Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/040111

CLR Properties in Complex-Valued Metric Space

Harpreet Kaur[1] *and Saurabh Manro*[2,*]

[1] Departmen t of Mathematics, Desh Bhagat University, Mandi Gobindgarh, Punjab, India
[2] B.No. 33, H.No. 223, Peer Khana Road, Near Tiwari Di Kothi, Khanna-141401, District- Ludhiana (Punjab), India

Received: 3 May 2015, Revised: 8 Aug. 2015, Accepted: 11 Aug. 2015
Published online: 1 Jan. 2016

Abstract: The aim of this paper is to establish some new common fixed point theorems for noncompatible maps in complex-valued metric space by using common $(E.A)$ property and its variants. As a consequence, a multitude of common fixed point theorems existing in the literature are sharpened and enriched.

Keywords: Complex-valued metric space, property $(E.A)$, common property $(E.A)$, CLR_{PQ} property, $JCLR_{PQ}$ property, CLR_P property, common fixed point.

1 Introduction

The Banach fixed point theorem for contraction map has been generalized and extended in many directions. This theorem has many applications, but suffers from one drawback that it require map to be continuous throughout the domain. It has been known since the paper of Kannan [6] that there exist discontinuous maps having fixed points, however these maps are continuous at the fixed point. Recently fixed point theory for discontinuous and noncompatible maps has attracted much attention. Aamri et al. [1] generalized the concepts of non compatibility by defining property $(E.A)$ which allows replacing the completeness requirement of the space to a more natural condition of closedness of the range as well as relaxes the continuity of one or more maps and containment of the range of one map into the range of other which is utilized to construct the sequence of joint iterates. Liu et al. [7] introduced the notion of common property $(E.A)$ which contains property $(E.A)$. On the other hand the concept of the common limit in the range (CLR) property introduced by Sintunavarat and Kumam [13] do not require even closedness of range for the existence of the common fixed point. Imdad et al. [4], Manro et al. [8] and Chauhan et al. [3] introduced the concept of CLR_{PQ} property, CLR_P property and $JCLR_{PQ}$ property respectively and utilized the same to prove common fixed point theorems. The aim of this paper is to establish some new common fixed point theorems for noncompatible maps using these new properties in complex-valued metric space, introduced by Azam et al. [2] which is more general than classical metric space. Recently, Sastry et al. [12] proved that every complex-valued metric space is metrizable and hence is not real generalizations of metric spaces. But indeed it is a metric space and it is well known that complex numbers have many applications in Control theory, Fluid dynamics, Dynamic equations, Electromagnetism, Signal analysis, Quantum mechanics, Relativity, Geometry, Fractals, Analytic number theory, Algebraic number theory etc. For more details about complex valued metric spaces, one can refers to ths papers [9,10,12,14,15]. Our improvement in this paper is four fold:
(i) the containment of ranges amongst the involved maps is removed;
(ii) the continuity requirement of maps is not used;
(iii) the completeness / closedness of the whole space or any of its range space is removed;
(iv) minimal type contractive condition used. As a consequence, a multitude of common fixed point theorems existing in the literature are sharpened and enriched.

2 Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$, recall a natural partial order relation $\preceq$ on $\mathbb{C}$ as follows:
$z_1 \preceq z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$,
$z_1 \prec z_2$ if and only if $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

* Corresponding author e-mail: sauravmanro@hotmail.com

Definition 1(2). *Let X be a nonempty set such that the map $d : X \times X \to \mathbb{C}$ satisfies the following conditions:*
(C_1) $0 \preceq d(x,y)$ *for all* $x,y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$;
(C_2) $d(x,y) = d(y,x)$ *for all* $x,y \in X$;
(C_3) $d(x,y) \preceq d(x,z) + d(z,y)$ *for all* $x,y,z \in X$.
Then d is called a complex-valued metric on X, and (X,d) is called a complex-valued metric space.

Example 1 *Define complex-valued metric $d : X \times X \to \mathbb{C}$ by $d(z_1,z_2) = e^{3i}|z_1 - z_2|$. Then (X,d) is a complex-valued metric space.*

Definition 2.[2] *Let (X,d) complex-valued metric space and $x \in X$. Then sequence $\{x_n\}$ sequence is*
(i) *convergent if for every $0 \prec c \in \mathbb{C}$, there is a natural number N such that $d(x_n,x) \prec c$, for all $n > N$. We write it as $lim_{n\to\infty} x_n = x$.*
(ii) *a Cauchy sequence, if for every $0 \prec c \in \mathbb{C}$, there is a natural number N such that $d(x_n,x_m) \prec c$, for all $n,m > N$.*

Lemma 1.[2,14] *Let (X,d) be a complex valued metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ converges to x if and only if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Definition 3.[8] *A pair of self-maps A and S of a complex-valued metric space (X,d) are weakly compatible if $ASx = SAx$ for all $x \in X$ at which $Ax = Sx$.*

Example 2[15] *Define complex-valued metric $d : X \times X \to \mathbb{C}$ by $d(z_1,z_2) = e^{ia}|z_1 - z_2|$, where a is any real constant. Then (X,d) is a complex-valued metric space. Suppose self maps A and S be defined as: $Az = 2e^{i\frac{\pi}{4}}$ if $Re(z) \neq 0$, $Az = 3e^{i\frac{\pi}{3}}$ if $Re(z) = 0$, and $Sz = 2e^{i\frac{\pi}{4}}$ if $Re(z) \neq 0$, $Az = 4e^{i\frac{\pi}{6}}$ if $Re(z) = 0$,*
Then maps A and S are weakly compatible at all $z \in \mathbb{C}$. with $Re(z) \neq 0$.

Definition 4.[15] *A pair of self maps A and S on a complex-valued metric space (X,d) satisfies the property $(E.A)$ if there exist a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = z$ for some $z \in X$.*

The class of maps satisfying property $(E.A)$ contain the class of compatible (Jungck [5]) as well as the class of noncompatible maps.

Example 3 *Let $X = \mathbb{C}$ and d be any complex-valued metric. Define self maps A and S by $Az = z^2$ and $Sz = z$, for all $z \in X$. Consider a sequence in X as $\{x_n\} = \{\frac{1}{n}\}$ where $n = 1,2,3,\ldots$ then $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = 0$. Hence, the pair (A,S) satisfies property $(E.A)$ for the sequences $\{x_n\}$ in X.*

Definition 5.[7] *Two pairs of self maps (A,S) and (B,T) on a complex-valued metric space (X,d) satisfy common property $(E.A)$ if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n =$ $lim_{n\to\infty} By_n = lim_{n\to\infty} Ty_n = p$ for some $p \in X$.*

Clearly, common property $(E.A)$ contains property $(E.A)$.

Definition 6.[13] *A pair of self-maps A and S on a complex-valued metric space (X,d) satisfies the common limit in the range of S property (CLR_S) if there exist a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = Sz$ for some $z \in X$.*

Example 4 *Let (X,d) be any complex-valued metric space. Define self maps A and S by $Az = z^2$ and $Sz = \frac{1}{z}$ for all $z \in X$. Consider a sequence in X as $\{x_n\} = \{\frac{1}{n}\}$ where $n = 1,2,3,\ldots$ then $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = 0 = S(0)$. Hence, the pair (A,S) satisfies common limit in the range of S property (CLR_S) for the sequences $\{x_n\}$ in X.*

With a view to extend the (CLR_S) property to two pair of self maps, Imdad et. al. [4] defined the (CLR_{PQ}) property (with respect to maps P and Q) as follows:

Definition 7.[4] *Two pairs (A,P) and (B,Q) of complex-valued metric space (X,d) satisfy the (CLR_{PQ}) property (with respect to maps P and Q) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Px_n$ $= lim_{n\to\infty} Qy_n = lim_{n\to\infty} By_n = z$ where $z \in P(X) \cap Q(X)$.*

Example 5 *Let $X = \mathbb{C}$ and d be any complex-valued metric. Define self maps A,B,P and Q on X by $Az = \frac{z}{3}, Bz = \frac{-z}{3}$, $Pz = \frac{z}{2}, Qz = \frac{z}{2}$ for all $z \in X$. Then with sequences $\{x_n\} = \{\frac{1}{n}\}$ and $\{y_n\} = \{\frac{-1}{n}\}$ in X, $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Px_n =$ $lim_{n\to\infty} Qy_n = lim_{n\to\infty} By_n = 0$. This shows that the pairs (A,P) and (B,Q) share the common limit in the range of P and Q property.*

Remark 1 *In view of the preceding example notice that when the pairs (A,P) and (B,Q) share the common property $(E.A)$ and $P(X)$ as well as $Q(X)$ are closed subsets of X, then the pairs also share the CLR_{PQ} property.*

Definition 8.[3] *Two pairs of self maps (A,P) and (B,Q) of a complex-valued metric space (X,d) satisfy the $(JCLR_{PQ})$ property (with respect to mappings P and Q) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Px_n$ $= lim_{n\to\infty} Qy_n = lim_{n\to\infty} By_n = Pz = Qz$ where $z \in X$.*

Manro et. al. [8] defined the following:

Definition 9(8). *The pairs (A,P) and (B,Q) on a complex-valued metric space (X,d) share common limit in the range of P (CLR_P) property if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Px_n$ $= lim_{n\to\infty} Qy_n = lim_{n\to\infty} By_n = Pz$ for some $z \in X$.*

If $A = B$ and $P = Q$, then the above definition implies (CLR_P) property due to Sintunavarat et al. [13]. Also notice that the preceding definition implies the common property $(E.A)$ but the converse implication is not true in general.

3 Main Results

Theorem 1. *Let A,B,P and Q be four self maps in complex-valued metric space (X,d) satisfying conditions:*

(1) *pairs (A,P) and (B,Q) satisfies the common property $(E.A)$;*

(2)

$$\begin{aligned} d(Ax,By)^2 \preceq \phi(&d(Px,Ax)d(Qy,By), d(Px,By)d(Qy,Ax),\\ &d(Px,Ax)d(Px,By), d(Qy,Ax)d(Qy,By),\\ &d(Px,Qy)^2, d(Px,Ax)d(Qy,Ax),\\ &d(Qy,By)d(Px,By), d(Px,By)d(Px,Ax),\\ &d(Px,Qy)d(Qy,Ax), d(Px,Qy)d(Qy,By)) \end{aligned}$$

for all $x,y \in X$ where the function $\phi : [0,\infty)^{10} \to [0,\infty)$ satisfies the conditions:
(a) ϕ is upper semi-continuous and non-decreasing in each coordinate variable,
(b) for all $t \succ 0$,
$\phi(0,0,0,0,0,t,0,0,0,0) \prec t$,
$\phi(0,0,0,0,0,0,t,0,0,0) \prec t$,
$\phi(0,t,0,0,t,0,0,0,t,0) \prec t$;

(3) *PX and QX are closed subspace of X.*

Then pairs (A,P) and (B,Q) have coincidence point. Further if (A,P) and (B,Q) be weakly compatible pairs then A,B,P and Q have a unique common fixed point in X.

Proof. In view of (1), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Px_n$ $= lim_{n\to\infty}Qy_n = lim_{n\to\infty}By_n = z$ for some $z \in X$.

Since PX is a closed subset of X, therefore, there exists a point $u \in X$ such that $z = Pu$.

We claim that $Au = z$. Suppose not, then by (2), take $x = u, y = y_n$,

$$\begin{aligned} d(Au,By_n)^2 \preceq \phi(&d(Pu,Au)d(Qy_n,By_n),\\ &d(Pu,By_n)d(Qy_n,Au), d(Pu,Au)d(Pu,By_n),\\ &d(Qy_n,Au)d(Qy_n,By_n), d(Pu,Qy_n)^2,\\ &d(Pu,Au)d(Qy_n,Au), d(Qy_n,By_n)d(Pu,By_n),\\ &d(Pu,By_n)d(Pu,Au), d(Pu,Qy_n)d(Qy_n,Au),\\ &d(Pu,Qy_n)d(Qy_n,By_n)), \end{aligned}$$

taking $n \to \infty$, we get

$$\begin{aligned} d(Au,z)^2 \preceq \phi(&d(z,Au)d(z,z), d(z,z)d(z,Au),\\ &d(z,Au)d(z,z), d(z,Au)d(z,z),\\ &d(z,z)^2, d(z,Au)d(z,Au),\\ &d(z,z)d(z,z),\\ &d(z,z)d(z,Au), d(z,z)d(z,Au),\\ &d(z,z)d(z,z)), \end{aligned}$$

$$\begin{aligned} d(Au,z)^2 &\preceq \phi(0,0,0,0,0,d(z,Au)^2,0,0,0,0)\\ &\prec d(z,Au)^2, \end{aligned}$$

Therefore, $Au = z = Pu$ which shows that u is a coincidence point of the pair (A,P). Since QX is also a closed subset of X, therefore in QX and hence there exists $v \in X$ such that $Qv = z = Au = Pu$. Now, by taking $x = u, y = v$ in (2) we can easily show that $Bv = z$. Therefore, $Bv = z = Qv$ which shows that v is a coincidence point of the pair (B,Q). Since the pairs (A,P) and (B,Q) are weakly compatible and $Au = Pu, Bv = Qv$, therefore,

$$Az = APu = PAu = Pz,$$
$$Bz = BQv = QBv = Qz.$$

Next, we claim that $Az = z$. Suppose not, then again by using inequality (2), take $x = u$ and $y = v$, we have

$$\begin{aligned} d(Au,Bv)^2 \preceq \phi(&d(Pu,Au)d(Qv,Bv),\\ &d(Pu,Bv)d(Qv,Au), d(Pu,Au)d(Pu,Bv),\\ &d(Qv,Au)d(Qv,Bv), d(Pu,Qv)^2,\\ &d(Pu,Au)d(Qv,Au),\\ &d(Qv,Bv)d(Pu,Bv), d(Pu,Bv)d(Pu,Au),\\ &d(Pu,Qv)d(Qv,Au), d(Pu,Qv)d(Qv,Bv)) \end{aligned}$$

$$\begin{aligned} d(Au,z)^2 &\preceq \phi(0,d(Au,z)^2,0,0,d(Au,z)^2,0,0,0,d(Au,z)^2,0)\\ &\prec d(Au,z)^2, \end{aligned}$$

which give a contradiction. Hence, $Az = z = Pz$. Similarly, one can prove that $Bz = Qz = z$. Hence, $Az = Bz = Pz = Qz$, and z is common fixed point of A,B,P and Q. The uniqueness of common fixed point is an easy consequence of inequality (2).

Next we attempt to drop closedness of range of maps and relax containment of two subspaces to one subspace by replacing property $(E.A)$ by a weaker condition CLR_P property in Theorem 1.

Theorem 2. *Let A,B,P and Q be four self maps in complex-valued metric space (X,d) satisfying condition to (2) of Theorem 1 and*
(4) (A,P) and (B,Q) shares the CLR_P property (CLR_Q property),
(5) $AX \subset QX$ (or $BX \subset PX$).
Then pairs (A,P) and (B,Q) have coincidence point. Further if (A,P) and (B,Q) be weakly compatible pair then self maps A,B,P and Q have a unique common fixed point in X.

Proof. As the pairs (A,P) and (B,Q) share the common limit in the range of P property, that is there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Px_n =$
$lim_{n\to\infty}Qy_n = lim_{n\to\infty}By_n = Pz$ for some $z \in X$.
Firstly, we assert that $Az = Pz$. Suppose not, then by (2), we have

$$\begin{aligned} d(Az,By_n)^2 \preceq \phi(&d(Pz,Az)d(Qy_n,By_n),\\ &d(Pz,By_n)d(Qy_n,Az),\\ &d(Pz,Az)d(Pz,By_n), d(Qy_n,Az)d(Qy_n,By_n),\\ &d(Pz,Qy_n)^2, d(Pz,Az)d(Qy_n,Az),\\ &d(Qy_n,By_n)d(Pz,By_n), d(Pz,By_n)d(Pz,Az),\\ &d(Pz,Qy_n)d(Qy_n,Az), d(Pz,Qy_n)d(Qy_n,By_n)) \end{aligned}$$

taking $n \to \infty$, we get

$$d(Az,Pz)^2 \preceq \phi(0,0,0,0,0,d(Pz,Az)^2,0,0,0,0) \prec d(Az,Pz)^2.$$

This gives a contradiction, hence, $Az = Pz$ which shows that z is a coincidence point of the pair (A,P).
Since $AX \subset QX$, there exist $v \in X$ such that $Az = Qv$. Secondly, we assert that $Bv = Qv$. Suppose not, then by (2), we get

$$\begin{aligned} d(Az,Bv)^2 \preceq \phi(&d(Pz,Az)d(Qv,Bv),\\ &d(Pz,Bv)d(Qv,Az),d(Pz,Az)d(Pz,Bv),\\ &d(Qv,Az)d(Qv,Bv),d(Pz,Qv)^2,\\ &d(Pz,Az)d(Qv,Az),\\ &d(Qv,Bv)d(Pz,Bv),d(Pz,Bv)d(Pz,Az),\\ &d(Pz,Qv)d(Qv,Az),d(Pz,Qv)d(Qv,Bv)) \end{aligned}$$

$$\begin{aligned} d(Qv,Bv)^2 &\preceq \phi(0,0,0,0,0,0,d(Qv,Bv)^2,0,0,0)\\ &\prec d(Qv,Bv)^2, \end{aligned}$$

a contradiction, hence, $Bv = Qv$ which shows that v is a coincidence point of the pair (B,Q).

Thus, we have $u = Qv = Bv = Az = Pz$.

Since the pairs (A,P) and (B,Q) are weakly compatible, this gives,

$$Au = APu = PAu = AAu = PPu = Pu,$$

$$Bu = BQv = QBv = QQv = BBv = Qu.$$

Finally, we assert that $Au = u$. Suppose not, again by (2), we have

$$\begin{aligned} d(Au,Bv)^2 \preceq \phi(&d(Pu,Au)d(Qv,Bv),\\ &d(Pu,Bv)d(Qv,Au),d(Pu,Au)d(Pu,Bv),\\ &d(Qv,Au)d(Qv,Bv),d(Pu,Qv)^2,\\ &d(Pu,Au)d(Qv,Au),\\ &d(Qv,Bv)d(Pu,Bv),d(Pu,Bv)d(Pu,Au),\\ &d(Pu,Qv)d(Qv,Au),d(Pu,Qv)d(Qv,Bv)), \end{aligned}$$

$$\begin{aligned} d(Au,u)^2 &\preceq \phi(0,d(Au,u)^2,0,0,d(Au,u)^2,0,0,0,d(Au,u)^2,0)\\ &\prec d(Au,u)^2, \end{aligned}$$

a contraction, hence, $Au = u = Pu$ which gives, u is common fixed point of A and P.

Similarly, one can easily prove that $Bu = u = Qu$, that is u is common fixed point of B and Q. Therefore u is common fixed point of A,P,B and Q. The uniqueness of common fixed point is an easy consequence of inequality (2).

Now we attempt to drop containment of subspaces by using weaker condition $JCLR_{PQ}$ property/CLR_{PQ} property in Theorem 1.

Theorem 3. *Let A,B,P and Q be four self maps in a complex-valued metric space (X,d) satisfying condition (2) of Theorem 1 and*
(6) (A,P) and (B,Q) satisfy $JCLR_{PQ}$ property.
Then pairs (A,P) and (B,Q) have coincidence point. Further if (A,P) and (B,Q) be weakly compatible pairs then A,B,P and Q have a unique common fixed point in X.

Proof. As the pairs (A,P) and (B,Q) satisfy the $JCLR_{PQ}$ property, that is, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Px_n$ $= lim_{n\to\infty}Qy_n = lim_{n\to\infty}By_n = Pz = Qz$ for some $z \in X$. Firstly, we assert that $Az = Pz$. Suppose not, then by (2), we have

$$\begin{aligned} d(Az,By_n)^2 \preceq \phi(&d(Pz,Az)d(Qy_n,By_n),\\ &d(Pz,By_n)d(Qy_n,Az),(Pz,Az)d(Pz,By_n),\\ &d(Qy_n,Az)d(Qy_n,By_n),d(Pz,Qy_n)^2,\\ &d(Pz,Az)d(Qy_n,Az),d(Qy_n,By_n)d(Pz,By_n),\\ &d(Pz,By_n)d(Pz,Az),d(Pz,Qy_n)d(Qy_n,Az),\\ &d(Pz,Qy_n)d(Qy_n,By_n)), \end{aligned}$$

taking $n \to \infty$, we get

$$\begin{aligned} d(Az,Pz)^2 &\preceq \phi(0,0,0,0,0,d(Pz,Az)^2,0,0,0,0)\\ &\prec d(Az,Pz)^2. \end{aligned}$$

a contradiction, hence, $Az = Pz$ which shows that z is a coincidence point of the pair (A,P).

Secondly, we assert that $Bz = Qz$. Suppose not, then again by (2), we get

$$\begin{aligned} d(Az,Bz)^2 \preceq \phi(&d(Pz,Az)d(Qz,Bz),\\ &d(Pz,Bz)d(Qz,Az),d(Pz,Az)d(Pz,Bz),\\ &d(Qz,Az)d(Qz,Bz),d(Pz,Qz)^2,\\ &d(Pz,Az)d(Qz,Az),\\ &d(Qz,Bz)d(Pz,Bz),d(Pz,Bz)d(Pz,Az),\\ &d(Pz,Qz)d(Qz,Az),d(Pz,Qz)d(Qz,Bz)), \end{aligned}$$

$$\begin{aligned} d(Qz,Bz)^2 &\preceq \phi(0,0,0,0,0,0,d(Qz,Bz)^2,0,0,0)\\ &\prec d(Qz,Bz)^2, \end{aligned}$$

a contradiction again, hence, $Bz = Qz$ which shows that z is a coincidence point of the pair (B,Q). Thus, we have $Qz = Bz = Az = Pz$. Now, we assume that $u = Qz = Bz = Az = Pz$. Since the pairs (A,P) and (B,Q) are weakly compatible, this gives,

$$Au = APz = PAz = AAz = PPz = Pu,$$

$$Bu = BQz = QBz = QQz = BBz = Qu.$$

Finally, we assert that $Au = u$. Suppose not, again by (2), we have

$$\begin{aligned} d(Au,Bz)^2 \preceq \phi(&d(Pu,Au)d(Qz,Bz),\\ &d(Pu,Bz)d(Qz,Au), d(Pu,Au)d(Pu,Bz),\\ &d(Qz,Au)d(Qz,Bz), d(Pu,Qz)^2,\\ &d(Pu,Au)d(Qz,Au),\\ &d(Qz,Bz)d(Pu,Bz), d(Pu,Bz)d(Pu,Au),\\ &d(Pu,Qz)d(Qz,Au), d(Pu,Qz)d(Qz,Bz)), \end{aligned}$$

$$\begin{aligned} d(Au,u)^2 &\preceq \phi(0, d(Au,u)^2, 0, 0, d(Au,u)^2, 0, 0, 0, d(Au,u)^2, 0)\\ &\prec d(Au,u)^2, \end{aligned}$$

a contradiction, hence, $Au = u = Pu$ which gives, u is common fixed point of A and P. Similarly, by taking $x = z$ and $y = u$ in (2), one can easily prove that $Bu = u = Qu$, that is u is common fixed point of B and Q, Therefore u is common fixed point of A,P,B and Q. The uniqueness of common fixed point is an easy consequence of inequality (2).

Taking $B = A$ and $P = Q$ in Theorem 2, we get following result:

Corollary 1 *Let A and P be two self-maps in complex-valued metric space (X,d) satisfying conditions:* (7)

$$\begin{aligned} d(Ax,Ay)^2 \preceq \phi(&d(Px,Ax)d(Py,Ay), d(Px,Ay)d(Py,Ax),\\ &d(Px,Ax)d(Px,Ay), d(Py,Ax)d(Py,Ay),\\ &d(Px,Py)^2, d(Px,Ax)d(Py,Ax),\\ &d(Py,Ay)d(Px,Ay), d(Px,Ay)d(Px,Ax),\\ &d(Px,Py)d(Py,Ax), d(Px,Py)d(Py,Ay)) \end{aligned}$$

(8) *(A,P) satisfies CLR_{AP} property.*
Then pair A and P has coincidence point in X. Further if pair (A,P) be weakly compatible pair then A and P have a unique common fixed point in X.

Now we attempt to relax $JCLR_{PQ}$ property by weaker condition CLR_{PQ} property.

Theorem 4. *Let A,B,P and Q be four self maps in complex-valued metric space (X,d) satisfying condition (2) of Theorem 1 and*
(9) *(A,P) and (B,Q) satisfies CLR_{PQ} property.*
Then pairs (A,P) and (B,Q) have coincidence point. Further if (A,P) and (B,Q) be weakly compatible pair then A,B,P and Q have a unique common fixed point in X.

Proof. As the pairs (A,P) and (B,Q) satisfy the CLR_{PQ} property, that is, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Px_n = lim_{n\to\infty}Qy_n = lim_{n\to\infty}By_n = z$ where $z \in P(X)\cap Q(X)$. Since $z \in X$, there exist a point $u \in X$ such that $Pu = z$. Firstly, we assert that $Au = Pu$. Suppose not, again by (2), we have

$$\begin{aligned} d(Au,By_n)^2 \preceq \phi(&d(Pu,Au)d(Qy_n,By_n),\\ &d(Pu,By_n)d(Qy_n,Au),\\ &d(Pu,Au)d(Pu,By_n), d(Qy_n,Au)d(Qy_n,By_n),\\ &d(Pu,Qy_n)^2, d(Pu,Au)d(Qy_n,Au),\\ &d(Qy_n,By_n)d(Pu,By_n), d(Pu,By_n)d(Pu,Au),\\ &d(Pu,Qy_n)d(Qy_n,Au), d(Pu,Qy_n)d(Qy_n,By_n)), \end{aligned}$$

taking $n \to \infty$, we get

$$\begin{aligned} d(Au,Pu)^2 &\preceq \phi(0,0,0,0,0, d(Pu,Au)^2, 0,0,0,0)\\ &\prec d(Au,Pu)^2, \end{aligned}$$

a contradiction, hence, $Au = Pu = z$ which shows that u is a coincidence point of the pair (A,P). Also, as $z \in QX$, there exist a point $v \in X$ such that $Qv = z$. Secondly, we assert that $Bv = Qv$. Suppose not, then by (2), we get

$$\begin{aligned} d(Au,Bv)^2 \preceq \phi(&d(Pu,Au)d(Qv,Bv),\\ &d(Pu,Bv)d(Qv,Au), d(Pu,Au)d(Pu,Bv),\\ &d(Qv,Au)d(Qv,Bv), d(Pu,Qv)^2,\\ &d(Pu,Au)d(Qv,Au),\\ &d(Qv,Bv)d(Pu,Bv), d(Pu,Bv)d(Pu,Au),\\ &d(Pu,Qv)d(Qv,Au), d(Pu,Qv)d(Qv,Bv)) \end{aligned}$$

$$\begin{aligned} d(Qv,Bv)^2 &\preceq \phi(0,0,0,0,0,0, d(Qv,Bv)^2, 0,0,0)\\ &\prec d(Qv,Bv)^2, \end{aligned}$$

a contradiction, hence, $Bv = Qv = z$ which shows that v is a coincidence point of the pair (B,Q). Thus, we have $z = Qv = Bv = Au = Pu$. Since the pairs (A,P) and (B,Q) are weakly compatible, this gives,

$$Az = APu = PAu = AAu = PPu = Pz,$$

$$Bz = BQv = QBv = QQv = BBv = Qz.$$

Finally, we assert that $Az = z$. Suppose not, then again by (2), we have

$$\begin{aligned} d(Az,Bv)^2 \preceq \phi(&d(Pz,Az)d(Qv,Bv),\\ &d(Pz,Bv)d(Qv,Az), d(Pz,Az)d(Pz,Bv),\\ &d(Qv,Az)d(Qv,Bv), d(Pz,Qv)^2,\\ &d(Pz,Az)d(Qv,Az),\\ &d(Qv,Bv)d(Pz,Bv), d(Pz,Bv)d(Pz,Az),\\ &d(Pz,Qv)d(Qv,Az), d(Pz,Qv)d(Qv,Bv)) \end{aligned}$$

$$\begin{aligned} d(Az,z)^2 &\preceq \phi(0, d(Az,z)^2, 0, 0, d(Az,z)^2, 0,0,0, d(Az,z)^2, 0)\\ &\prec d(Az,z)^2, \end{aligned}$$

a contradiction, hence, $Az = z = Pz$ which gives, z is common fixed point of A and P. Similarly, by taking $x = u$ and $y = z$ in (2), one can easily prove that $Bz = z = Qz$, that is z is common fixed point of B and Q. Therefore z is common fixed point of A,P,B and Q. The uniqueness of common fixed point is an easy consequence of inequality (2).

Now, we give example in support of our main result Theorem 3.

Example 6 *Let $X = \mathbb{C}$ and d be any complex-valued metric. Define self maps A, B, P and Q on X by $Az = \frac{z}{3}, Bz = \frac{-z}{3}, Pz = \frac{z}{6}, Qz = \frac{z}{2}$ for all $z \in X$. Then with sequences $\{x_n\} = \{\frac{1}{n}\}$ and $\{y_n\} = \{\frac{-1}{n}\}$ in X, $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Px_n$*
$= lim_{n\to\infty}Qy_n = lim_{n\to\infty}By_n = P(0)$.

This shows that the pairs (A,P) and (B,Q) share the common limit in the range of P property. Also, $AX \subset QX$ and $BX \subset PX$. Maps A, B, P and Q satisfy condition (2). Thus, the pairs (A,P) and (B,Q) satisfy all conditions of Theorem 3 and $z = 0$ is a common fixed point of A, B, P and Q.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] M. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions. J. Math. Anal. Appl., **270**, 181-188 (2002).

[2] A. Azam, B. Fisher and M. Khan, Common fixed point theorems in complex valued metric spaces, Numerical Functional Analysis and Optimization, **32**(3), 243-253 (2011).

[3] S. Chauhan, W. Sintunavarat and P. Kumam, Common fixed point theorems for weakly compatible mappings in fuzzy metric spaces using (JCLR) property, Applied Mathematics, **3**(9), 976-982 (2012).

[4] M. Imdad, B. D. Pant and S. Chauhan, Fixed point theorems in Menger spaces using the CLR_{ST} property and applications, Journal of Nonlinear Analysis and Optimization, **3**(2), 225-237 (2012).

[5] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci., **9**, 771-779 (1986).

[6] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc., **60**, 71-76 (1968).

[7] W. Lui, J. Wu and Z. Li, Common fixed points of single-valued and multivalued maps, Int. J. Math. Math. Sci, **19**, 3045-3055 (2005).

[8] S. Manro, S. S. Bhatia, S. Kumar and P. Kumam, Sumitra, Weakly compatible mappings along with CLR_S property in fuzzy metric spaces, Journal Nonlinear Analysis and Application, 1-12 (2013).

[9] S. Manro, Some common fixed point theorems in complex valued metric spaces using implicit relation, International Journal of Analysis and Applications, **2**(1), 62-70 (2013).

[10] S. Manro and K. Das, Some common fixed point theorems using CLRg property in Complex valued metric spaces, Gazi University Journal of Science, **27**(2), 735-738 (2014).

[11] H.K. Pathak, S.M. Kang and J.H. Baek, Weak compatible mappings of type (A) and common fixed points in Menger spaces, Comm. Korean Math. Soc., **10**(1), 67-83 (1995).

[12] K. P. R Sastry and G. A Naidu, Tadesse Bekeshie, Metrizability of complex valued metric spaces and remarks on fixed point theorems in complex valued metric spaces, International Journal of Mathematical Archive, **3**(7), 2686-2690 (2012).

[13] W. Sintunavarat and P. Kumam, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, J. Appl. Math., Volume 2011, Article ID 637958, 14 pages.

[14] W. Sintunavarat and P. Kumam, Generalized common fixed point theorems in complex valued metric spaces with applications, Journal of Inequalities and Applications, doi:10.1186/1029-242X-2012-84.

[15] R. K. Verma and H.K. Pathak, Common fixed point theorems using property (E.A) in complex valued metric spaces, Thai Journal of Mathematics, **11**(2), 347-355 (2013).

Harpreet Kaur having more than 8 years of teaching experience as a Lecturer. Currently purusing Ph.D. degree from Department of Mathematics, Desh Bhagat University, Mandigobindgarh, Punjab, India. She has published 04 research papers and 03 research papers are accepted in international journals. She has attended various national and international conferences. Her research interests are fixed point theory and applications.

Saurabh Manro received the PhD degree in Non-linear Analysis at Thapar University, Patiala (India). He is referee of several international journals in the frame of pure and applied mathematics. His main research interests are: fixed point theory, fuzzy mathematics, game theory, optimization theory, differential geometry and applications, geometric dynamics and applications.