1}{2}(b-a) + \left| u - \frac{a+b}{2} \right| \right] \int_a^b \|A(t)\|_p dt, \\ \left[\frac{(u-a)^{\beta+1} + (b-u)^{\beta+1}}{\beta+1} \right]^{\frac{1}{\beta}} \left(\int_a^b \|A(t)\|_p^{\alpha} \right)^{\frac{1}{\alpha}}, \\ \text{for } \alpha, \beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \left[\frac{1}{4}(b-a)^2 + \left(u - \frac{a+b}{2} \right)^2 \right] \sup_{t \in [a,b]} \|A(t)\|_p, \end{cases} \tag{3}$$

for all $u \in [a,b]$, an Ostrowski type inequality.

Further inspiration comes from S. Dragomir [7] recent work on Grüss inequalities:

For two continuous functions $A, B : [a,b] \to \mathscr{B}(H)$ we define the noncommutative Cebysev fractional

$$D(A,B) := (b-a) \int_a^b A(t) B(t) dt - \int_a^b A(t) dt \int_a^b B(t) dt. \tag{4}$$

If $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, let $A : [a,b] \to \mathscr{B}_p(H)$, $B : [a,b] \to \mathscr{B}_q(H)$ be strongly differentiable functions on the interval (a,b), then

$$\|D(A,B)\|_1 \leq D\left(\int_a^b \|A'(u)\|_p du, \int_a^b \|B'(u)\|_q du \right) \leq$$

$$\frac{1}{4}(b-a)^2 \int_a^b \|A'(u)\|_p du \int_a^b \|B'(u)\|_q du.$$

In this article we generalize [3], Ch. 8,9 for several Banach algebra $\mathscr{B}_p(H)$ valued functions, in the sense of developing fractional Ostrowski and Grüss type inequalities. Now our left and right generalized Caputo fractional derivatives are for Banach space valued functions and our integrals are of Bochner type [8]. Applications finish the article.

2 Vectorial background fractional calculus

Here all come from [9].

We need

Definition 1._([9], p. 106) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$._

We define the left generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{a+;g}^{\alpha} f \right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x) - g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1} \right)^{(n)} (g(t)) dt, \tag{5}$$

$\forall x \in [a,b]$, where Γ is the gamma function. The last integral is of Bochner type ([8] and [10], pp. 422-428).

If $\alpha \notin \mathbb{N}$, by Theorem 4.10 ([9], p. 98), we have that $\left(D_{a+;g}^{\alpha} f \right) \in C([a,b], X)$.

We set

$$D_{a+;g}^{n} f(x) := \left(\left(f \circ g^{-1} \right)^{n} \circ g \right)(x) \in C([a,b], X), \quad n \in \mathbb{N}, \tag{6}$$

$$D_{a+;g}^{0} f(x) = f(x), \quad \forall x \in [a,b].$$

When $g = id$, then

$$D_{a+;g}^{\alpha} f = D_{a+;id}^{\alpha} f = D_{*a}^{\alpha} f, \tag{7}$$

the usual left X-valued Caputo fractional derivative, see [5], Ch. 1.

We also need

Definition 2.([9], p. 107) *Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b],X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$.*
We define the right generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{b-;g}^{\alpha} f\right)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (g(t) - g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{8}$$

$\forall\, x \in [a,b]$. The last integral is of Bochner type.
If $\alpha \notin \mathbb{N}$, by Theorem 4.11 ([9], p. 101), we have that $\left(D_{b-;g}^{\alpha} f\right) \in C([a,b],X)$.
We set

$$D_{b-;g}^{n} f(x) := (-1)^n \left(\left(f \circ g^{-1}\right)^n \circ g\right)(x) \in C([a,b],X), \;\; n \in \mathbb{N}, \tag{9}$$

$$D_{b-;g}^{0} f(x) := f(x), \;\; \forall\, x \in [a,b].$$

When $g = id$, then

$$D_{b-;g}^{\alpha} f(x) = D_{b-;id}^{\alpha} f(x) = D_{b-}^{\alpha} f, \tag{10}$$

the usual right X-valued Caputo fractional derivative, see [9], Ch. 2.

We mention the following generalized fractional Taylor formulae with integral remainders over Banach spaces.

Theorem 3.([9], p. 107) *Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, $a \le x \le b$. Then*

$$\begin{aligned} f(x) = f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) + \\ \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D_{a+;g}^{\alpha} f\right)(t)\, dt = \\ f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) + \\ \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} \left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz. \end{aligned} \tag{11}$$

We also mention

Theorem 4.([9], p. 108) *Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, $a \le x \le b$. Then*

$$\begin{aligned} f(x) = f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) + \\ \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha} f\right)(t)\, dt = \\ f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) + \\ \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{\alpha-1} \left(\left(D_{b-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz. \end{aligned} \tag{12}$$

If $0 < \alpha \le 1$, then the sums in (11), (12) disappear.
Also in (11), (12), we have that $\left\|\left\|\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right\|\right\|_{\infty,[g(a),g(b)]}$,
$\left\|\left\|\left(D_{b-;g}^{\alpha} f\right) \circ g^{-1}\right\|\right\|_{\infty,[g(a),g(b)]} < \infty$.

3 Banach algebras background

All here come from [11].

We need

Definition 3.(*[11], p. 245) A complex algebra is a vector space A over the complex filed $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{13}$$

$$(x+y)z = xz + yz, \; x(y+z) = xy + xz, \tag{14}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{15}$$

for all x, y and z in A and for all scalars α.

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \|y\| \quad (x \in A, \, y \in A) \tag{16}$$

and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{17}$$

and

$$\|e\| = 1, \tag{18}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

*Remark.*Commutativity of A will be explicited stated when needed.

There exists at most one $e \in A$ that satisfies (17).

Inequality (16) makes multiplication to be continuous, more precisely left and right continuous, see [11], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [11], p. 247-248, § 10.3.

We also make

*Remark.*Next we mention about integration of A-valued functions, see [11], p. 259, § 10.22:

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [11], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \, d\mu = \int_Q x f(p) \, d\mu(p) \tag{19}$$

and

$$\left(\int_Q f \, d\mu \right) x = \int_Q f(p) x \, d\mu(p). \tag{20}$$

The Bochner integrals we will involve in our article follow (19) and (20). Also, let $f \in C([a,b], X)$, where $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [9], p. 3, f is Bochner integrable.

4 p-Schatten norms background

In this basic section all come from [6].

Let $(H,\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\mathscr{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathscr{B}(H)$ is of trace class if

$$\|A\|_1 := \sum_{i\in I}\langle |A|\, e_i, e_i\rangle < \infty. \tag{21}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthornormal basis $\{e_i\}_{i\in I}$. We denote by $\mathscr{B}_1(H)$ the set of trace class operators in $\mathscr{B}(H)$.

We define the trace of a trace class operator $A\in\mathscr{B}_1(H)$ to be

$$tr(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{22}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (22) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 5. *We have:*

(i) If $A\in\mathscr{B}_1(H)$ then $A^*\in\mathscr{B}_1(H)$ and

$$tr(A^*) = \overline{tr(A)}; \tag{23}$$

(ii) If $A\in\mathscr{B}_1(H)$ and $T\in\mathscr{B}(H)$, then $AT,\, TA\in\mathscr{B}_1(H)$ and

$$tr(AT) = tr(TA) \quad \text{and} \quad |tr(AT)| \le \|A\|_1\|T\|; \tag{24}$$

(iii) $tr(\cdot)$ is a bounded linear functional on $\mathscr{B}_1(H)$ with $\|tr\| = 1$;

(iv) If $A,B\in\mathscr{B}_2(H)$ then $AB,\ BA\in\mathscr{B}_1(H)$ and $tr(AB) = tr(BA)$;

(v) $\mathscr{B}_{fin}(H)$ (finite rank operators) is a dense subspace of $\mathscr{B}_1(H)$.

An operator $A\in\mathscr{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathscr{B}_p(H)$, $1\le p<\infty$ if the p-Schatten norm is finite [12, p. 60-64]

$$\|A\|_p := \left[tr\left(|A|^p\right)\right]^{\frac{1}{p}} < \infty,$$

$|A|^p$ is an operator notation and not a power.

For $1<p<q<\infty$ we have that

$$\mathscr{B}_1(H)\subset\mathscr{B}_p(H)\subset\mathscr{B}_q(H)\subset\mathscr{B}(H) \tag{25}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{26}$$

For $p\ge 1$ the functional $\|\cdot\|_p$ is a norm on the $*$-ideal $\mathscr{B}_p(H)$, which is a Banach algebra, and $\left(\mathscr{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [12, p. 60-64], for $p\ge 1$,

$$\|A\|_p = \|A^*\|_p,\ A\in\mathscr{B}_p(H) \tag{27}$$

$$\|AB\|_p \le \|A\|_p\|B\|_p,\ A,B\in\mathscr{B}_p(H) \tag{28}$$

and

$$\|AB\|_p \le \|A\|_p\|B\|,\ \ \|BA\|_p \le \|B\|\,\|A\|_p,\ A\in\mathscr{B}_p(H),\, B\in\mathscr{B}(H). \tag{29}$$

This implies that

$$\|CAB\|_p \le \|C\|\,\|A\|_p\|B\|,\ A\in\mathscr{B}_p(H),\, B,C\in\mathscr{B}(H). \tag{30}$$

In terms of p-Schatten norm we have the Hölder inequality for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$:

$$(|tr(AB)|\le)\,\|AB\|_1 \le \|A\|_p\|B\|_q,\ \ A\in\mathscr{B}_p(H),\, B\in\mathscr{B}_q(H). \tag{31}$$

For the theory of trace functionals and their applications the interested reader is referred to [12,13].

For some classical trace inequalities see [14,15,16], which are continuations of the work of Bellman [17].

5 Main results

We start with 1-Schatten norm weighted mixed generalized fractional Ostrowski type inequalities involving several functions taking values in the Banach algebra $\mathscr{B}_2(H) \subset \mathscr{B}(H)$:

Theorem 6. *Let the $*$-ideal $\mathscr{B}_2(H)$, which $(\mathscr{B}_2(H), \|\cdot\|_2)$ is a Banach algebra; $x_0 \in [a,b] \subset \mathbb{R}$, $\alpha > 0$, $n = \lceil \alpha \rceil$; $A_i \in C^n([a,b], \mathscr{B}_2(H))$, $i = 1,...,r \in \mathbb{N} - \{1\}$; $g \in C^1([a,b])$, strictly increasing such that $g^{-1} \in C^n([g(a), g(b)])$, with $(A_i \circ g^{-1})^{(k)}(g(x_0)) = 0$, $k = 1,...,n-1$; $i = 1,...,r$.*

Then

1) it holds,

$$\Phi(A_1,...,A_r)(x_0) :=$$

$$\sum_{i=1}^{r}\left[\int_a^b \left(\prod_{\substack{j=1\\j\neq i}}^{r} A_j(x)\right) A_i(x)\,dx - \left(\int_a^b \left(\prod_{\substack{j=1\\j\neq i}}^{r} A_j(x)\right) dx\right) A_i(x_0)\right] = \tag{32}$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} A_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]\right.$$

$$\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} A_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]\right],$$

2) for $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, with $\alpha > \frac{1}{\delta}$ we have that

$$\|\Phi(A_1,...,A_r)(x_0)\|_1 \leq \frac{1}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}} \tag{33}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\,(g(x_0)-g(x))^{\alpha-\frac{1}{\delta}}\,dx+\right.$$

$$\left.\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\,(g(x)-g(x_0))^{\alpha-\frac{1}{\delta}}\,dx\right],$$

3) if $\alpha \geq 1$, we obtain

$$\|\Phi(A_1,...,A_r)(x_0)\|_1 \leq \frac{1}{\Gamma(\alpha)} \tag{34}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\,(g(x_0)-g(x))^{\alpha-1}\,dx+\right.$$

$$\left.\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\,(g(x)-g(x_0))^{\alpha-1}\,dx\right],$$

and

4)

$$\|\Phi(A_1,...,A_r)(x_0)\|_1 \leq \frac{1}{\Gamma(\alpha+1)}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_2\left(g(x_0)-g(x)\right)^{\alpha}dx+\right.$$

$$\left.\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_2\left(g(x)-g(x_0)\right)^{\alpha}dx\right]. \tag{35}$$

*Proof.*Since $\left(A_i\circ g^{-1}\right)^{(k)}\left(g(x_0)\right)=0$, $k=1,\dots,n-1$; $i=1,\dots,r$, we have by Theorem 3 that

$$A_i(x)-A_i(x_0)=\frac{1}{\Gamma(\alpha)}\int_{g(x_0)}^{g(x)}\left(g(x)-z\right)^{\alpha-1}\left(\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right)(z)\,dz, \tag{36}$$

$\forall\, x\in[x_0,b]$,
and by Theorem 4 that

$$A_i(x)-A_i(x_0)=\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(x_0)}\left(z-g(x)\right)^{\alpha-1}\left(\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right)(z)\,dz, \tag{37}$$

$\forall\, x\in[a,x_0]$; for all $i=1,\dots,r$.

Let multiplying (36) and (37) with $\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_j(x)\right)$ we get, respectively,

$$\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_j(x)\right)A_i(x)-\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_j(x)\right)A_i(x_0)=$$

$$\frac{\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_j(x)\right)}{\Gamma(\alpha)}\int_{g(x_0)}^{g(x)}\left(g(x)-z\right)^{\alpha-1}\left(\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right)(z)\,dz, \tag{38}$$

$\forall\, x\in[x_0,b]$,
and

$$\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_j(x)\right)A_i(x)-\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_j(x)\right)A_i(x_0)=$$

$$\frac{\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_j(x)\right)}{\Gamma(\alpha)}\int_{g(x)}^{g(x_0)}\left(z-g(x)\right)^{\alpha-1}\left(\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right)(z)\,dz, \tag{39}$$

$\forall\, x\in[a,x_0]$; for all $i=1,\dots,r$.
Adding (38) and (39) as separate groups, we obtain

$$\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_j(x)\right)A_i(x)-\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_j(x)\right)A_i(x_0)=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}A_j(x)\right)\int_{g(x_0)}^{g(x)}\left(g(x)-z\right)^{\alpha-1}\left(\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right)(z)\,dz, \tag{40}$$

$\forall\, x \in [x_0, b]$,

and

$$\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) A_i(x) - \sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) A_i(x_0) =$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right)\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1}\left(\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right)(z)\,dz, \tag{41}$$

$\forall\, x \in [a, x_0]$.

Next we integrate (40) and (41) with respect to $x \in [a,b]$. We have

$$\sum_{i=1}^{r}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) A_i(x)\,dx - \sum_{i=1}^{r}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) dx\right) A_i(x_0) =$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right)\left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1}\left(\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right)(z)\,dz\right) dx\right], \tag{42}$$

and

$$\sum_{i=1}^{r}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) A_i(x)\,dx - \sum_{i=1}^{r}\left(\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) dx\right) A_i(x_0) =$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right)\left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1}\left(\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right)(z)\,dz\right) dx\right]. \tag{43}$$

Finally, adding (42) and (43) we obtain the useful identity

$$\Phi(A_1,\ldots,A_r)(x_0) :=$$

$$\sum_{i=1}^{r}\left[\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) A_i(x)\,dx - \left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right) dx\right) A_i(x_0)\right] = \tag{44}$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right)\left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1}\left(\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right)(z)\,dz\right) dx\right]\right.$$

$$\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right)\left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1}\left(\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right)(z)\,dz\right) dx\right]\right].$$

Next we take the 1-Schatten norm in (44) and we get:

$$\|\Phi(A_1,\ldots,A_r)(x_0)\|_1 \leq \frac{1}{\Gamma(\alpha)}$$

$$\sum_{i=1}^{r}\left[\left\|\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} A_j(x)\right)\left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1}\left(\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right)(z)\,dz\right) dx\right\|_1\right.$$

$$+\left\| \int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} A_j(x) \right) \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left(\left(D_{x_0+;g}^{\alpha} A_i \right) \circ g^{-1} \right)(z)\, dz \right) dx \right\|_1 \Bigg] \leq$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[\int_{a}^{x_0} \left\| \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} A_j(x) \right) \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left(\left(D_{x_0-;g}^{\alpha} A_i \right) \circ g^{-1} \right)(z)\, dz \right) \right\|_1 dx \right. \tag{45}$$

$$\left. + \int_{x_0}^{b} \left\| \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} A_j(x) \right) \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left(\left(D_{x_0+;g}^{\alpha} A_i \right) \circ g^{-1} \right)(z)\, dz \right) \right\|_1 dx \right] \leq$$

(by using the p-Schatten norm and Hölder's type inequality (31) for $p = q = 2$)

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[\int_{a}^{x_0} \left\| \prod_{\substack{j=1 \\ j\neq i}}^{r} A_j(x) \right\|_2 \left\| \int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left(\left(D_{x_0-;g}^{\alpha} A_i \right) \circ g^{-1} \right)(z)\, dz \right\|_2 dx \right.$$

$$\left. + \int_{x_0}^{b} \left\| \prod_{\substack{j=1 \\ j\neq i}}^{r} A_j(x) \right\|_2 \left\| \int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left(\left(D_{x_0+;g}^{\alpha} A_i \right) \circ g^{-1} \right)(z)\, dz \right\|_2 dx \right] \leq \tag{46}$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[\int_{a}^{x_0} \prod_{\substack{j=1 \\ j\neq i}}^{r} \left\| A_j(x) \right\|_2 \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left\| \left(\left(D_{x_0-;g}^{\alpha} A_i \right) \circ g^{-1} \right)(z) \right\|_2 dz \right) dx \right.$$

$$\left. + \int_{x_0}^{b} \prod_{\substack{j=1 \\ j\neq i}}^{r} \left\| A_j(x) \right\|_2 \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left\| \left(\left(D_{x_0+;g}^{\alpha} A_i \right) \circ g^{-1} \right)(z) \right\|_2 dz \right) dx \right].$$

We have proved, so far, that

$$\left\| \Phi(A_1,\ldots,A_r)(x_0) \right\|_1 \leq \frac{1}{\Gamma(\alpha)}$$

$$\sum_{i=1}^{r} \left[\int_{a}^{x_0} \prod_{\substack{j=1 \\ j\neq i}}^{r} \left\| A_j(x) \right\|_2 \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left\| \left(\left(D_{x_0-;g}^{\alpha} A_i \right) \circ g^{-1} \right)(z) \right\|_2 dz \right) dx + \right.$$

$$\left. \int_{x_0}^{b} \prod_{\substack{j=1 \\ j\neq i}}^{r} \left\| A_j(x) \right\|_2 \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left\| \left(\left(D_{x_0+;g}^{\alpha} A_i \right) \circ g^{-1} \right)(z) \right\|_2 dz \right) dx \right] =: (\xi). \tag{47}$$

Let now $\gamma, \delta > 1$ such that $\frac{1}{\gamma} + \frac{1}{\delta} = 1$, and we apply the usual Hölder's inequality in (47); $\alpha > \frac{1}{\delta}$. Then we have

$$\left\| \Phi(A_1,\ldots,A_r)(x_0) \right\|_1 \leq (\xi) \leq \frac{1}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\sum_{i=1}^{r} \left[\int_{a}^{x_0} \prod_{\substack{j=1 \\ j\neq i}}^{r} \left\| A_j(x) \right\|_2 (g(x_0)-g(x))^{\frac{\gamma(\alpha-1)+1}{\gamma}} \left(\int_{g(x)}^{g(x_0)} \left\| \left(\left(D_{x_0-;g}^{\alpha} A_i \right) \circ g^{-1} \right)(z) \right\|_2^{\delta} dz \right)^{\frac{1}{\delta}} dx + \right.$$

$$\left. \int_{x_0}^{b} \prod_{\substack{j=1 \\ j\neq i}}^{r} \left\| A_j(x) \right\|_2 (g(x)-g(x_0))^{\frac{\gamma(\alpha-1)+1}{\gamma}} \left(\int_{g(x_0)}^{g(x)} \left\| \left(\left(D_{x_0+;g}^{\alpha} A_i \right) \circ g^{-1} \right)(z) \right\|_2^{\delta} dz \right)^{\frac{1}{\delta}} dx \right] \tag{48}$$

$$\leq \frac{1}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_2\,(g(x_0)-g(x))^{\alpha-\frac{1}{\delta}}dx+\right.$$

$$\left.\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_2\,(g(x)-g(x_0))^{\alpha-\frac{1}{\delta}}dx\right], \tag{49}$$

proving (33).

If $\alpha \geq 1$ we obtain

$$\|\Phi(A_1,...,A_r)(x_0)\|_1 \leq (\xi) \leq \frac{1}{\Gamma(\alpha)}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_2\,(g(x_0)-g(x))^{\alpha-1}dx+\right.$$

$$\left.\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_2\,(g(x)-g(x_0))^{\alpha-1}dx\right], \tag{50}$$

proving (34).

At last we derive

$$\|\Phi(A_1,...,A_r)(x_0)\|_1 \leq (\xi) \leq \frac{1}{\Gamma(\alpha+1)}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_2\,(g(x_0)-g(x))^{\alpha}dx+\right.$$

$$\left.\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_2\,(g(x)-g(x_0))^{\alpha}dx\right], \tag{51}$$

proving (35).

The theorem is proved.

When $r = 2$ we obtain the following operator related Ostrowski type fractional inequalities.

Theorem 7. *Let $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, and let the $*$-ideals $\mathscr{B}_p(H)$, $\mathscr{B}_q(H)$, for which $\left(\mathscr{B}_p(H),\|\cdot\|_p\right)$, $\left(\mathscr{B}_q(H),\|\cdot\|_q\right)$ are Banach algebras; $x_0 \in [a,b] \subset \mathbb{R}$, $\alpha > 0$, $n = \lceil\alpha\rceil$; $A_1 \in C^n([a,b],\mathscr{B}_p(H))$, $A_2 \in C^n([a,b],\mathscr{B}_q(H))$; $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a),g(b)])$, with $\left(A_i \circ g^{-1}\right)^{(k)}(g(x_0)) = 0$, $k = 1,...,n-1$; $i = 1,2$. Then*

1) it holds

$$\Phi(A_1,A_2)(x_0) := \int_a^b A_2(x)A_1(x) + \int_a^b A_1(x)A_2(x)\,dx-$$

$$\left(\int_a^b A_2(x)\,dx\right)A_1(x_0) - \left(\int_a^b A_1(x)\,dx\right)A_2(x_0) =$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left[\int_a^{x_0}A_2(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(\left(D_{x_0-;g}^{\alpha}A_1\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]+\right.$$

$$\left[\int_{x_0}^{b}A_2(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D_{x_0+;g}^{\alpha}A_1\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]+$$

$$\left[\int_a^{x_0} A_1(x)\left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1}\left(\left(D_{x_0-;g}^{\alpha}A_2\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]+$$

$$\left[\int_{x_0}^{b} A_1(x)\left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1}\left(\left(D_{x_0+;g}^{\alpha}A_2\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]\Bigg\}, \tag{52}$$

2) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, with $\alpha>\frac{1}{\delta}$, we have that

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq \frac{1}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\Bigg\{\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\|A_2(x)\|_q\,(g(x_0)-g(x))^{\alpha-\frac{1}{\delta}}\,dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_2(x)\|_q\,(g(x)-g(x_0))^{\alpha-\frac{1}{\delta}}\,dx\right]+ \tag{53}$$

$$\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\|A_1(x)\|_p\,(g(x_0)-g(x))^{\alpha-\frac{1}{\delta}}\,dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_1(x)\|_p\,(g(x)-g(x_0))^{\alpha-\frac{1}{\delta}}\,dx\right]\Bigg\},$$

3) if $\alpha\geq 1$, we obtain

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq \frac{1}{\Gamma(\alpha)}$$

$$\Bigg\{\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\|A_2(x)\|_q\,(g(x_0)-g(x))^{\alpha-1}\,dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_2(x)\|_q\,(g(x)-g(x_0))^{\alpha-1}\,dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\|A_1(x)\|_p\,(g(x_0)-g(x))^{\alpha-1}\,dx\right]+ \tag{54}$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_1(x)\|_p\,(g(x)-g(x_0))^{\alpha-1}\,dx\right]\Bigg\},$$

and

4)

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq \frac{1}{\Gamma(\alpha+1)}$$

$$\Bigg\{\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\|A_2(x)\|_q\,(g(x_0)-g(x))^{\alpha}\,dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_2(x)\|_q\,(g(x)-g(x_0))^{\alpha}\,dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\|A_1(x)\|_p\,(g(x_0)-g(x))^{\alpha}\,dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_1(x)\|_p\,(g(x)-g(x_0))^{\alpha}\,dx\right]\Bigg\}. \tag{55}$$

Proof. Here we have that (acting as in the proof of Theorem 6 for $r=2$)

$$\Phi(A_1,A_2)(x_0) := \int_a^b A_2(x)A_1(x) + \int_a^b A_1(x)A_2(x)\,dx -$$

$$\left(\int_a^b A_2(x)\,dx\right)A_1(x_0) - \left(\int_a^b A_1(x)\,dx\right)A_2(x_0) \overset{(32)}{=}$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left[\int_a^{x_0} A_2(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(\left(D^{\alpha}_{x_0-;g}A_1\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]+\right.$$

$$\left[\int_{x_0}^b A_2(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D^{\alpha}_{x_0+;g}A_1\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]+$$

$$\left[\int_a^{x_0} A_1(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(\left(D^{\alpha}_{x_0-;g}A_2\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]+$$

$$\left.\left[\int_{x_0}^b A_1(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D^{\alpha}_{x_0+;g}A_2\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]\right\}. \tag{56}$$

Therefore it holds by taking the 1-Schatten norm that

$$\left\|\Phi(A_1,A_2)(x_0)\right\|_1 = \left\|\int_a^b A_2(x)A_1(x) + \int_a^b A_1(x)A_2(x)\,dx -\right.$$

$$\left.\left(\int_a^b A_2(x)\,dx\right)A_1(x_0) - \left(\int_a^b A_1(x)\,dx\right)A_2(x_0)\right\|_1 \leq$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left[\left\|\int_a^{x_0} A_2(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(\left(D^{\alpha}_{x_0-;g}A_1\right)\circ g^{-1}\right)(z)\,dz\right)dx\right\|_1\right]+\right.$$

$$\left[\left\|\int_{x_0}^b A_2(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D^{\alpha}_{x_0+;g}A_1\right)\circ g^{-1}\right)(z)\,dz\right)dx\right\|_1\right]+$$

$$\left[\left\|\int_a^{x_0} A_1(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(\left(D^{\alpha}_{x_0-;g}A_2\right)\circ g^{-1}\right)(z)\,dz\right)dx\right\|_1\right]+$$

$$\left.\left[\left\|\int_{x_0}^b A_1(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D^{\alpha}_{x_0+;g}A_2\right)\circ g^{-1}\right)(z)\,dz\right)dx\right\|_1\right]\right\} \leq \tag{57}$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left[\int_a^{x_0}\left\|A_2(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(\left(D^{\alpha}_{x_0-;g}A_1\right)\circ g^{-1}\right)(z)\,dz\right)\right\|_1 dx\right]+\right.$$

$$\left[\int_{x_0}^b\left\|A_2(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D^{\alpha}_{x_0+;g}A_1\right)\circ g^{-1}\right)(z)\,dz\right)\right\|_1 dx\right]+$$

$$\left[\int_a^{x_0}\left\|A_1(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(\left(D^{\alpha}_{x_0-;g}A_2\right)\circ g^{-1}\right)(z)\,dz\right)\right\|_1 dx\right]+$$

$$\left.\left[\int_{x_0}^b\left\|A_1(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D^{\alpha}_{x_0+;g}A_2\right)\circ g^{-1}\right)(z)\,dz\right)\right\|_1 dx\right]\right\} \leq \tag{58}$$

(by using the p-Schatten norm and Hölder's type inequality (31) for $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$)

$$\frac{1}{\Gamma(\alpha)}\left\{\left[\int_a^{x_0}\left\|A_2(x)\right\|_q\left\|\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(\left(D^{\alpha}_{x_0-;g}A_1\right)\circ g^{-1}\right)(z)\,dz\right)\right\|_p dx\right]+\right.$$

$$\left[\int_{x_0}^b\left\|A_2(x)\right\|_q\left\|\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D^{\alpha}_{x_0+;g}A_1\right)\circ g^{-1}\right)(z)\,dz\right)\right\|_p dx\right]+$$

$$\left[\int_a^{x_0} \|A_1(x)\|_p \left\| \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left(\left(D^{\alpha}_{x_0-;g} A_2 \right) \circ g^{-1} \right)(z)\, dz \right) \right\|_q dx \right] +$$

$$\left[\int_{x_0}^{b} \|A_1(x)\|_p \left\| \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left(\left(D^{\alpha}_{x_0+;g} A_2 \right) \circ g^{-1} \right)(z)\, dz \right) \right\|_q dx \right] \Bigg\} \leq \tag{59}$$

$$\frac{1}{\Gamma(\alpha)} \Bigg\{ \left[\int_a^{x_0} \|A_2(x)\|_q \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left\| \left(\left(D^{\alpha}_{x_0-;g} A_1 \right) \circ g^{-1} \right)(z) \right\|_p dz \right) dx \right] +$$

$$\left[\int_{x_0}^{b} \|A_2(x)\|_q \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left\| \left(\left(D^{\alpha}_{x_0+;g} A_1 \right) \circ g^{-1} \right)(z) \right\|_p dz \right) dx \right] +$$

$$\left[\int_a^{x_0} \|A_1(x)\|_p \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left\| \left(\left(D^{\alpha}_{x_0-;g} A_2 \right) \circ g^{-1} \right)(z) \right\|_q dz \right) dx \right] +$$

$$\left[\int_{x_0}^{b} \|A_1(x)\|_p \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left\| \left(\left(D^{\alpha}_{x_0+;g} A_2 \right) \circ g^{-1} \right)(z) \right\|_q dz \right) dx \right] \Bigg\}. \tag{60}$$

We have proved, so far, that

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq$$

$$\frac{1}{\Gamma(\alpha)} \Bigg\{ \left[\int_a^{x_0} \|A_2(x)\|_q \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left\| \left(\left(D^{\alpha}_{x_0-;g} A_1 \right) \circ g^{-1} \right)(z) \right\|_p dz \right) dx \right] +$$

$$\left[\int_{x_0}^{b} \|A_2(x)\|_q \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left\| \left(\left(D^{\alpha}_{x_0+;g} A_1 \right) \circ g^{-1} \right)(z) \right\|_p dz \right) dx \right] +$$

$$\left[\int_a^{x_0} \|A_1(x)\|_p \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left\| \left(\left(D^{\alpha}_{x_0-;g} A_2 \right) \circ g^{-1} \right)(z) \right\|_q dz \right) dx \right] +$$

$$\left[\int_{x_0}^{b} \|A_1(x)\|_p \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left\| \left(\left(D^{\alpha}_{x_0+;g} A_2 \right) \circ g^{-1} \right)(z) \right\|_q dz \right) dx \right] \Bigg\} =: (\lambda). \tag{61}$$

Let now $\gamma, \delta > 1$ such that $\frac{1}{\gamma} + \frac{1}{\delta} = 1$, and we apply the usual Hölder's inequality in (61); $\alpha > \frac{1}{\delta}$. Then we have that

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq (\lambda) \leq \frac{1}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\Bigg\{ \left[\int_a^{x_0} \|A_2(x)\|_q (g(x_0)-g(x))^{\frac{\gamma(\alpha-1)+1}{\gamma}} \left(\int_{g(x)}^{g(x_0)} \left\| \left(\left(D^{\alpha}_{x_0-;g} A_1 \right) \circ g^{-1} \right)(z) \right\|_p^{\delta} dz \right)^{\frac{1}{\delta}} dx \right] +$$

$$\left[\int_{x_0}^{b} \|A_2(x)\|_q (g(x)-g(x_0))^{\frac{\gamma(\alpha-1)+1}{\gamma}} \left(\int_{g(x_0)}^{g(x)} \left\| \left(\left(D^{\alpha}_{x_0+;g} A_1 \right) \circ g^{-1} \right)(z) \right\|_p^{\delta} dz \right)^{\frac{1}{\delta}} dx \right] +$$

$$\left[\int_a^{x_0} \|A_1(x)\|_p (g(x_0)-g(x))^{\frac{\gamma(\alpha-1)+1}{\gamma}} \left(\int_{g(x)}^{g(x_0)} \left\| \left(\left(D^{\alpha}_{x_0-;g} A_2 \right) \circ g^{-1} \right)(z) \right\|_q^{\delta} dz \right)^{\frac{1}{\delta}} dx \right] + \tag{62}$$

$$\left[\int_{x_0}^{b} \|A_1(x)\|_p (g(x)-g(x_0))^{\frac{\gamma(\alpha-1)+1}{\gamma}} \left(\int_{g(x_0)}^{g(x)} \left\| \left(\left(D^{\alpha}_{x_0+;g} A_2 \right) \circ g^{-1} \right)(z) \right\|_q^{\delta} dz \right)^{\frac{1}{\delta}} dx \right] \Bigg\}$$

$$\leq \frac{1}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\Bigg\{ \left[\left\| \left\| \left(D^{\alpha}_{x_0-;g} A_1 \right) \circ g^{-1} \right\|_p \right\|_{\delta,[g(a),g(x_0)]} \int_a^{x_0} \|A_2(x)\|_q (g(x_0)-g(x))^{\alpha-\frac{1}{\delta}} dx \right] +$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_2(x)\|_q(g(x)-g(x_0))^{\alpha-\frac{1}{\delta}}dx\right]+ \tag{63}$$

$$\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(a),g(x_0)]}\int_{a}^{x_0}\|A_1(x)\|_p(g(x_0)-g(x))^{\alpha-\frac{1}{\delta}}dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_1(x)\|_p(g(x)-g(x_0))^{\alpha-\frac{1}{\delta}}dx\right]\Bigg\},$$

proving (53).

If $\alpha\geq 1$, we obtain

$$\|\Phi(A_1,A_2)(x_0)\|_1\leq(\lambda)\leq\frac{1}{\Gamma(\alpha)}$$

$$\Bigg\{\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{L_1([g(a),g(x_0)])}\int_{a}^{x_0}\|A_2(x)\|_q(g(x_0)-g(x))^{\alpha-1}dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_2(x)\|_q(g(x)-g(x_0))^{\alpha-1}dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{L_1([g(a),g(x_0)])}\int_{a}^{x_0}\|A_1(x)\|_p(g(x_0)-g(x))^{\alpha-1}dx\right]+ \tag{64}$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_1(x)\|_p(g(x)-g(x_0))^{\alpha-1}dx\right]\Bigg\},$$

proving (54).

At last we derive

$$\|\Phi(A_1,A_2)(x_0)\|_1\leq(\lambda)\leq\frac{1}{\Gamma(\alpha+1)}$$

$$\Bigg\{\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\|A_2(x)\|_q(g(x_0)-g(x))^{\alpha}dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_1\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_2(x)\|_q(g(x)-g(x_0))^{\alpha}dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\|A_1(x)\|_p(g(x_0)-g(x))^{\alpha}dx\right]+$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_1(x)\|_p(g(x)-g(x_0))^{\alpha}dx\right]\Bigg\}, \tag{65}$$

proving (55).

The theorem is proved.

We continue with p-Schatten norm weighted mixed generalized fractional Ostrowski type inequalities involving several functions taking values in the Banach algebra $\mathscr{B}_p(H)\subset\mathscr{B}(H)$, $p>1$.

Theorem 8. *Let the $*$-ideal $\mathscr{B}_p(H)$, which $\left(\mathscr{B}_p(H),\|\cdot\|_p\right)$, $p>1$, is a Banach algebra; $x_0\in[a,b]\subset\mathbb{R}$, $\alpha>0$, $n=\lceil\alpha\rceil$; $A_i\in C^n([a,b],\mathscr{B}_p(H))$, $i=1,...,r\in\mathbb{N}-\{1\}$; $g\in C^1([a,b])$, strictly increasing such that $g^{-1}\in C^n([g(a),g(b)])$, with $\left(A_i\circ g^{-1}\right)^{(k)}(g(x_0))=0$, $k=1,...,n-1$; $i=1,...,r$. Let $\Phi(A_1,...,A_r)(x_0)$ be as in (32).*

Then

1) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, with $\alpha>\frac{1}{\delta}$ we have that

$$\|\Phi(A_1,...,A_r)(x_0)\|_p\leq\frac{1}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_p(g(x_0)-g(x))^{\alpha-\frac{1}{\delta}}dx+\right.$$

$$\left.\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_p(g(x)-g(x_0))^{\alpha-\frac{1}{\delta}}dx\right], \tag{66}$$

2) if $\alpha\geq 1$, we obtain

$$\left\|\Phi(A_1,...,A_r)(x_0)\right\|_p\leq\frac{1}{\Gamma(\alpha)}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_p\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_p(g(x_0)-g(x))^{\alpha-1}dx+\right.$$

$$\left.\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_p\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_p(g(x)-g(x_0))^{\alpha-1}dx\right], \tag{67}$$

and

3)

$$\left\|\Phi(A_1,...,A_r)(x_0)\right\|_p\leq\frac{1}{\Gamma(\alpha+1)}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_p(g(x_0)-g(x))^{\alpha}dx+\right.$$

$$\left.\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_p(g(x)-g(x_0))^{\alpha}dx\right]. \tag{68}$$

Proof. As similar to the proof of Theorem 6 is omitted. Use of (28).

We make

Remark. (to Theorem 6)

i) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, with $\alpha>\frac{1}{\delta}$; case of inequality (33):

Call and assume

$$M_1(f_1,...,f_r):= \tag{69}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|\left(D_{x_0-;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{\delta,[g(a),g(x_0)]},\ \sup_{x_0\in[a,b]}\left\|\left\|\left(D_{x_0+;g}^{\alpha}A_i\right)\circ g^{-1}\right\|_2\right\|_{\delta,[g(x_0),g(b)]}\right\}<+\infty.$$

Then

$$\left\|\Phi(A_1,...,A_r)(x_0)\right\|_1\leq\text{Right hand side (33)}\leq$$

$$\frac{M_1(f_1,...,f_r)(g(b)-g(a))^{\alpha-\frac{1}{\delta}}}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}\sum_{i=1}^{r}\left(\int_a^b\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_2dx\right). \tag{70}$$

ii) Case of inequality (35):

Call and assume

$$M_2(f_1,...,f_r):= \tag{71}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0-;g}A_i\right)\circ g^{-1}\right\|_2\right\|_{\infty,[g(a),g(x_0)]},\ \sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0+;g}A_i\right)\circ g^{-1}\right\|_2\right\|_{\infty,[g(x_0),g(b)]}\right\}<+\infty.$$

Then

$$\left\|\Phi\left(A_1,...,A_r\right)(x_0)\right\|_1\leq \text{Right hand side (35)}\ \leq$$

$$\frac{M_2\left(f_1,...,f_r\right)\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\sum_{i=1}^{r}\left(\int_a^b\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j\left(x\right)\right\|_2dx\right). \tag{72}$$

We make

Remark.(to Theorem 7)
i) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, with $\alpha>\frac{1}{\delta}$; case of inequality (53):
Call and assume

$$N_1\left(f_1,f_2\right):=$$

$$\max\left\{\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0-;g}A_1\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(a),g(x_0)]},\ \sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0+;g}A_1\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(x_0),g(b)]},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0-;g}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(a),g(x_0)]},\ \sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0+;g}A_2\right)\circ g^{-1}\right\|_q\right\|_{\delta,[g(x_0),g(b)]}\right\}<+\infty. \tag{73}$$

Then

$$\left\|\Phi\left(A_1,A_2\right)(x_0)\right\|_1\leq \text{right hand side (53)}\ \leq$$

$$\frac{N_1\left(f_1,f_2\right)\left(g\left(b\right)-g\left(a\right)\right)^{\alpha-\frac{1}{\delta}}}{\Gamma\left(\alpha\right)\left(\gamma\left(\alpha-1\right)+1\right)^{\frac{1}{\gamma}}}\left[\int_a^b\left\|A_1\left(x\right)\right\|_pdx+\int_a^b\left\|A_2\left(x\right)\right\|_qdx\right]. \tag{74}$$

ii) Case of inequality (55):
Call and assume

$$N_2\left(f_1,f_2\right):=$$

$$\max\left\{\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0-;g}A_1\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(a),g(x_0)]},\ \sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0+;g}A_1\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(x_0),g(b)]},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0-;g}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(a),g(x_0)]},\ \sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0+;g}A_2\right)\circ g^{-1}\right\|_q\right\|_{\infty,[g(x_0),g(b)]}\right\}<+\infty. \tag{75}$$

Then

$$\left\|\Phi\left(A_1,A_2\right)(x_0)\right\|_1\leq \text{right hand side (55)}\ \leq$$

$$\frac{N_2\left(f_1,f_2\right)\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\left[\int_a^b\left\|A_1\left(x\right)\right\|_pdx+\int_a^b\left\|A_2\left(x\right)\right\|_qdx\right]. \tag{76}$$

We also make

Remark.(to Theorem 8)
i) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, with $\alpha>\frac{1}{\delta}$; case of inequality (66):
Call and assume

$$R_1\left(f_1,...,f_r\right):= \tag{77}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0-;g}A_i\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(a),g(x_0)]},\ \sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0+;g}A_i\right)\circ g^{-1}\right\|_p\right\|_{\delta,[g(x_0),g(b)]}\right\}<+\infty.$$

Then ($p>1$)

$$\left\|\Phi\left(A_1,...,A_r\right)(x_0)\right\|_p\leq \text{right hand side (66)}\ \leq$$

$$\frac{R_1(f_1,...,f_r)(g(b)-g(a))^{\alpha-\frac{1}{\delta}}}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}\sum_{i=1}^{r}\left(\int_a^b\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_p\,dx\right). \tag{78}$$

ii) Case of inequality (68):
Call and assume

$$R_2(f_1,...,f_r):= \tag{79}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0-;g}A_i\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(a),g(x_0)]},\ \sup_{x_0\in[a,b]}\left\|\left\|\left(D^{\alpha}_{x_0+;g}A_i\right)\circ g^{-1}\right\|_p\right\|_{\infty,[g(x_0),g(b)]}\right\}<+\infty.$$

Then $(p>1)$

$$\|\Phi(A_1,...,A_r)(x_0)\|_p\le \text{Right hand side (68)}\le$$

$$\frac{R_2(f_1,...,f_r)(g(b)-g(a))^{\alpha}}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\left(\int_a^b\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_p\,dx\right). \tag{80}$$

Next come 1-Schatten norm generalized fractional Grüss type inequalities involving several functions taking values in the Banach algebra $\mathscr{B}_2(H)\subset\mathscr{B}(H)$:

Theorem 9. *Let the $*$-ideal $\mathscr{B}_2(H)$, which $(\mathscr{B}_2(H),\|\cdot\|_2)$ is a Banach algebra; $0<\alpha<1$, and $A_i\in C^1([a,b],\mathscr{B}_2(H))$, $i=1,...,r\in\mathbb{N}-\{1\}$; $g\in C^1([a,b])$, strictly increasing such that $g^{-1}\in C^1([g(a),g(b)])$. Here $M_1(f_1,...,f_r)$ is as in (69), and $M_2(f_1,...,f_r)$ is as in (71). Denote by*

$$\Delta(A_1,...,A_r):=\int_a^b\Phi(A_1,...,A_r)(x_0)\,dx_0=$$

$$\sum_{i=1}^{r}\left[(b-a)\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)A_i(x)\,dx-\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}A_j(x)\right)dx\right)\left(\int_a^b A_i(x)\,dx\right)\right], \tag{81}$$

where $\Phi(A_1,...,A_r)(x_0)$ as in (32).
Then
(i) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, with $\alpha>\frac{1}{\delta}$ we have that

$$\|\Delta(A_1,...,A_r)\|_1\le\frac{M_1(f_1,...,f_r)(g(b)-g(a))^{\alpha-\frac{1}{\delta}}(b-a)}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\sum_{i=1}^{r}\left(\int_a^b\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\,dx\right), \tag{82}$$

and
(ii)

$$\|\Delta(A_1,...,A_r)\|_1\le\frac{M_2(f_1,...,f_r)(g(b)-g(a))^{\alpha}(b-a)}{\Gamma(\alpha+1)}$$

$$\sum_{i=1}^{r}\left(\int_a^b\prod_{\substack{j=1\\j\neq i}}^{r}\|A_j(x)\|_2\,dx\right). \tag{83}$$

Proof.(i) By (81) we have that

$$\left\|\Delta\left(A_1,\ldots,A_r\right)\right\|_1 \leq \int_a^b \left\|\Phi\left(A_1,\ldots,A_r\right)(x)\right\|_1 dx \overset{(70)}{\leq}$$

$$\frac{M_1\left(f_1,\ldots,f_r\right)\left(g(b)-g(a)\right)^{\alpha-\frac{1}{\delta}}(b-a)}{\Gamma(\alpha)\left(\gamma(\alpha-1)+1\right)^{\frac{1}{\gamma}}}\sum_{i=1}^{r}\left(\int_a^b \prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_2 dx\right), \tag{84}$$

proving (82).

(ii) Also it holds

$$\left\|\Delta\left(A_1,\ldots,A_r\right)\right\|_1 \leq \int_a^b \left\|\Phi\left(A_1,\ldots,A_r\right)(x)\right\|_1 dx \overset{(72)}{\leq}$$

$$\frac{M_2\left(f_1,\ldots,f_r\right)\left(g(b)-g(a)\right)^{\alpha}(b-a)}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\left(\int_a^b \prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_2 dx\right), \tag{85}$$

proving (83).

The theorem is proved.

When $r=2$ we obtain the following operator related Grüss type fractional inequalities.

Theorem 10. *Let $p,q>1: \frac{1}{p}+\frac{1}{q}=1$, and let the $*$-ideals $\mathscr{B}_p(H)$, $\mathscr{B}_q(H)$, for which $\left(\mathscr{B}_p(H),\|\cdot\|_p\right)$, $\left(\mathscr{B}_q(H),\|\cdot\|_q\right)$ are Banach algebras; $0<\alpha<1$, $A_1\in C^1\left([a,b],\mathscr{B}_p(H)\right)$, $A_2\in C^1\left([a,b],\mathscr{B}_q(H)\right)$; $g\in C^1([a,b])$, strictly increasing, such that $g^{-1}\in C^1\left([g(a),g(b)]\right)$. Here $N_1(f_1,f_2)$ is as in (73), and $N_2(f_1,f_2)$ is as in (75).*

Denote by

$$\Delta\left(A_1,A_2\right) := \int_a^b \Phi\left(A_1,A_2\right)(x_0)\,dx_0 =$$

$$(b-a)\left(\int_a^b A_2(x)A_1(x)\,dx+\int_a^b A_1(x)A_2(x)\,dx\right)- \tag{86}$$

$$\left(\int_a^b A_2(x)\,dx\right)\left(\int_a^b A_1(x)\,dx\right)-\left(\int_a^b A_1(x)\,dx\right)\left(\int_a^b A_2(x)\,dx\right),$$

where $\Phi\left(A_1,A_2\right)(x_0)$ as in (52).

Then

(i) for $\gamma,\delta>1: \frac{1}{\gamma}+\frac{1}{\delta}=1$, with $\alpha>\frac{1}{\delta}$, we have that

$$\left\|\Delta\left(A_1,A_2\right)\right\|_1 \leq \frac{N_1\left(f_1,f_2\right)\left(g(b)-g(a)\right)^{\alpha-\frac{1}{\delta}}(b-a)}{\Gamma(\alpha)\left(\gamma(\alpha-1)+1\right)^{\frac{1}{\gamma}}}$$

$$\left[\int_a^b \left\|A_1(x)\right\|_p dx+\int_a^b \left\|A_2(x)\right\|_q dx\right], \tag{87}$$

and

(ii)

$$\left\|\Delta\left(A_1,A_2\right)\right\|_1 \leq \frac{N_2\left(f_1,f_2\right)\left(g(b)-g(a)\right)^{\alpha}(b-a)}{\Gamma(\alpha+1)}$$

$$\left[\int_a^b \left\|A_1(x)\right\|_p dx+\int_a^b \left\|A_2(x)\right\|_q dx\right]. \tag{88}$$

Proof.(i) By (86) we have that

$$\|\Delta(A_1,A_2)\|_1 \leq \int_a^b \|\Phi(A_1,A_2)(x)\|_1 dx \overset{(74)}{\leq}$$

$$\frac{N_1(f_1,f_2)(g(b)-g(a))^{\alpha-\frac{1}{\delta}}(b-a)}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}\left[\int_a^b \|A_1(x)\|_p dx + \int_a^b \|A_2(x)\|_q dx\right], \quad (89)$$

proving (87).

(ii) Also we get

$$\|\Delta(A_1,A_2)\|_1 \leq \int_a^b \|\Phi(A_1,A_2)(x)\|_1 dx \overset{(76)}{\leq}$$

$$\frac{N_2(f_1,f_2)(g(b)-g(a))^{\alpha}(b-a)}{\Gamma(\alpha+1)}\left[\int_a^b \|A_1(x)\|_p dx + \int_a^b \|A_2(x)\|_q dx\right], \quad (90)$$

proving (88). The theorem is proved.

We continue with p-Schatten norm generalized fractional Grüss type inequalities involving several functions takin values in the Banach algebra $\mathscr{B}_p(H) \subset \mathscr{B}(H)$, $p>1$.

Theorem 11.*Let the $*$-ideal $\mathscr{B}_p(H)$, which $\left(\mathscr{B}_p(H), \|\cdot\|_p\right)$, $p>1$, is a Banach algebra, $0<\alpha<1$. Here $A_i \in C^1([a,b], \mathscr{B}_p(H))$, $i=1,\dots,r \in \mathbb{N}-\{1\}$; $g \in C^1([a,b])$, strictly increasing such that $g^{-1} \in C^1([g(a),g(b)])$. Here $R_1(f_1,\dots,f_r)$ is as in (77), and $R_2(f_1,\dots,f_r)$ is as in (79). Furthermore $\Delta(A_1,\dots,A_r)$ is as in (81) with $\Phi(A_1,\dots,A_r)(x_0)$ as in (32). Then*

(i) for $\gamma,\delta>1: \frac{1}{\gamma}+\frac{1}{\delta}=1$, with $\alpha>\frac{1}{\delta}$ we have that

$$\|\Delta(A_1,\dots,A_r)\|_p \leq \frac{R_1(f_1,\dots,f_r)(g(b)-g(a))^{\alpha-\frac{1}{\delta}}(b-a)}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\sum_{i=1}^{r}\left(\int_a^b \prod_{\substack{j=1\\ j\neq i}}^{r} \|A_j(x)\|_p dx\right), \quad (91)$$

and

(ii)

$$\|\Delta(A_1,\dots,A_r)\|_p \leq \frac{R_2(f_1,\dots,f_r)(g(b)-g(a))^{\alpha}(b-a)}{\Gamma(\alpha+1)}$$

$$\sum_{i=1}^{r}\left(\int_a^b \prod_{\substack{j=1\\ j\neq i}}^{r} \|A_j(x)\|_p dx\right). \quad (92)$$

Proof.(i) By (81) we have

$$\|\Delta(A_1,\dots,A_r)\|_p \leq \int_a^b \|\Phi(A_1,\dots,A_r)(x)\|_p dx \overset{(78)}{\leq}$$

$$\frac{R_1(f_1,\dots,f_r)(g(b)-g(a))^{\alpha-\frac{1}{\delta}}(b-a)}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}\sum_{i=1}^{r}\left(\int_a^b \prod_{\substack{j=1\\ j\neq i}}^{r} \|A_j(x)\|_p dx\right), \quad (93)$$

proving (91).

(ii) Also we have

$$\|\Delta(A_1,\dots,A_r)\|_p \leq \int_a^b \|\Phi(A_1,\dots,A_r)(x)\|_p dx \overset{(80)}{\leq}$$

$$\frac{R_2(f_1,...,f_r)(g(b)-g(a))^{\alpha}(b-a)}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\left(\int_a^b\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_p\,dx\right), \tag{94}$$

proving (92). The theorem is proved.

6 Applications

We start with special Ostrowski type inequalities. We give

Corollary 1.*(to Theorem 6) All as in Theorem 6, with $g(t)=t$, $\forall\, t\in[a,b]$. Then*
1) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, $\alpha>\frac{1}{\delta}$ we have that

$$\|\Phi(A_1,...,A_r)(x_0)\|_1\leq\frac{1}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\sum_{i=1}^{r}\left[\left\|\|(D_{x_0-}^{\alpha}A_i)\|_2\right\|_{\delta,[a,x_0]}\int_a^{x_0}\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_2(x_0-x)^{\alpha-\frac{1}{\delta}}dx+\right. \tag{95}$$

$$\left.\left\|\|(D_{*x_0}^{\alpha}A_i)\|_2\right\|_{\delta,[x_0,b]}\int_{x_0}^{b}\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_2(x-x_0)^{\alpha-\frac{1}{\delta}}dx\right],$$

2) if $\alpha\geq1$, we obtain

$$\|\Phi(A_1,...,A_r)(x_0)\|_1\leq\frac{1}{\Gamma(\alpha)}$$

$$\sum_{i=1}^{r}\left[\left\|\|(D_{x_0-}^{\alpha}A_i)\|_2\right\|_{L_1([a,x_0])}\int_a^{x_0}\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_2(x_0-x)^{\alpha-1}dx+\right.$$

$$\left.\left\|\|(D_{*x_0}^{\alpha}A_i)\|_2\right\|_{L_1([x_0,b])}\int_{x_0}^{b}\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_2(x-X_0)^{\alpha-1}dx\right], \tag{96}$$

and
3)

$$\|\Phi(A_1,...,A_r)(x_0)\|_1\leq\frac{1}{\Gamma(\alpha+1)}$$

$$\sum_{i=1}^{r}\left[\left\|\|(D_{x_0-}^{\alpha}A_i)\|_2\right\|_{\infty,[a,x_0]}\int_a^{x_0}\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_2(x_0-x)^{\alpha}dx+\right.$$

$$\left.\left\|\|(D_{*x_0}^{\alpha}A_i)\|_2\right\|_{\infty,[x_0,b]}\int_{x_0}^{b}\prod_{\substack{j=1\\ j\neq i}}^{r}\|A_j(x)\|_2(x-X_0)^{\alpha}dx\right]. \tag{97}$$

We present

Corollary 2.(*to Theorem 7*) *All as in Theorem 7, with* $g(t) = e^t$, $\forall\, t \in [a,b]$. *Then*

1) for $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, *with* $\alpha > \frac{1}{\delta}$, *we have that*

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq \frac{1}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\left\{ \left[\left\| \left\| \left(D^{\alpha}_{x_0-;e^t} A_1 \right) \circ \log \right\|_p \right\|_{\delta,[e^a,e^{x_0}]} \int_a^{x_0} \|A_2(x)\|_q \left(e^{x_0} - e^x\right)^{\alpha - \frac{1}{\delta}} dx \right] + \right.$$

$$\left[\left\| \left\| \left(D^{\alpha}_{x_0+;e^t} A_1 \right) \circ \log \right\|_p \right\|_{\delta,\left[e^{x_0},e^b\right]} \int_{x_0}^b \|A_2(x)\|_q \left(e^x - e^{x_0}\right)^{\alpha - \frac{1}{\delta}} dx \right] + \tag{98}$$

$$\left[\left\| \left\| \left(D^{\alpha}_{x_0-;e^t} A_2 \right) \circ \log \right\|_q \right\|_{\delta,[e^a,e^{x_0}]} \int_a^{x_0} \|A_1(x)\|_p \left(e^{x_0} - e^x\right)^{\alpha - \frac{1}{\delta}} dx \right] +$$

$$\left. \left[\left\| \left\| \left(D^{\alpha}_{x_0+;e^t} A_2 \right) \circ \log \right\|_q \right\|_{\delta,\left[e^{x_0},e^b\right]} \int_{x_0}^b \|A_1(x)\|_p \left(e^x - e^{x_0}\right)^{\alpha - \frac{1}{\delta}} dx \right] \right\},$$

2) if $\alpha \geq 1$, *we obtain*

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq \frac{1}{\Gamma(\alpha)}$$

$$\left\{ \left[\left\| \left\| \left(D^{\alpha}_{x_0-;e^t} A_1 \right) \circ \log \right\|_p \right\|_{L_1([e^a,e^{x_0}])} \int_a^{x_0} \|A_2(x)\|_q \left(e^{x_0} - e^x\right)^{\alpha - 1} dx \right] + \right.$$

$$\left[\left\| \left\| \left(D^{\alpha}_{x_0+;e^t} A_1 \right) \circ \log \right\|_p \right\|_{L_1\left(\left[e^{x_0},e^b\right]\right)} \int_{x_0}^b \|A_2(x)\|_q \left(e^x - e^{x_0}\right)^{\alpha - 1} dx \right] +$$

$$\left[\left\| \left\| \left(D^{\alpha}_{x_0-;e^t} A_2 \right) \circ \log \right\|_q \right\|_{L_1([e^a,e^{x_0}])} \int_a^{x_0} \|A_1(x)\|_p \left(e^{x_0} - e^x\right)^{\alpha - 1} dx \right] + \tag{99}$$

$$\left. \left[\left\| \left\| \left(D^{\alpha}_{x_0+;e^t} A_2 \right) \circ \log \right\|_q \right\|_{L_1\left(\left[e^{x_0},e^b\right]\right)} \int_{x_0}^b \|A_1(x)\|_p \left(e^x - e^{x_0}\right)^{\alpha - 1} dx \right] \right\},$$

and

3)

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq \frac{1}{\Gamma(\alpha+1)}$$

$$\left\{ \left[\left\| \left\| \left(D^{\alpha}_{x_0-;e^t} A_1 \right) \circ \log \right\|_p \right\|_{\infty,[e^a,e^{x_0}]} \int_a^{x_0} \|A_2(x)\|_q \left(e^{x_0} - e^x\right)^{\alpha} dx \right] + \right.$$

$$\left[\left\| \left\| \left(D^{\alpha}_{x_0+;e^t} A_1 \right) \circ \log \right\|_p \right\|_{\infty,\left[e^{x_0},e^b\right]} \int_{x_0}^b \|A_2(x)\|_q \left(e^x - e^{x_0}\right)^{\alpha} dx \right] +$$

$$\left[\left\| \left\| \left(D^{\alpha}_{x_0-;e^t} A_2 \right) \circ \log \right\|_q \right\|_{\infty,[e^a,e^{x_0}]} \int_a^{x_0} \|A_1(x)\|_p \left(e^{x_0} - e^x\right)^{\alpha} dx \right] +$$

$$\left. \left[\left\| \left\| \left(D^{\alpha}_{x_0+;e^t} A_2 \right) \circ \log \right\|_q \right\|_{\infty,\left[e^{x_0},e^b\right]} \int_{x_0}^b \|A_1(x)\|_p \left(e^x - e^{x_0}\right)^{\alpha} dx \right] \right\}. \tag{100}$$

We continue with

Corollary 3.*(to Theorem 8) All as in Theorem 8, eith* $g(t)=\log t$, $\forall\, t\in[a,b]\subset\mathbb{R}_{+}-\{0\}$. *Then*

1) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, *with* $\alpha>\frac{1}{\delta}$ *we have that*

$$\left\|\Phi\left(A_1,\ldots,A_r\right)(x_0)\right\|_p\le\frac{1}{\Gamma(\alpha)\left(\gamma(\alpha-1)+1\right)^{\frac{1}{\gamma}}}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D^{\alpha}_{x_0-;\log}A_i\right)\circ e^t\right\|_p\right\|_{\delta,[\log a,\log x_0]}\int_a^{x_0}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_p\left(\log\frac{x_0}{x}\right)^{\alpha-\frac{1}{\delta}}dx+\right.$$

$$\left.\left\|\left\|\left(D^{\alpha}_{x_0+;\log}A_i\right)\circ e^t\right\|_p\right\|_{\delta,[\log x_0,\log b]}\int_{x_0}^{b}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_p\left(\log\frac{x}{x_0}\right)^{\alpha-\frac{1}{\delta}}dx\right],\tag{101}$$

2) if $\alpha\ge1$, *we obtain*

$$\left\|\Phi\left(A_1,\ldots,A_r\right)(x_0)\right\|_p\le\frac{1}{\Gamma(\alpha)}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D^{\alpha}_{x_0-;\log}A_i\right)\circ e^t\right\|_p\right\|_{L_1([\log a,\log x_0])}\int_a^{x_0}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_p\left(\log\frac{x_0}{x}\right)^{\alpha-1}dx+\right.$$

$$\left.\left\|\left\|\left(D^{\alpha}_{x_0+;\log}A_i\right)\circ e^t\right\|_p\right\|_{L_1([\log x_0,\log b])}\int_{x_0}^{b}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_p\left(\log\frac{x}{x_0}\right)^{\alpha-1}dx\right],\tag{102}$$

and

3)

$$\left\|\Phi\left(A_1,\ldots,A_r\right)(x_0)\right\|_p\le\frac{1}{\Gamma(\alpha+1)}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D^{\alpha}_{x_0-;\log}A_i\right)\circ e^t\right\|_p\right\|_{\infty,[\log a,\log x_0]}\int_a^{x_0}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_p\left(\log\frac{x_0}{x}\right)^{\alpha}dx+\right.$$

$$\left.\left\|\left\|\left(D^{\alpha}_{x_0+;\log}A_i\right)\circ e^t\right\|_p\right\|_{\infty,[\log x_0,\log b]}\int_{x_0}^{b}\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_p\left(\log\frac{x}{x_0}\right)^{\alpha}dx\right].\tag{103}$$

We continue with special Grüss type inequalities.

Corollary 4.*(to Theorem 9) All as in Theorem 9, with* $g(t)=t$, $\forall\, t\in[a,b]$. *Then*

(i) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, $\alpha>\frac{1}{\delta}$ *we have that*

$$\left\|\Delta\left(A_1,\ldots,A_r\right)\right\|_1\le\frac{M_1\left(f_1,\ldots,f_r\right)(b-a)^{\alpha+\frac{1}{\gamma}}}{\Gamma(\alpha)\left(\gamma(\alpha-1)+1\right)^{\frac{1}{\gamma}}}\sum_{i=1}^{r}\left(\int_a^b\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_2dx\right),\tag{104}$$

and

(ii)

$$\left\|\Delta\left(A_1,\ldots,A_r\right)\right\|_1\le\frac{M_2\left(f_1,\ldots,f_r\right)(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\left(\int_a^b\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|A_j(x)\right\|_2dx\right).\tag{105}$$

Next comes

Corollary 5.(*to Theorem 10*) *All as in Theorem 10, with $g(t) = e^t$, $\forall\, t \in [a,b]$. Then*

(i) for $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, $\alpha > \frac{1}{\delta}$, we have that

$$\|\Delta(A_1, A_2)\|_1 \leq \frac{N_1(f_1, f_2)\left(e^b - e^a\right)^{\alpha - \frac{1}{\delta}}(b-a)}{\Gamma(\alpha)\left(\gamma(\alpha-1)+1\right)^{\frac{1}{\gamma}}}$$

$$\left[\int_a^b \|A_1(x)\|_p\, dx + \int_a^b \|A_2(x)\|_q\, dx\right], \tag{106}$$

and

(ii)

$$\|\Delta(A_1, A_2)\|_1 \leq \frac{N_2(f_1, f_2)\left(e^b - e^a\right)^{\alpha}(b-a)}{\Gamma(\alpha+1)}$$

$$\left[\int_a^b \|A_1(x)\|_p\, dx + \int_a^b \|A_2(x)\|_q\, dx\right]. \tag{107}$$

We finish with

Corollary 6.(*to Theorem 11*) *All as in Theorem 11, with $g(t) = \log t$, $\forall\, t \in [a,b] \subset \mathbb{R}_+ - \{0\}$. Then*

(i) for $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, $\alpha > \frac{1}{\delta}$ we have that

$$\|\Delta(A_1, \ldots, A_r)\|_p \leq \frac{R_1(f_1, \ldots, f_r)\left(\log \frac{b}{a}\right)^{\alpha - \frac{1}{\delta}}(b-a)}{\Gamma(\alpha)\left(\gamma(\alpha-1)+1\right)^{\frac{1}{\gamma}}}$$

$$\sum_{i=1}^{r}\left(\int_a^b \prod_{\substack{j=1\\ j\neq i}}^{r} \|A_j(x)\|_p\, dx\right), \tag{108}$$

and

(ii)

$$\|\Delta(A_1, \ldots, A_r)\|_p \leq \frac{R_2(f_1, \ldots, f_r)\left(\log \frac{b}{a}\right)^{\alpha}(b-a)}{\Gamma(\alpha+1)}$$

$$\sum_{i=1}^{r}\left(\int_a^b \prod_{\substack{j=1\\ j\neq i}}^{r} \|A_j(x)\|_p\, dx\right). \tag{109}$$

References

[1] A. Ostrowski, Über die absolutabweichung einer differentiabaren funcktion von ihrem integralmittelwert, *Comment. Math. Helv.* **10**, 226-227 (1938).

[2] P. Čebyšev, Sur les expressions approximatives des intégrales définies par les aures prises entre les mêmes limites, *Proc. Math. Soc. Charkov* **2**, 93-98 (1882).

[3] G. A. Anastassiou, *Intelligent comparisons: analytic inequalities*, Springer, Heidelberg, New York, 2016.

[4] G. A. Anastassiou, *Fractional differentiation inequalities*, Research Monograph, Springer, New York, 2009.

[5] G. A. Anastassiou, *Advances on fractional inequalities*, Research Monograph, Springer, New York, 2011.

[6] S. S. Dragomir, p-Schatten norm inequalities of Ostrowski's type, *RGMIA Res. Rep. Coll.* **24**, Art. 108, 19 pp. (2021).

[7] S. S. Dragomir, *p-Schatten norm inequalities of Grüss type*, *RGMIA Res. Rep. Coll.* **24**, Art. 115, 16 pp. (2021).

[8] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.

[9] G. A. Anastassiou, *Intelligent computations: abstract fractional calculus, inequalities, approximations*, Springer, Heidelberg, New York, 2018.

[10] C. Aliprantis and K. Border, *Infinite dimensional analysis*, Third edition, Springer, New York, 2006.

[11] W. Rudin, *Functional analysis*, Second Edition, McGraw-Hill, Inc., New York, 1991.
[12] V. A. Zagrebvov, *Gibbs semigroups, operator theory: advances and applications*, Birkhauser, 2019.
[13] B. Simon, *Trace ideals and their applications*, Cambridge University Press, Cambridge, 1979.
[14] D. Chang, A matrix trace inequality for products of Hermitian matrices, *J. Math. Anal. Appl.* **237**, 721-725 (1999).
[15] I. D. Coop, On matrix trace inequalities and related topics for products of Hermitian matrix, *J. Math. Anal. Appl.* **188**, 999-1001 (1994).
[16] H. Neudecker, A matrix trace inequality, *J. Math. Anal. Appl.* **166**, 302-303 (1992).
[17] R. Bellman, *Some inequalities for positive definite matrices*, in E. F. Beckenbach (Ed.), General Inequalities 2, Proceedings of the 2nd International Conference on General Inequalities, Birkhauser, Basel, 1980.