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Symmetry Groups of a Class of Spherical Folding Tilings

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We classify the group of symmetries of all dihedral folding tilings by spherical triangles and spherical parallelograms, obtained in [2], [3] and [4]. The transitivity classes of isogonality and isohedrality are also determined, see Table 3.1.

Keywords: Dihedral tilings, isometric foldings, symmetry groups.

1 Introduction

A *spherical folding tiling*, or *f-tiling* for short, is an edge-to-edge decomposition of the sphere by geodesic polygons, such that all vertices are of even valency and the sums of alternating angles around each vertex are π. A f-tiling τ is said *dihedral* if every tile of τ is congruent to either two fixed sets T and Q. In this case T and Q are the prototiles of τ.

F-tilings are related to the theory of isometric foldings of Riemannian manifolds. See [11] for the foundations of this subject.

Isometric foldings can be seen as locally isometries which send piecewise geodesic segments into piecewise geodesic segments of the same length. These maps are continuous but not necessarily differentiable. The points where they fail to be differentiable are called *singular points*. For surfaces, the singularity set gives rise to a *two-coloured graph* whose vertices fulfill the angle-folding relation, i.e., each vertex is of even valency and the sum of alternating angles is π. For a topological view of this theory see [12].

In [10], Lawrence and Spingarn show that the angle-folding relation is generalized for isometric foldings of the Euclidian space $\mathbb{R}^d$. Farran *et al.* [9] present a study which involves a partition of a surface into polygons.

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A full range of problems and methods associated with tilings and patterns is presented by Grnbaum and Shephard [5]. They address the problem of tiling two-dimensional space with congruent tiles [6].

The complete classification of monohedral tilings of the sphere by triangles (which obviously includes the monohedral triangular f-tilings [1]) was given by Yukako Ueno and Yoshio Agaoka [13]. This classification was partially done by D. Sommerville [7], and an outline of the proof was provided by H. Davies [8].

The classification of all dihedral folding tilings by spherical triangles and spherical parallelograms was obtained in [2], [3] and [4].

Let τ denote a spherical f-tiling. A spherical isometry σ is a *symmetry* of τ if σ maps every tile of τ into a tile of τ. The set of all symmetries of τ is a group under composition of maps, denoted by $G(\tau)$. Here, we classify the group of symmetries of the referred class of spherical f-tilings.

We shall say that the tiles T and T' of τ are in the same *transitivity class* if the symmetry group $G(\tau)$ contains a transformation that maps T into T'. If all the tiles of τ form one transitivity class we say that τ is *tile-transitive* or *isohedral*. If there are k transitivity classes of tiles, then τ is k*-isohedral*. On the other hand, if $G(\tau)$ contains operations that map every vertex of τ into any other vertex, then we say that the vertices form one transitivity class or that τ is *isogonal*. If there are k transitivity classes of vertices, then τ is k*-isogonal*. In this paper we also determine the transitivity classes of isogonality and isohedrality.

In Figure 1.1 we present a complete list of all dihedral f-tilings, whose prototiles are a spherical triangle T and a spherical paralelelogram Q. A detailed study of the f-tilings is included in [2], [3] and [4]. Only one element of each class or family is given. They consists of:

- *A family of square antiprisms* $(\mathcal{A}_\alpha)_{\alpha\in[\alpha_0,\,\pi[}$, in which T is an isosceles triangle iff $\alpha \in \{\alpha_0, 2\pi/3\}$, where $\alpha_0 = \arccos(1-\sqrt{2}) \approx 114.47^\circ$ and α is internal angle of Q;
- For each $k \geq 2$ *a family of* $2k$*-polygonal radially elongated dipyramids*, $\mathcal{R}^k_{\alpha_1\alpha_2}$;
- A *class of f-tilings* $\mathcal{I}^k$ $(k \geq 2)$, in which Q is a square and T is a scalene triangle. We illustrate $\mathcal{I}^2$;
- *A class of f-tilings* $\mathcal{J}^k$ $(k \geq 2)$. Q is a spherical quadrangle with all congruent angles and with distinct pairs of congruent opposite sides. T is a scalene triangle. We consider $k = 2$;
- *F-tilings* $\mathcal{U}_i$, $i = 1, 2, 3, 4$, with the same prototiles. Q has all congruent sides and distinct pairs of angles. T is an isosceles triangle (note: there exists one another element of the form $\mathcal{R}^3_{\phi_1\phi_2}$ with such prototiles);
- For each $k \geq 3$ *a family of f-tilings* $\mathcal{M}^k_\alpha$ $(\pi/k < \alpha < 2\pi/k)$, in which Q has distinct pairs of angles and T is scalene. In Figure we take the minimum value of k;
- *Two classes of f-tilings* $\mathcal{E}^k$ *and* $\mathcal{S}^{k-1}$ $(k \geq 3)$ such that Q has all congruent sides and distinct pairs of angles and T is a scalene triangle. We illustrate $\mathcal{E}^3$ and $\mathcal{S}^3$;

- *F-tilings* $\mathcal{G}_i$, $i = 1, 2, 3$, with the same prototiles. Q has all congruent sides and distinct pairs of angles. T is scalene;
- For each $k \geq 1$ *a family of f-tilings* $\mathcal{D}_\beta^k$ $(0 < \beta < \pi/(2k))$ in which Q has distinct pairs of angles and distinct pairs of sides and T is isosceles. In Figure $\mathcal{D}_\beta^2$ is illustrated;
- *A class of f-tilings* $\mathcal{T}^k$ $(k \geq 2)$ in which Q has distinct pairs of angles and distinct pairs of sides and T is scalene. We illustrate $\mathcal{T}^2$;
- *A f-tiling* $\mathcal{M}$ such that Q has distinct pairs of angles and distinct pairs of sides and T is scalene.

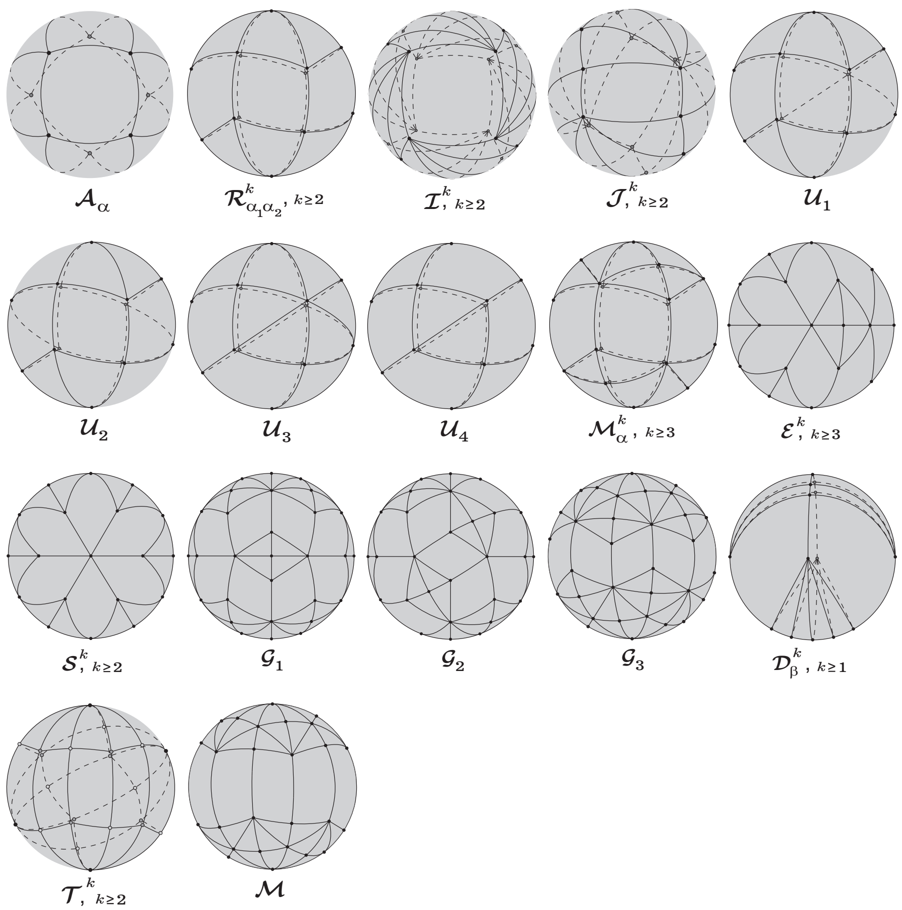


Figure 1.1: Dihedral f-tilings of the sphere by triangles and parallelograms.

2 Preliminaries

It is well known that any spherical isometry is either a reflection, a rotation, or a glide-reflection, which consists of reflecting through some spherical great circle and then rotating around the line orthogonal to the great circle and containing the origin.

The following trivial Lemma is a matter of a great import on what follows.

Lemma 2.1. *Let v and v' be vertices of a spherical f-tiling τ, and let σ be a symmetry of τ, such that $\sigma(v) = v'$. Then, every symmetry of τ that sends v into v' is composition of σ with a symmetry of τ fixing v'.*

On the other hand, the isometries that fix v' are exactly the rotations around the line containing $\pm v'$ and the reflections through the great circles by $\pm v'$.

On what follows R^x_θ, R^y_θ and R^z_θ denote the rotations through an angle θ around the xx axis, yy axis and zz axis, respectively. The reflections on the coordinate planes xy, xz and yz are denoted, respectively, by ρ^{xy}, ρ^{xz} and ρ^{yz}. It follows that: $R^x_\theta \rho^{xy} = \rho^{xy} R^x_{-\theta}$, $R^x_\theta R^y_\pi = R^y_\pi R^x_{-\theta}$, $\rho^{xy} R^z_\theta = R^z_\theta \rho^{xy}$ and $\rho^{xy}\rho^{yz} = \rho^{yz}\rho^{xy} = R^y_\pi$. Besides, $2k$ is the smallest positive integer such that $(\rho^{xy} R^z_{\pi/k})^{2k} = id$.

The nth dihedral group D_n (group of symmetries of the planar regular n-gon) consists of n rotations and n symmetries (reflections). If a is a rotation of order n and b is a symmetry, then $\langle\, a, b \,:\, a^n = 1,\, b^2 = 1,\, ba = a^{n-1}b \,\rangle$ is a group presentation for D_n. Moreover, the elements 1, a, $\ldots$, a^{n-1}, b, ab, $\ldots$, $a^{n-1}b$ are pairwise disjoints.

3 Symmetry Groups

Here we determine the group of symmetries of the mentioned class of spherical tilings. The number of transitivity classes of tiles and vertices of each tiling is indicated. We consider separately the families involved in Figure 1.1.

Antiprisms $\mathcal{A}_\alpha$ – Described in [2]

Figure 3.2 illustrates some antiprisms $\mathcal{A}_\alpha$, $\arccos(1-\sqrt{2}) = \alpha_0 \leq \alpha < \pi$, where α is the internal angle of the spherical square Q.

Firstly, suppose that $\alpha_0 < \alpha < \pi$. If $\alpha = 2\pi/3$ (Figure 3.2-III), then the prototile T is an isosceles triangle. If $\alpha \neq 2\pi/3$ (Figure 3.2-II and Figure 3.2-IV), then T is scalene. In any case we observe that the unique symmetry of $\mathcal{A}_\alpha$ fixing a vertex v of the tiling must be the identity map. By Lemma 2.1, $G(\mathcal{A}_\alpha)$ contains at most 8 symmetries.

However $< R^x_{\pi/2} >$ is a subgroup of $G(\mathcal{A}_\alpha)$ of order 4. On the other hand, the rotation $R^z_\pi \in G(\mathcal{A}_\alpha) \setminus < R^x_{\pi/2} >$. And so $G(\mathcal{A}_\alpha)$ has exactly 8 elements.

Now, if $a = R^x_{\pi/2}$ and $b = R^z_\pi$ then $a^4 = 1$, $b^2 = 1$ and $a^3 b = ba$, where 1 is the identity element. And so $G(\mathcal{A}_\alpha)$ is isomorphic to the octic group D_4. Since all the vertices

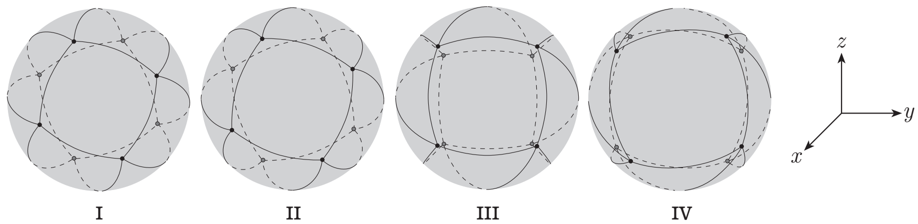


Figure 3.2: Square antiprisms.

of $\mathcal{A}_\alpha$ form one transitivity class then $\mathcal{A}_\alpha$ is isogonal. On the other hand, $\mathcal{A}_\alpha$ is 2-isohedral.

Consider now $\alpha = \alpha_0$. The prototile T is an isosceles triangle of angles $\pi/2$, $\pi - \alpha_0$ and $\pi/2$ as illustrated in Figure 3.2-I. The symmetries of $\mathcal{A}_{\alpha_0}$ that fix a vertex v belonging to a certain tile Q of the tiling is either the identity map or the reflection through the unique great circle containing v and the other vertex of Q opposite to v. By Lemma 2.1, $G(\mathcal{A}_{\alpha_0})$ contains at most 16 symmetries.

Similarly to the previous case $G(\mathcal{A}_{\alpha_0})$ contains a subgroup S isomorphic to D_4, generated by $R^x_{\pi/2}$ and R^z_π. On the other hand, $\phi = \rho^{yz} R^x_{\pi/4} = R^x_{\pi/4}\rho^{yz}$ obtained by reflecting on the plane yz followed by a rotation of $\pi/4$ around the xx axis is also a symmetry of $\mathcal{A}_{\alpha_0}$. Since ϕ has order 8, then $\phi \notin S$ (otherwise S would be abelian). It follows that $\{a\,\phi : a \in S\}$ and S are disjoint, and so $G = G(\mathcal{A}_{\alpha_0})$ has exactly 16 elements. Now, one has

$$\phi^7 R^z_\pi = \rho^{yz} R^x_{7\pi/4} R^z_\pi = \rho^{yz} R^x_{7\pi/4}\rho^{xz}\rho^{yz} = \rho^{yz}\rho^{xz} R^x_{\pi/4}\rho^{yz} = R^z_\pi R^x_{\pi/4}\rho^{yz} = R^z_\pi\phi,$$

and so G is isomorphic to D_8, generated by ϕ and R^z_π. Finally, $\mathcal{A}_{\alpha_0}$ is isogonal and 2-isohedral.

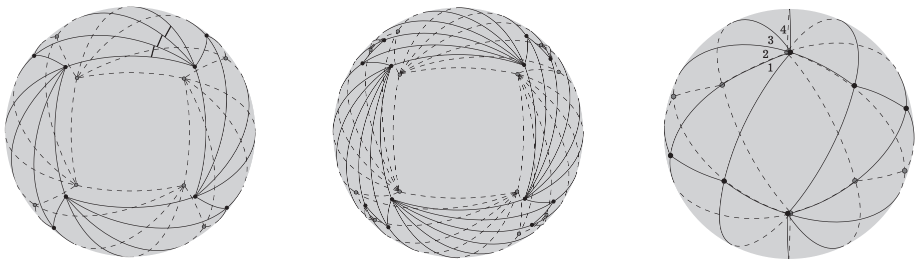
$\mathcal{I}^k$, $\mathcal{J}^k$, $k \geq 2$ – Described in [2]

The f-tiling $\mathcal{I}^k$ ($k \geq 2$) contains 2 spherical squares and $8(2k-1)$ triangles, see Figure 3.3.

Similarly to the previous case $G(\mathcal{I}^k)$ contains a subgroup of order 8 generated by $R^x_{\pi/2}$ and R^z_π.

Now, the cyclic sequence of angles around a vertex v belonging to the quadrangle contains $2k-1$ angles δ, and it is given by $(\alpha, \delta, \ldots, \delta, \gamma, \beta)$. As the transitivity classes of the triangles with angle δ in v are pairwise disjoints, then $G(\mathcal{I}^k)$ has no more elements.

And so, up to an isomorphism, $G(\mathcal{I}^k)$ is the 4th dihedral group. It follows that there are $2k-1$ transitivity classes of triangles (each one with 8 triangles) and one transitivity class of quadrangles. Hence $\mathcal{I}^k$ is $2k$-isohedral. Finally, $\mathcal{I}^k$ is k-isogonal.



$\mathcal{I}^2$ $\mathcal{I}^3$ $\mathcal{J}^2$

Figure 3.3: f-tilings $\mathcal{I}^2$, $\mathcal{I}^3$ and $\mathcal{J}^2$.

Concerning to the f-tilings $\mathcal{J}^k$ ($k \geq 2$), the triangles numbered from 1 to $2k$ ($k = 2$ in Figure 3.3) are in distinct transitivity classes of tiles. On the other hand, the spherical isometries $1 = Id$, R_π^x, R_π^y and R_π^z are symmetries of $\mathcal{J}^k$. Since, $\mathcal{J}^k$ is composed of $8k$ triangles, then $G(\mathcal{J}^k)$ has no more elements. Up to an isomorphism, $G(\mathcal{J}^k)$ is the Klein 4-group. $\mathcal{J}^k$ is $(2k+1)$-isohedral and $(k+1)$-isogonal.

$\mathcal{R}^k_{\alpha_1\alpha_2}$, $k \geq 2$ – Described in [2], [3] and [4]

Firstly, we consider the case when the prototile Q is equiangular. A 3D representation for $k = 4$ is illustrated in Figure 3.4-I. Let α be the internal angle of Q.

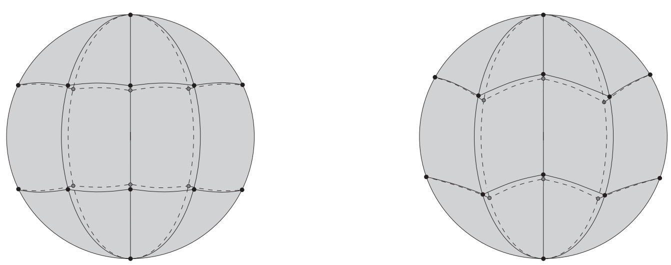


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Figure 3.4: f-tilings $\mathcal{R}^4_\alpha$ and $\mathcal{R}^4_{\alpha_1\alpha_2}$.

Any symmetry of $\mathcal{R}^k_\alpha$ fixes $(0,0,1)$ or maps $(0,0,1)$ into $(0,0,-1)$. The symmetries that fix $(0,0,1)$ are generated, for instance, by the rotation $R^z_{\pi/k}$ of order $2k$ and the reflection ρ^{yz}, giving rise to a subgroup S of $G(\mathcal{R}^k_\alpha)$ isomorphic to D_{2k}. To obtain the symmetries that sends $(0,0,1)$ into $(0,0,-1)$ it is enough to compose each element of S with ρ^{xy}. Now, since ρ^{xy} commutes with $R^z_{\pi/k}$ and ρ^{yz}, then ρ^{xy} commutes with all elements of S. And so, the map defined by $\psi \mapsto (0,\psi)$ and $\rho^{xy}\psi \mapsto (1,\psi)$, $\psi \in S$ is an isomorphism between $G(\mathcal{R}^k_\alpha)$ and $D_1 \times D_{2k}$. It follows immediately that $\mathcal{R}^k_\alpha$ is 2-isohedral and 2-isogonal.

Consider now that Q has distinct pairs of congruent opposite angles, say α_1 and α_2. A 3D representation for $k = 4$ is illustrated in Figure 3.4-II.

In this case the group of symmetries that fix $(0, 0, 1)$ is precisely the kth dihedral group D_k generated by $R^z_{2\pi/k}$ and ρ^{yz}. In fact, neither the reflections on the vertical great circles bisecting triangles nor the rotations of the form $R^z_{(2n+1)\pi/k}$ ($n \in \mathbb{Z}$) are symmetries of $\mathcal{R}^k_{\alpha_1\alpha_2}$.

The map $a = R^z_{\pi/k}\rho^{xy} = \rho^{xy}R^z_{\pi/k}$ is a symmetry of $\mathcal{R}^k_{\alpha_1\alpha_2}$ that maps $(0, 0, 1)$ into $(0, 0, -1)$ allowing us to get the symmetries that map $(0, 0, 1)$ into $(0, 0, -1)$. Now, one has

$$a^{2k-1}\rho^{yz} = R^z_{(2k-1)\pi/k}\rho^{xy}\rho^{yz} = R^z_{(2k-1)\pi/k}R^y_\pi = R^y_\pi R^z_{\pi/k} = \rho^{yz}\rho^{xy}R^z_{\pi/k} = \rho^{yz}a.$$

On the other hand, a has order $2k$ and $\rho^{yz} \notin < a >$. It follows that a and ρ^{yz} generate $G(\mathcal{R}^k_{\alpha_1\alpha_2})$. And so it is isomorphic to D_{2k}. Finally, $\mathcal{R}^k_{\alpha_1\alpha_2}$ has two transitivity classes of tiles and three transitivity classes of vertices, which means that $\mathcal{R}^k_{\alpha_1\alpha_2}$ is 2-isohedral and 3-isogonal.

$\mathcal{S}^k$, $k \geq 2$ and $\mathcal{E}^k$, $k \geq 3$ – Described in [3]

Firstly, we consider the tilings $\mathcal{S}^k$, $k \geq 2$. A 3D representation for $k = 3$ is illustrated in Figure 3.5-I.

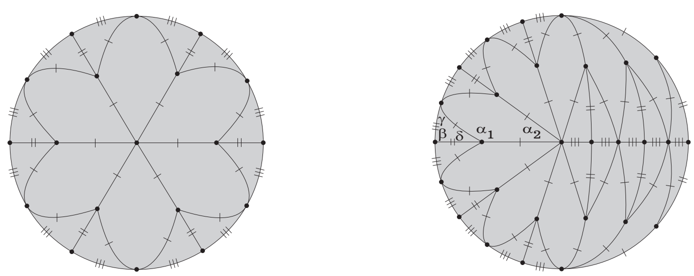

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Figure 3.5: f-tilings $\mathcal{S}^3$ and $\mathcal{E}^4$.

Any symmetry of $\mathcal{S}^k$ fixes $(1, 0, 0)$ or maps $(1, 0, 0)$ into $(-1, 0, 0)$. The symmetries that fix $(1, 0, 0)$ are generated by the rotation $R^x_{\pi/k}$ and the reflection ρ^{xy}, giving rise to D_{2k}. The symmetries that maps $(1, 0, 0)$ into $(-1, 0, 0)$ are obtained composing each one of these elements with ρ^{yz}. Since ρ^{yz} commutes with $R^x_{\pi/k}$ and ρ^{xy}, we conclude that $G(\mathcal{S}^k)$ is isomorphic to $D_1 \times D_{2k}$. It follows immediately that $\mathcal{S}^k$ is 2-isohedral and 4-isogonal.

Now, we shall consider the tilings $\mathcal{E}^k$, $k \geq 3$. A 3D representation for $k = 4$ is illustrated in Figure 3.5-II. $\mathcal{E}^k$ has exactly four vertices surrounded by the cyclic sequence $(\gamma, \gamma, \alpha_2, \alpha_2, \alpha_2, \dots)$. Namely, $(1, 0, 0)$, $(0, 0, 1)$, $(-1, 0, 0)$ and $(0, 0, -1)$.

The isometries ρ^{xy}, ρ^{yz}, ρ^{xy} and ρ^{yz} are the non-identity symmetries of $\mathcal{E}^k$ that leave fixed $(1,0,0)$, $(0,0,1)$, $(-1,0,0)$ and $(0,0,-1)$, respectively. On the other hand, the isometies $\rho^{xz}R^y_{\pi/2}$, ρ^{yz} and $\rho^{xz}R^y_{3\pi/2}$ are symmetries of $\mathcal{E}^k$ that map, $(1,0,0)$ into $(0,0,1)$, $(-1,0,0)$ and $(0,0,-1)$, respectively. By Lemma 2.1 $G(\mathcal{E}^k)$ has exactly eight isometries:

$$id,\ \rho^{xy},\ \rho^{xz}R^y_{\pi/2},\ \rho^{yz}\rho^{xz}R^y_{\pi/2},\ \rho^{yz},\ \rho^{xy}\rho^{yz},\ \rho^{xz}R^y_{3\pi/2} \text{ and } \rho^{yz}\rho^{xz}R^y_{3\pi/2}.$$

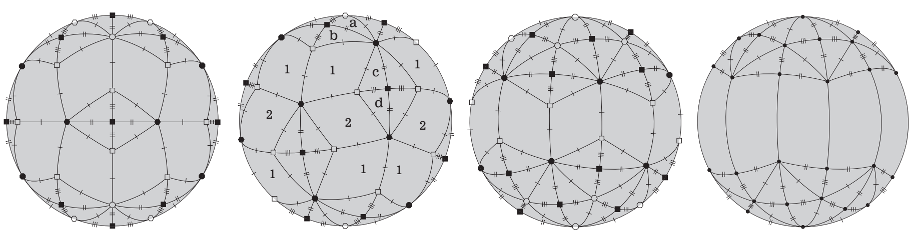
It is a straightforward exercise to show that $G(\mathcal{E}^k)$ is isomorphic to D_4 and it is generated by $\rho^{xz}R^y_{\pi/2}$ and ρ^{xy}.

There are $k-2$ transitivity classes of quadrangles with 8 elements and one class with 4 elements. Concerning to triangles, one gets $2k-3$ transitivity classes, each one contains 8 elements. Hence $\mathcal{E}^k$ is $(3k-4)$ - isohedral.

The vertices surrounded by the cyclic sequence $(\gamma,\gamma,\alpha_2,\alpha_2,\alpha_2,\dots)$ are in the same transitivity class. The vertices enclosed by $(\alpha_2,\gamma,\gamma,\alpha_2,\gamma,\gamma)$ form $k-2$ transitivity classes, each one with 4 vertices. Related to the vertices $(\alpha_1,\alpha_1,\delta,\delta)$ one obtain $k-1$ transitivity classes, one of them has 4 vertices and $k-2$ classes have 8 vertices. Finally, the vertices surrounded by $(\beta,\beta,\beta,\beta)$ are in $k-1$ distinct transitivity classes, one of them has 2 vertices and the remaining classes have 4 vertices. And so, $\mathcal{E}^k$ is $1+(k-2)+(k-1)+(k-1)=(3k-3)$ - isogonal.

$\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ – Described in [3] and $\mathcal{M}$ – Described in [4]

The prototiles of $\mathcal{G}_i$ $(i=1,2,3)$ are a spherical rhombus with pairs of opposite angles $(\alpha_1,\alpha_2)=(2\pi/3,2\pi/5)$ and a spherical triangle with angles $(\beta,\gamma,\delta)=(\pi/2,\pi/3,\pi/5)$. 3D representations are illustrated in Figure 3.6.



$\mathcal{G}_1$ $\mathcal{G}_2$ $\mathcal{G}_3$ $\mathcal{M}$

Figure 3.6: f-tilings $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{M}$.

The reflections ρ^{xy}, ρ^{xz} and ρ^{yz} are symmetries of $\mathcal{G}_1$. On the other hand, for a rhombus Q in the first octant, the unique symmetry that leaves Q in the first octant is the identity map. And so $G(\mathcal{G}_1)$ is generated by ρ^{xy}, ρ^{xz} and ρ^{yz}. It follows that $G(\mathcal{G}_1)$ is isomorphic to $D_1 \times D_1 \times D_1$.

Now, there are 2 transitivity classes of rhombus and 7 transitivity classes of triangles. Hence $\mathcal{G}_1$ is 9-isohedral. On the other hand, numbering the vertices of the first octant, we conclude that $\mathcal{G}_1$ is 11-isogonal.

Consider now the tiling $\mathcal{G}_2$. Here we must observe that the cyclic sequence $(\gamma, \gamma, \gamma, \gamma, \gamma, \gamma)$ encloses exactly two vertices: $(0, 0, 1)$ and $(0, 0, -1)$. Similarly to the case considered in $\mathcal{R}^k_{\alpha_1\alpha_2}$ (with $k = 3$), it can be seen that $G(\mathcal{G}_2)$ is isomorphic to D_6, generated by $\rho^{xy} R^z_{\pi/3}$ and ρ^{yz}.

One gets two transitivity classes of rhombi. The identification is done in Figure 3.6 (rhombi labelled by 1 and 2). Representative elements of transitivity classes of triangles are labelled by a, b, c and d, with 12 triangles each transitivity class. Therefore $G(\mathcal{G}_2)$ is 6-isohedral. Concerning to vertices, it can be seen that $G(\mathcal{G}_2)$ is 7-isogonal.

Finally, using similar procedures, we conclude that $G(\mathcal{G}_3)$ is isomorphic to D_{10}, generated by $\rho^{xy} R^z_{\pi/5}$ and ρ^{yz}. The tiling $\mathcal{M}$ (Figure 3.6 on the right) is obtained by deleting a pair of opposite sides of the prototile Q of $\mathcal{G}_3$ and preserving the angle folding relation. Moreover, $G(\mathcal{M}) = G(\mathcal{G}_3)$. It follows that $\mathcal{G}_3$ is 5-isohedral and 6-isogonal while $\mathcal{M}$ is 4-isohedral and 5-isogonal.

$\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ and $\mathcal{U}_4$ – Described in [3]

The angles of the prototiles Q and T of the tilings $\mathcal{U}_i$ ($i = 1, 2, 3, 4$) are $(\alpha_1, \alpha_2) = (2\pi/3, \pi/2)$ and $(\beta, \gamma, \gamma) = (\pi/2, \pi/3, \pi/3)$. Bisecting the rhombus Q by α_1 one gets two triangles congruent to T. In Figure 3.7 3D representations are illustrated (we have chosen these positions since they fit better our purposes).

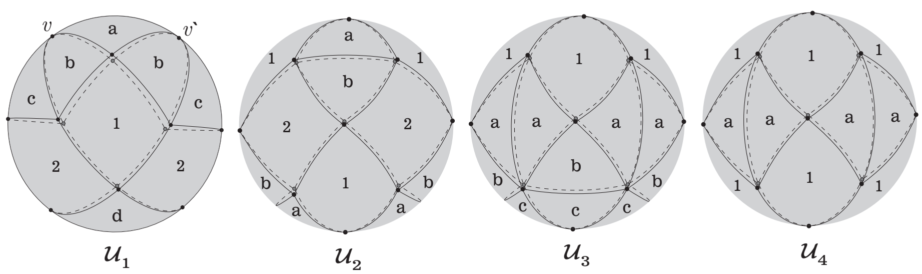


Figure 3.7: f-tilings $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ and $\mathcal{U}_4$.

The tiling $\mathcal{U}_1$ has exactly two vertices (v and v') surrounded by the cyclic sequence $(\gamma, \gamma, \gamma, \gamma, \gamma, \gamma)$. Now, the symmetries of $\mathcal{U}_1$ that fix v (and v') are the identity map and the reflection ρ^{yz}. On the other hand, ρ^{xz} is a symmetry of $\mathcal{U}_1$ sending v into v'. By Lemma 2.1 $G(\mathcal{U}_1)$ has exactly four elements. It follows immediately that $G(\mathcal{U}_1)$ is isomorphic to the Klein 4-group.

There are two transitivity classes of quadrangles (1 and 2) and four transitivity classes of triangles (a, b, c and d). Hence $\mathcal{U}_1$ is 6-isohedral. Besides, $\mathcal{U}_1$ is 6-isogonal.

Consider the tiling $\mathcal{U}_2$. The vertices of $\mathcal{U}_2$ surrounded by $(\alpha_2, \beta, \alpha_2, \beta,)$ are $(0,0,1)$ and $(0,0,-1)$; besides, the symmetries of $\mathcal{U}_2$ that fix these vertices are id, ρ^{yz}, ρ^{xz} and $R^z_\pi = \rho^{yz}\rho^{xz}$. Now, $\rho^{xy}R^z_{\pi/2}$ is a symmetry of $\mathcal{U}_2$ mapping $(0,0,1)$ into $(0,0,-1)$. It follows that $G(\mathcal{U}_2)$ is isomorphic to D_4, generated by $\rho^{xy}R^z_{\pi/2}$ and ρ^{yz}. $\mathcal{U}_2$ is 4-isohedral (the identifications are made in Figure) and 3-isogonal.

Taking in account the type of vertices of the tiling $\mathcal{U}_3$, we observe that any symmetry of $\mathcal{U}_3$ must fix $(0,0,1)$. Among the rotations, with this property, we have the maps id, $R^z_{\pi/2}$, R^z_π and $R^z_{3\pi/2}$; concerning to reflections we have the maps ρ^{yz} , ρ^{xz}, $\rho^{yz}R^z_{\pi/2}$ and $\rho^{yz}R^z_{3\pi/2}$. Up to an isomorphism, $G(\mathcal{U}_3) = D_4$. It is generated by $R^z_{\pi/2}$ and ρ^{yz}. $\mathcal{U}_3$ is 5-isogonal and 4-isohedral.

Finally, using similar procedures to the ones considered in the tilings of the form $\mathcal{R}^k_\alpha$ (with $k = 2$), we conclude that $G(\mathcal{U}_4)$ is isomorphic to $D_1 \times D_4$. Besides, $\mathcal{U}_4$ is 2-isohedral and 3-isogonal.

$\mathcal{M}^k_\alpha, k \geq 3$ – Described in [3] and [4], $\mathcal{D}^k_\beta, k \geq 1$ – Described in [4] and $\mathcal{T}^k, k \geq 2$ – Described in [4]

Figure 3.8 illustrates 3D representations of $\mathcal{M}^4_\alpha$, $\mathcal{D}^2_\beta$ and $\mathcal{T}^2$, for some α and β. A similar study to the one used in the tilings $\mathcal{R}^k_{\alpha_1\alpha_2}$ shows that $G(\mathcal{M}^k_{\alpha_2})$ is isomorphic to D_{2k}. $\mathcal{M}^k_{\alpha_2}$ is 3-isohedral and 4-isogonal.

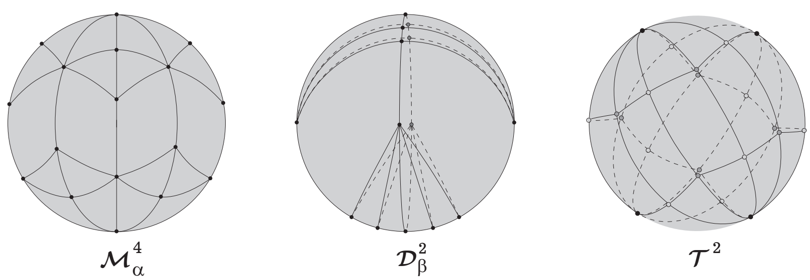


Figure 3.8: f-tilings $\mathcal{M}^4_\alpha$, $\mathcal{D}^2_\beta$ and $\mathcal{T}^2$.

The prototiles of $\mathcal{D}^k_\beta$, are an isosceles triangle of angles (β, γ, γ), with $\gamma = \pi/2$, and a spherical parallelogram of distinct pairs of opposite angles, (α_1, α_2), with $\alpha_2 = \pi/2$ and $\alpha_1 + k\beta = \pi$. $\mathcal{D}^k_\beta$ is composed of four quadrangles and $8k$ triangles. In Figure 3.8 a 3D representation for $k = 2$ is illustrated, as mentioned before.

It is a straightforward exercise to show that any symmetry of $\mathcal{D}^k_\beta$ fixes $(0,0,1)$ or maps $(0,0,1)$ into $(0,0,-1)$. $G(\mathcal{D}^k_\beta)$ contains exactly four symmetries fixing $(0,0,1)$. Namely, id, ρ^{yz}, ρ^{xz} and R^z_π. The spherical isometry $\phi = R^x_\pi R^z_{\pi/2}$, explicitly defined by $\phi(x,y,z) = (-y,-x,-z)$, is a symmetry of $\mathcal{D}^k_\beta$ sending $(0,0,1)$ into $(0,0,-1)$. By Lemma 2.1 $G(\mathcal{D}^k_\beta)$ has 8 elements. Since $\phi^4 = id$, $(\rho^{yz})^2 = id$ and $\rho^{yz}\phi = \phi^3\rho^{yz}$, then

$G(\mathcal{D}_\beta^k)$ is isomorphic to D_4. Finally, there are k transitivity classes of triangles and one transitivity class of quadrangles. And so $\mathcal{D}_\beta^k$ is $k+1$-isohedral. Besides, $\mathcal{D}_\beta^k$ is $k+2$-isogonal (the vertices surrounded by $(\gamma,\gamma,\gamma,\gamma)$) are distributed by k transitivity classes).

Finally, we shall consider the tilings $\mathcal{T}^k$, $k \geq 2$. In Figure 3.8 a 3D representation for $k = 2$ is illustrated. $\mathcal{T}^k$ has exactly four vertices (in bold) surrounded by the cyclic sequence $(\alpha_1, \alpha_1, \delta, \delta, \ldots, \delta)$ (δ appears $2k$ times). The unique symmetry of $\mathcal{T}^k$ fixing one of these vertices is the identity map. By Lemma 2.1 $G(\mathcal{T}^k)$ contains at most 4 symmetries. However, the isometries id, R_π^x, R_π^y and $R_\pi^z = R_\pi^x R_\pi^y$ are symmetries of $\mathcal{T}^k$. It follows that $G(\mathcal{T}^k)$ is isomorphic to the Klein 4-group. $\mathcal{T}^k$ has $2k+2$ transitivity classes of quadrangles (each one has two elements) and $2k$ transitivity classes of triangles (each one with four elements). Therefore $\mathcal{T}^k$ is $4k+2$-isohedral. Besides, $\mathcal{T}^k$ is $2k+2$-isogonal.

We have proved the following result:

Proposition 3.1. *The symmetry groups of the dihedral f-tilings by spherical triangles and spherical parallelograms are dihedral groups or direct products of dihedral groups as indicated in Table 3.1. The index of isogonality and isohedrality is also disclosed.*

F-Tiling	Symmetry Group	isohedrality-classes	isogonality-classes
$\mathcal{A}_{\alpha_0}$	D_8	2	1
$\mathcal{A}_\alpha, \alpha \neq \alpha_0$	D_4	2	1
$\mathcal{I}^k, k \geq 2$	D_4	$2k$	k
$\mathcal{J}^k, k \geq 2$	$D_1 \times D_1$	$2k+1$	$k+1$
${}^E\mathcal{R}_\alpha^k, k \geq 2$	$D_1 \times D_{2k}$	2	2
${}^*\mathcal{R}_{**}^k, k \geq 2$	D_{2k}	2	3
$\mathcal{S}^k, k \geq 2$	$D_1 \times D_{2k}$	2	4
$\mathcal{E}^k, k \geq 3$	D_4	$3k-4$	$3k-3$
$\mathcal{G}_1$	$D_1 \times D_1 \times D_1$	9	11
$\mathcal{G}_2$	D_6	6	7
$\mathcal{G}_3$	D_{10}	5	6
$\mathcal{M}$	D_{10}	4	5
$\mathcal{U}_1$	$D_1 \times D_1$	6	6
$\mathcal{U}_2$	D_4	4	3
$\mathcal{U}_3$	D_4	4	5
$\mathcal{U}_4$	$D_1 \times D_4$	2	3
$\mathcal{M}_{\alpha_2}^k, k \geq 3$	D_{2k}	3	4
$\mathcal{D}_\beta^k, k \geq 1$	D_4	$k+1$	$k+2$
$\mathcal{T}^k, k \geq 2$	$D_1 \times D_1$	$4k+2$	$2k+2$

Table 3.1: Group of Symmetries and Transitivity Classes

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