Applied Mathematics & Information Sciences **5**(3) (2011), 635-654
– An International Journal

On the fine spectrum of the operator Δ_v over the sequence spaces c and $l_p, (1 < p < \infty)$

A. M. Akhmedov[1] and S. R. El-Shabrawy[2]

Baku State University, Faculty of Mech. & Math., Z. Khalilov Str., 23, AZ 1148, Baku, Azerbaijan

Email Address: [1]*akhmedovali@rambler.ru*, [2]*srshabrawy@yahoo.com*

Received Jan. 11, 2011; Revised April 20, 2011; Accepted July 02, 2011

The purpose of this paper is threefold: first to mainly review several recent results concerning the fine spectrum of the operator Δ_v over the sequence spaces c and l_p, where $1 < p < \infty$; second to provide some new results concerning the residual spectrum and the continuous spectrum of the operator Δ_v over the sequence spaces c and l_p; and third to modify the definition of the operator Δ_v and to determine the fine spectrum of the modified operator over the sequence spaces c and l_p, where $1 < p < \infty$. Also, it may be helpful to provide some comments and examples to support our results.

Keywords: Spectrum of an operator, Generalized difference operator, The sequence spaces c and l_p.

1 Preliminaries, background and notations

By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called a *sequence space*. We shall write $l_\infty,\ c,\ c_0$ and bv for the spaces of all bounded, convergent, null and bounded variation sequences, respectively. Also by l_1, l_p and bv_p we denote the spaces of all absolutely summable sequences, *p*-absolutely summable sequences and *p*-bounded variation sequences, respectively.

A triangle is a lower triangular matrix with all of the principal diagonal elements nonzero. Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N} = \{0, 1, 2, ...\}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A : \lambda \to \mu$ if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of *x*, is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \qquad (n \in \mathbb{N}). \tag{1.1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By (λ, μ), we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus,

$A \in (\lambda, \mu)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. We use the convention that any term with negative subscript is equal to naught.

We recall some basic concepts of spectral theory which are needed for our investigation [see 15, pp. 370-372].

Let X be a Banach space and $T : X \to X$ be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in X : y = Tx,\ x \in X\}.$$

By $B(X)$, we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T we associate the operator

$$T_\lambda = T - \lambda I, \tag{1.2}$$

where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse which is linear, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1}, \tag{1.3}$$

and call it the *resolvent operator* of T.

Many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ in the complex plane such that T_λ^{-1} exists. The boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ's the domain of T_λ^{-1} is dense in X, to name just a few aspects.

Definition 1.1. Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. A *regular value* λ of T is a complex number such that:
(R1) T_λ^{-1} exists,
(R2) T_λ^{-1} is bounded,
(R3) T_λ^{-1} is defined on a set which is dense in *X*.

The *resolvent set* of T, denoted by $\rho(T, X)$, is the set of all regular values λ of T. Its complement $\sigma(T, X) = \mathbb{C}\backslash\rho(T, X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The *point* (*discrete*) *spectrum* $\sigma_p(T, X)$ is the set such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T, X)$ is called an *eigenvalue* of *T*.

The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_λ^{-1} exists and satisfies (R3) but not (R2), that is, T_λ^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_λ^{-1} exists (and may be bounded or not) but does not satisfy (R3), that is, the domain of T_λ^{-1} is not dense in X.

Hence if $(T - \lambda I)x = \theta$ for some $x \neq \theta$, then $\lambda \in \sigma_p(T, X)$, by definition, that is, λ is an eigenvalue of T. The vector *x* is then called an *eigenvector* of T corresponding to the eigenvalue λ.

From now on, we should note that the index *p* has different meanings in the notation of the spaces l_p, l_p^* and the point spectrums $\sigma_p(\Delta_v, l_p)$, $\sigma_p(\Delta_v^*, l_p^*)$ which occur in theorems given in Sections 2 and 3.

Several authors have studied the spectrum and fine spectrum of linear operators defined by some particular limitation matrices over some sequence spaces. We summarize the knowledge in the existing literature concerned with the spectrum and the fine spectrum. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c has been studied by Altay and Başar [5]. Akhmedov and Başar [1,2] have studied the fine spectrum of the difference operator Δ over the sequence spaces l_p and bv_p, where $1 \leq p < \infty$. Note that the sequence space bv_p was studied by Başar and Altay [8] and Akhmedov and Başar [2]. Malafosse [17] has studied the spectrum and the fine spectrum of the difference operator Δ over the space s_r, where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by

$$\|x\|_{s_r} = \sup_{k \in \mathbb{N}} \frac{|x_k|}{r^k}, \qquad (r > 0).$$

The fine spectrum of the Zweier matrix operator Z^s over the sequence spaces l_1 and bv has been examined by Altay and Karakuş [7]. The fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces c_0 and c has been studied by Altay and Başar [6]. Also, the fine spectrum of the operator $B(r, s)$ over the sequence spaces l_p and bv_p, where $1 < p < \infty$ has been determined by Bilgiç and Furkan [9]. The fine spectrum of the generalized difference operator $B(r, s, t)$ over the sequence spaces c_0 and c has been studied by Furkan et al. [12]. Also, the fine spectrum of the operator $B(r, s, t)$ over the sequence spaces l_p and bv_p, where $1 < p < \infty$ has been determined by Furkan et al. [13]. The fine spectrum of the operator Δ_v over the sequence spaces c_0 and l_1 has been studied by Srivastava and Kumar [19,20]. Also, the fine spectrum of the operator Δ_v over the sequence space c has been examined by Akhmedov and El-Shabrawy [4]. Recently, El-shabrawy [11] has studied the fine spectrum of the operator Δ_v over the sequence space l_p, where $1 < p < \infty$. Panigrahi and Srivastava [18] have studied the fine spectrum of the generalized second order difference operator Δ_{uv}^2 over the sequence space c_0. The fine spectrum of the generalized difference operator $\Delta_{a,b}$ over the sequence spaces c has been studied by Akhmedov and El-Shabrawy [3].

Now, we may give:

Lemma 1.1. *(cf. [21, p. 6]). The matrix* $A = (a_{nk})$ *gives rise to a bounded linear operator*

$T \in B(c)$ from c to itself if and only if

1. *the rows of A are in l_1 and their l_1 norms are bounded,*
2. *the columns of A are in c,*
3. *the sequence of row sums of A is in c.*

The operator norm of T is the supremum of the l_1 norms of the rows.

Lemma 1.2. *[10, p. 253]. The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(l_1)$ from l_1 to itself if and only if the supremum of l_1 norms of the columns of A is bounded.*

Lemma 1.3. *[10, p. 245]. The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(l_\infty)$ from l_∞ to itself if and only if the supremum of l_1 norms of the rows of A is bounded.*

Lemma 1.4. *[14, p. 59]. T has a dense range if and only if T^* is one to one.*

The rest of this paper is organized as follows. Next, in Section 2 we mainly review several recent results concerning the fine spectrum of operator Δ_v over the sequence spaces c and l_p, where $1 < p < \infty$. Also, some new results are obtained. In Section 3 we modify the definition of the operator Δ_v and determine the fine spectrum of the modified operator over the sequence spaces c and l_p, where $1 < p < \infty$. Finally, Section 4 presents our conclusions.

2 The spectrum of the operator Δ_v on c and l_p, $1 < p < \infty$

The generalized difference operator Δ_v has been defined by Srivastava and Kumar [19]. The generalized difference operator Δ_v is represented by the matrix

$$\Delta_v = \begin{pmatrix} v_0 & 0 & 0 & \cdots \\ -v_0 & v_1 & 0 & \cdots \\ 0 & -v_1 & v_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.1}$$

where, the sequence (v_k) is assumed to be either constant or strictly decreasing sequence of positive real numbers satisfying

$$\lim_{k\to\infty} v_k = L > 0 \tag{2.2}$$

and

$$\sup_k v_k \le 2L. \tag{2.3}$$

Note That, if (v_k) is a constant sequence, say $v_k = L \neq 0$ for all $k \in \mathbb{N}$, then the operator Δ_v is reduced to the operator $B(r, s)$ with $r = L$, $s = -L$ and the results for the spectrum and fine spectrum of the operator Δ_v on the sequence spaces c and l_p follow immediately from the corresponding results in [6,9]. Then, throughout Sections 2.1 and 2.2, we consider only the case when the sequence (v_k) is assumed to be a strictly decreasing sequence of positive real numbers satisfying Conditions (2.2) and (2.3).

The contents of this section are divided into three subsections. In Sections 2.1 and 2.2 we mainly review several recent results concerning the fine spectrum of the operator Δ_v on the sequence spaces $c \text{ and } l_p$, where $1 < p < \infty$. Also, we provide some new results concerning the residual spectrum and the continuous spectrum of the operator Δ_v on the sequence spaces $c \text{ and } l_p$. Finally, in Section 2.3 we give comments with some detailed examples.

2.1 The spectrum of the operator Δ_v on c

Akhmedov and El-Shabrawy [4] have studied the fine spectrum of the operator Δ_v on the sequence space c with the additional condition that $v_0 \neq 2L$. In this subsection we summarize the main results.

The bounded linearity of the operator Δ_v on c is given by the following theorem.

Theorem 2.1. *[4, Theorem 2.1] The generalized difference operator $\Delta_v : c \to c$ is a bounded linear operator with the norm $\|\Delta_v\|_c = v_0 + v_1$.*

The spectrum of the operator Δ_v on c is given by the following theorem.

Theorem 2.2. *[4, Theorem 2.2] $\sigma(\Delta_v, c) = \{\lambda \in \mathbb{C} : |\lambda - L| \leq L\}$.*

The following theorem gives the point spectrum of the operator Δ_v on c.

Theorem 2.3. *[4, Theorem 2.3] $\sigma_p(\Delta_v, c) = \emptyset$.*

It is known that if $T : c \to c$ is a bounded linear operator with matrix A, then the adjoint operator $T^* : c^* \to c^*$ acting on $\mathbb{C} \oplus l_1$ has a matrix representation of the form

$$\begin{pmatrix} \chi & 0 \\ B & A^t \end{pmatrix},$$

where χ is the limit of the sequence of row sums of *A* minus the sum of the limit of the columns of A, and B is the column vector whose *k*-th entry is the limit of the *k*-th column of A for each $k \in \mathbb{N}$. For $\Delta_v : c \to c$, the matrix $\Delta_v^* \in B(l_1)$ is of the form

$$\Delta_v^* = \begin{pmatrix} 0 & 0 \\ 0 & \Delta_v^t \end{pmatrix}.$$

It should be noted that the dual space c^* of c is isomorphic to the Banach space l_1 of absolutely summable sequences normed by $\|x\|_{l_1} = \sum_k |x_k|$.

The results concerning the point spectrum of the adjoint operator Δ_v^* of Δ_v are given by the following theorem.

Theorem 2.4. *[4, Theorem 2.4]*

i. $\{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup \{0\} \subseteq \sigma_p(\Delta_v^*, c^*)$,

ii. $\left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\} \subseteq \sigma_p(\Delta_v^*, c^*)$,

iii. $\sigma_p(\Delta_v^*, c^*) \subseteq \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\} \cup \{0\}$.

The following theorem gives some results on the residual spectrum of the operator Δ_v on c.

Theorem 2.5. *[4, Theorem 2.5]*

i. $\{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup \{0\} \subseteq \sigma_r(\Delta_v, c)$,

ii. $\left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\} \subseteq \sigma_r(\Delta_v, c)$,

iii. $\sigma_r(\Delta_v, c) \subseteq \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\} \cup \{0\}$.

For the continuous spectrum of the operator Δ_v on c, we have the following theorem.

Theorem 2.6. *[4, Theorem 2.6]*

i. $\sigma_c(\Delta_v, c) \subseteq \{\lambda \in \mathbb{C} : |\lambda - L| = L\} \setminus \{0\}$,

ii. $\sigma_c(\Delta_v, c) \subseteq \left[\{\lambda \in \mathbb{C} : |\lambda - L| \leq L\} \cap \left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{\lambda - v_k}{v_k}\right| \geq 1\right\}\right] \setminus \{0\}$.

Now we give the following example:

Example 2.1. Consider the sequence (v_k), where $v_k = \frac{(k+3)^2}{(k+2)^2+(k+3)^2}$, $k \in \mathbb{N}$. Clearly, (v_k) is a strictly decreasing sequence of positive real numbers satisfying

$$\lim_{k\to\infty} v_k = L = \tfrac{1}{2} > 0, \text{ and}$$
$$\sup_k v_k = \tfrac{9}{13} \leq 1 = 2L.$$

We can prove that $1 \in \sigma_p(\Delta_v^*, c^*)$. But $1 \notin \{\lambda \in \mathbb{C} : |\lambda - L| < |L|\} \cup \{0\}$ and $1 \notin \left\{\lambda \in \mathbb{C} : \sup_n \left|\frac{v_n - \lambda}{v_n}\right| < 1\right\}$.

On the other hand if $v_k = \frac{k+3}{2k+5}$, $k \in \mathbb{N}$, then $1 \in \left\{\lambda \in \mathbb{C} : \inf_n \left|\frac{v_n - \lambda}{v_n}\right| < 1\right\} \cup \{0\}$ and $1 \notin \sigma_p(\Delta_v^*, c^*)$.

From Example 2.1, we see that the equalities in Theorem 2.4 do not hold in general. But we give the following theorem for the point spectrum of the adjoint operator Δ_v^*.

Theorem 2.7. $\sigma_p(\Delta_v^*, c^*) = \{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup H \cup \{0\}$, *where*

$$H = \left\{ \lambda \in \mathbb{C} : |\lambda - L| = L, \ \sum_{k=2}^{\infty} \left| \prod_{i=0}^{k-2} \frac{\lambda - v_i}{v_i} \right| < \infty \right\}.$$

Proof. Suppose that $\Delta_v^* f = \lambda f$ for $f = (f_0, f_1, f_2, ...) \neq \theta$ in $c^* \cong l_1$. Then, by solving the system of equations

$$\begin{aligned} &(0) f_0 = \lambda f_0, \\ &v_0 f_1 - v_0 f_2 = \lambda f_1, \\ &v_1 f_2 - v_1 f_3 = \lambda f_2, \\ &\quad \vdots \\ &v_{k-2} f_{k-1} - v_{k-2} f_k = \lambda f_{k-1}, \\ &\quad \vdots \end{aligned}$$

we obtain

$$f_k = \frac{v_{k-2} - \lambda}{v_{k-2}} f_{k-1},$$

for all $k \geq 2$. If $f_0 \neq 0$, then $\lambda = 0$. So, $\lambda = 0$ is an eigenvalue with the corresponding eigenvector $f = (f_0, 0, 0, 0, ...)$, that is, $\lambda = 0 \in \sigma_p(\Delta_v^*, c^*)$. It is clear that, for all $k \in \mathbb{N}$, the vector $f = (0, f_1, ..., f_{k+1}, 0, 0, ...)$ is an eigenvector of the operator Δ_v^* corresponding to the eigenvalue $\lambda = v_k$, where $f_1 \neq 0$ and $f_{n+1} = \frac{v_{n-1} - \lambda}{v_{n-1}} f_n$, for all $n = 1, 2, 3, ..., k$. Thus $\{v_k : k \in \mathbb{N}\} \subseteq \sigma_p(\Delta_v^*, c^*)$. On the other hand if $\lambda \neq v_k$ for all $k \in \mathbb{N}$ and $\lambda \neq 0$, then we can see that $\sum_k |f_k| < \infty$ if $\lim_{k\to\infty} \left| \frac{f_{k+1}}{f_k} \right| = \left| \frac{\lambda - L}{L} \right| < 1$. Also, it can be proved that $H \subseteq \sigma_p(\Delta_v^*, c^*)$. Thus

$$\{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup H \cup \{0\} \subseteq \sigma_p(\Delta_v^*, c^*).$$

Conversely, it is easy to prove that if $\lambda \in \sigma_p(\Delta_v^*, c^*)$, then $\lambda \in \{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup H \cup \{0\}$. This completes the proof. □

Example 2.2. If $v_k = \frac{(k+3)^2}{(k+2)^2 + (k+3)^2}$, $k \in \mathbb{N}$, then we can easily see that $1 \in H$ and so $1 \in \sigma_p(\Delta_v^*, c^*)$. On the other hand, if $v_k = \frac{k+3}{2k+5}$, then we have $1 \notin H$ and $1 \notin \sigma_p(\Delta_v^*, c^*)$.

Also, we give the following results for the residual spectrum and the continuous spectrum of the operator Δ_v on c.

Theorem 2.8. $\sigma_r(\Delta_v, c) = \sigma_p(\Delta_v^*, c^*)$.

Proof. The proof follows immediately from the definition of the residual spectrum and Lemma 1.4. □

Theorem 2.9. $\sigma_r(\Delta_v, c) = \{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup H \cup \{0\}$.

Proof. The proof follows immediately from Theorem 2.7 and Theorem 2.8. □

Theorem 2.10. $\sigma_c(\Delta_v, c) = \sigma(\Delta_v, c) \backslash \sigma_p(\Delta_v^*, c^*)$.

Proof. The proof follows immediately from Theorem 2.3 and Theorem 2.8. □

Theorem 2.11. $\sigma_c(\Delta_v, c) = \{\lambda \in \mathbb{C} : |\lambda - L| = L\} \setminus (H \cup \{0\})$.

Proof. The proof follows immediately from Theorem 2.2, Theorem 2.7 and Theorem 2.10. □

2.2 The spectrum of the operator Δ_v on l_p, $(1 < p < \infty)$

The fine spectrum of the operator Δ_v over the sequence space l_p, where $1 < p < \infty$ has been studied by El-Shabrawy [11]. In this subsection we summarize the main results.

Theorem 2.12. *[11, Theorem 2.1] The generalized difference opeartor $\Delta_v : l_p \to l_p$ is a bounded linear operator and $2^{\frac{1}{p}} v_0 \leq \|\Delta_v\|_{l_p} \leq 2v_0$.*

Theorem 2.13. *[11, Theorem 2.2]* $\sigma(\Delta_v, l_p) = \{\lambda \in \mathbb{C} : |\lambda - L| \leq L\}$.

Theorem 2.14. *[11, Theorem 2.3]* $\sigma_p(\Delta_v, l_p) = \emptyset$.

If $T : l_p \to l_p$, where $1 < p < \infty$ is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^* : l_p^* \to l_p^*$ is defined by the transpose of the matrix A. It is well-known that the dual space l_p^* of l_p is isomorphic to l_q with $p^{-1} + q^{-1} = 1$.

Theorem 2.15. *[11, Theorem 2.4]*

i. $\{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup \{v_0\} \subseteq \sigma_p(\Delta_v^*, l_p^*)$,

ii. $\left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\} \subseteq \sigma_p(\Delta_v^*, l_p^*)$,

iii. $\sigma_p(\Delta_v^*, l_p^*) \subseteq \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\}$.

Theorem 2.16. *[11, Theorem 2.5]* $\sigma_r(\Delta_v, l_p) = \sigma_p(\Delta_v^*, l_p^*)$.

Theorem 2.17. *[11, Theorem 2.6]*

i. $\{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup \{v_0\} \subseteq \sigma_r(\Delta_v, l_p)$,

ii. $\left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\} \subseteq \sigma_r(\Delta_v, l_p)$, *iii.* $\sigma_r(\Delta_v, l_p) \subseteq \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\}$.

Theorem 2.18. *[11, Theorem 2.7]* $\sigma_c(\Delta_v, l_p) = \sigma(\Delta_v, l_p) \backslash \sigma_p(\Delta_v^*, l_p^*)$.

Theorem 2.19. *[11, Theorem 2.8]*

i. $\sigma_c(\Delta_v, l_p) \subseteq \{\lambda \in \mathbb{C} : |\lambda - L| = L\} \setminus \{v_0\}$,

ii. $\{\lambda \in \mathbb{C} : |\lambda - L| \leq L\} \cap \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{\lambda - v_k}{v_k}\right| \geq 1\right\} \subseteq \sigma_c(\Delta_v, l_p)$.

Consider the following example:

Example 2.3. Let $p = 2$ and consider the sequence (v_k), where $v_k = \frac{k+3}{2k+5}$, $k \in \mathbb{N}$. Clearly, (v_k) is a strictly decreasing sequence of positive real numbers satisfying

$$\lim_{k\to\infty} v_k = L = \tfrac{1}{2} > 0, \text{ and}$$
$$\sup_k v_k = \tfrac{3}{5} \leq 1 = 2L.$$

We can prove that $1 \in \sigma_p(\Delta_v^*, l_2^*)$. But, $1 \notin \{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup \{v_0\}$ and $1 \notin \left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\}$.

On the other hand, if $v_k = k + 3 - \sqrt{k^2 + 5k + 6}$, $k \in \mathbb{N}$ then (v_k) is a strictly decreasing sequence of positive real numbers and

$$\lim_{k\to\infty} v_k = L = \tfrac{1}{2} > 0, \text{ and}$$
$$\sup_k v_k = 3 - \sqrt{6} \leq 1 = 2L.$$

We can prove that $1 \notin \sigma_p(\Delta_v^*, l_2^*)$ and $1 \in \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\}$.

From Example 2.3, we note that the equalities in Theorem 2.15 do not hold in general. But we can similarly, as in Section 2.1, prove the following new result for the point spectrum of the adjoint operator Δ_v^*.

Theorem 2.20. $\sigma_p(\Delta_v^*, l_p^*) = \{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup H_1$, *where*

$$H_1 = \left\{\lambda \in \mathbb{C} : |\lambda - L| = L, \ \sum_{k=1}^{\infty} \left|\prod_{i=0}^{k-1} \frac{\lambda - v_i}{v_i}\right|^q < \infty\right\}.$$

Example 2.4. Let $p = 2$ and consider the sequence (v_k), where $v_k = \frac{k+3}{2k+5}$, $k \in \mathbb{N}$. We can prove that $1 \in H_1$ and so $1 \in \sigma_p(\Delta_v^*, l_2^*)$. On the other hand, if $v_k = k+3-\sqrt{k^2 + 5k + 6}$, $k \in \mathbb{N}$ then $1 \notin H_1$ and $1 \notin \sigma_p(\Delta_v^*, l_2^*)$.

Also, as in Section 2.1, it can be proved that the residual spectrum and the continuous spectrum of the operator Δ_v are given by the following theorems.

Theorem 2.21. $\sigma_r(\Delta_v, l_p) = \{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup H_1$.

Theorem 2.22. $\sigma_c(\Delta_v, l_p) \subseteq \{\lambda \in \mathbb{C} : |\lambda - L| = L\} \backslash H_1$.

2.3 Comments on the operator Δ_v

In this subsection we are going to show some ideas about changing the conditions on the sequence (v_k) in the fine spectrum of the operator Δ_v.

If the sequence (v_k) is assumed to be a sequence of positive real numbers (not necessarily strictly decreasing) satisfying Conditions (2.2) and (2.3), then we can have results similar to those in Sections 2.1 and 2.2. This means that the condition that the sequence (v_k) is strictly decreasing is not an effective condition. In the following examples we see that although the sequence (v_k) is not strictly decreasing, the residual spectrum and the continuous spectrum in addition to the spectrum and the point spectrum of the operator Δ_v are exactly determined.

Example 2.5. Consider the sequence (v_k), where $v_k = \frac{(k+2)^2}{(k+2)^2+(k+3)^2}$, $k \in \mathbb{N}$. Clearly, (v_k) is a sequence of positive real numbers satisfying

$$\lim_{k\to\infty} v_k = L = \tfrac{1}{2} > 0, \text{ and}$$
$$\sup_k v_k = \tfrac{1}{2} \leq 1 = 2L.$$

Then, Conditions (2.2) and (2.3) are satisfied. We can prove that *the operator* $\Delta_v : c \to c$ *is a bounded linear operator with the norm* $\|\Delta_v\|_c = 1$ and

$\sigma(\Delta_v, c) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$.

$\sigma_p(\Delta_v, c) = \emptyset$.

$\sigma_p(\Delta_v^*, c^*) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\} \cup \{0\}$.

$\sigma_r(\Delta_v, c) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\} \cup \{0\}$.

$\sigma_c(\Delta_v, c) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| = \frac{1}{2}\right\} \setminus \{0\}$.

Example 2.6. Let $p = 2$ and consider the sequence (v_k), where $v_k = \frac{k+2}{2k+5}$, $k \in \mathbb{N}$. Clearly, (v_k) is a sequence of positive real numbers satisfying

$$\lim_{k\to\infty} v_k = L = \tfrac{1}{2} > 0, \text{ and}$$
$$\sup_k v_k = \tfrac{1}{2} \leq 1 = 2L.$$

Then, Conditions (2.2) and (2.3) are satisfied. We can prove that the operator $\Delta_v : l_p \to l_p$ *is a bounded linear operator with the norm* $\|\Delta_v\|_{l_p} = 1$ and

$\sigma(\Delta_v, l_p) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$.

$\sigma_p(\Delta_v, l_p) = \emptyset$.

$\sigma_p(\Delta_v^*, l_p^*) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\}$.

$\sigma_r(\Delta_v, l_p) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\}$.

$\sigma_c(\Delta_v, l_p) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| = \frac{1}{2}\right\}$.

3 The spectrum of the modified operator Δ_v on c and $l_p,\ 1 < p < \infty$

In this section we modify the definition of the operator Δ_v, which is represented by the matrix

$$\Delta_v = \begin{pmatrix} v_0 & 0 & 0 & \cdots \\ -v_0 & v_1 & 0 & \cdots \\ 0 & -v_1 & v_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

by dropping the condition that the sequence (v_k) is strictly decreasing sequence of positive real numbers and replacing Condition (2.3) by another condition. That is, throughout this section, the sequence (v_k) is assumed to be a sequence of nonzero real numbers which is either constant or satisfying the conditions

$$\lim_{k\to\infty} v_k = L > 0 \tag{3.1}$$

and

$$\sup_k v_k \leq L. \tag{3.2}$$

We should indicate the reader that we use the same symbol for the operator Δ_v and its modification here, since they have the same matrix representation and the difference between them lies in the conditions on the sequence (v_k).

In this section we determine the spectrum, the point spectrum, the residual spectrum and the continuous spectrum of the modified operator Δ_v on the sequence spaces c and l_p, where $1 < p < \infty$.

3.1 The spectrum of the modified operator Δ_v on $l_p, 1 < p < \infty$

We begin with a theorem concerning the bounded linearity of the operator Δ_v on the sequence space l_p, where $1 < p < \infty$.

Theorem 3.1. *The operator $\Delta_v : l_p \to l_p$ is a bounded linear operator satisfying the inequalities*

$$2^{\frac{1}{p}} \sup_k |v_k| \leq \|\Delta_v\|_{l_p} \leq 2 \sup_k |v_k| \ .$$

Proof. The linearity of Δ_v is trivial and so is omitted. Let us take any $x = (x_k) \in l_p$.

Then, using Minkowski's inequality, we have

$$\begin{aligned}\|\Delta_v x\|_{l_p} &= \left(\sum_k |v_k x_k - v_{k-1}x_{k-1}|^p\right)^{\frac{1}{p}}\\ &\le \left(\sum_k |v_k x_k|^p\right)^{\frac{1}{p}} + \left(\sum_k |v_{k-1}x_{k-1}|^p\right)^{\frac{1}{p}}\\ &\le \sup_k |v_k| \left(\sum_k |x_k|^p\right)^{\frac{1}{p}} + \sup_k |v_k| \left(\sum_k |x_k|^p\right)^{\frac{1}{p}}\\ &= 2\left(\sup_k |v_k|\right)\|x\|_{l_p}.\end{aligned}$$

Then

$$\|\Delta_v\|_{l_p} \le 2\sup_k |v_k|.$$

Now, for each $k \in \mathbb{N}$, let $y = (y_n)$ be the sequence such that $y_k = 1$ and $y_n = 0$ for all $n \in \mathbb{N}\setminus\{k\}$. Then, for each $k \in \mathbb{N}$, we have

$$\|\Delta_v\|_{l_p} \ge \frac{\|\Delta_v y\|_{l_p}}{\|y\|_{l_p}} = (2|v_k|^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}|v_k|.$$

Thus

$$\|\Delta_v\|_{l_p} \ge 2^{\frac{1}{p}}\sup_k |v_k|.$$

This completes the proof. □

Now, we give the following lemma which is required in the proof of the next theorem.

Lemma 3.1. *[16, p. 174]. Let $1 < p < \infty$ and suppose $A \in (l_\infty, l_\infty) \cap (l_1, l_1)$. Then $A \in (l_p, l_p)$.*

Theorem 3.2. *Let $D = \{\lambda \in \mathbb{C} : |\lambda - L| \le |L|\}$ and $E = \{v_k : k \in \mathbb{N},\ |v_k - L| > |L|\}$. Then $\sigma(\Delta_v, l_p) = D \cup E$.*

Proof. First, we prove that $(\Delta_v - \lambda I)^{-1}$ exists and is in $B(l_p)$ for $\lambda \notin D \cup E$ and next the operator $\Delta_v - \lambda I$ is not invertible for $\lambda \in D \cup E$.

Let $\lambda \notin D \cup E$. Then, $|\lambda - L| > |L|$ and $\lambda \ne v_k$ for all $k \in \mathbb{N}$. So, $\Delta_v - \lambda I$ is triangle, and hence $(\Delta_v - \lambda I)^{-1}$ exists. We can calculate that

$$(\Delta_v - \lambda I)^{-1} = \begin{pmatrix} \frac{1}{(v_0-\lambda)} & 0 & 0 & \cdots\\ \frac{v_0}{(v_0-\lambda)(v_1-\lambda)} & \frac{1}{(v_1-\lambda)} & 0 & \cdots\\ \frac{v_0 v_1}{(v_0-\lambda)(v_1-\lambda)(v_2-\lambda)} & \frac{v_1}{(v_1-\lambda)(v_2-\lambda)} & \frac{1}{(v_2-\lambda)} & \cdots\\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then, the supremum of the l_1 norms of the columns of $(\Delta_v - \lambda I)^{-1}$ is $\sup_k R_k$, where

$$R_k = \frac{1}{|v_k - \lambda|} + \frac{|v_k|}{|v_k - \lambda|\,|v_{k+1} - \lambda|} + \frac{|v_k|\,|v_{k+1}|}{|v_k - \lambda|\,|v_{k+1} - \lambda|\,|v_{k+2} - \lambda|} + ... \ , \qquad k \in \mathbb{N}.$$

Since $\lim_{k\to\infty} \left|\frac{v_k}{v_{k+1}-\lambda}\right| = \left|\frac{L}{L-\lambda}\right| < 1$, then there exist $k_0 \in \mathbb{N}$ and $q_0 < 1$ such that $\left|\frac{v_k}{v_{k+1}-\lambda}\right| < q_0$ for all $k \geq k_0$. Then, for each $k \geq k_0 + 1$,

$$R_k \leq \frac{1}{|v_k - \lambda|}\left[1 + q_0 + q_0^2 + \ ... \ \right].$$

But, there exist $k_1 \in \mathbb{N}$ and a real number $q_1 < \frac{1}{|L|}$ such that $\frac{1}{|v_k - \lambda|} < q_1$ for all $k \geq k_1$. Then,

$$R_k \leq \frac{q_1}{1 - q_0},$$

for all $k > \max\{k_0, k_1\}$. Thus $\sup_k R_k < \infty$. This shows that $(\Delta_v - \lambda I)^{-1} \in (l_1, l_1)$. Similarly, we can prove that $(\Delta_v - \lambda I)^{-1} \in (l_\infty, l_\infty)$ and so $(\Delta_v - \lambda I)^{-1} \in (l_1, l_1) \cap (l_\infty, l_\infty)$. By Lemma 3.1, $(\Delta_v - \lambda I)^{-1} \in (l_p, l_p)$. This shows that $\sigma(\Delta_v, l_p) \subseteq D \cup E$.

Conversely, suppose that $\lambda \notin \sigma(\Delta_v, l_p)$. Then $(\Delta_v - \lambda I)^{-1} \in B(l_p)$. Since $(\Delta_v - \lambda I)^{-1}$-transform of the unit sequence $e_1 = (1,\ 0,\ 0,\ ...)$ is in l_p, we have $\lim_{k\to\infty} \left|\frac{v_k}{v_{k+1}-\lambda}\right|^p = \left|\frac{L}{L-\lambda}\right|^p \leq 1$ and $\lambda \neq v_k$ for all $k \in \mathbb{N}$. Then $\{\lambda \in \mathbb{C} : |\lambda - L| < |L|\} \subseteq \sigma(\Delta_v, l_p)$ and $\{v_k : k \in \mathbb{N}\} \subseteq \sigma(\Delta_v,\ l_p)$. But, $\sigma(\Delta_v, l_p)$ is compact set, and so it is closed. Then $D = \{\lambda \in \mathbb{C} : |\lambda - L| \leq |L|\} \subseteq \sigma(\Delta_v, l_p)$ and $E = \{v_k : k \in \mathbb{N},\ |v_k - L| > |L|\} \subseteq \sigma(\Delta_v, l_p)$. This completes the proof. □

The point spectrum of the operator Δ_v is given by the following theorem.

Theorem 3.3. $\sigma_p(\Delta_v, l_p) = E$.

Proof. Suppose $\Delta_v x = \lambda x$ for $x \neq \theta = (0,\ 0,\ 0,\ ...)$ in l_p. Then by solving the system of equations

$$\left.\begin{array}{c} v_0 x_0 = \lambda x_0 \\ -v_0 x_0 + v_1 x_1 = \lambda x_1 \\ -v_1 x_1 + v_2 x_2 = \lambda x_2 \\ \vdots \end{array}\right\}$$

we obtain

$$(v_0 - \lambda)x_0 = 0 \text{ and } -v_k x_k + (v_{k+1} - \lambda)x_{k+1} = 0, \text{ for all } k \in \mathbb{N}..$$

Hence, for all $\lambda \notin \{v_k : k \in \mathbb{N}\}$, we have $x_k = 0$ for all $k \in \mathbb{N}$, which contradicts our assumption. So, $\lambda \notin \sigma_p(\Delta_v, l_p)$. This shows that $\sigma_p(\Delta_v, l_p) \subseteq \{v_k : k \in \mathbb{N}\}$. Also, if $\lambda = L$, then we can easily prove that $\lambda \notin \sigma_p(\Delta_v, l_p)$. Thus $\sigma_p(\Delta_v, l_p) \subseteq \{v_k : k \in \mathbb{N}\} \setminus \{L\}$. Now, we will prove that

$$\lambda \in \sigma_p(\Delta_v, l_p) \text{ if and only if } \lambda \in E.$$

If $\lambda \in \sigma_p(\Delta_v, l_p)$, then $\lambda = v_j \neq L$ for some $j \in \mathbb{N}$ and there exists $x \in l_p,\ x \neq \theta$ such that $\Delta_v x = v_j x$. Then

$$\lim_{k\to\infty} \left|\frac{x_{k+1}}{x_k}\right|^p = \left|\frac{L}{L - v_j}\right|^p \leq 1.$$

But $\left|\frac{L}{L-v_j}\right|^p \neq 1$. Then $\lambda = v_j \in \{v_k : k \in \mathbb{N},\ |v_k - L| > |L|\} = E$. Thus $\sigma_p(\Delta_v, l_p) \subseteq E$.

Conversely, let $\lambda \in E$. Then there exists $j \in \mathbb{N}$, $\lambda = v_j \neq L$ and

$$\lim_{k\to\infty} \left|\frac{x_{k+1}}{x_k}\right|^p = \left|\frac{L}{L - v_j}\right|^p < 1,$$

that is, $x \in l_p$. Thus $E \subseteq \sigma_p(\Delta_v, l_p)$. This completes the proof. □

We give the following lemma which is required in the proof of the next theorem.

Lemma 3.2. *Let* $1 < p < \infty$ *and* $\lambda \in \{\lambda \in \mathbb{C} : |\lambda - L| = |L|\}$. *Then the series*

$$\sum_k \left|\frac{(v_0 - \lambda)(v_1 - \lambda)\ \dots\ (v_{k-1} - \lambda)}{v_0 v_1\ \dots\ v_{k-1}}\right|^p,$$

is not convergent series.

Proof. Let $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$ such that $|\lambda - L| = |L|$. Then

$$|\lambda|^2 = \lambda_1^2 + \lambda_2^2 = 2\lambda_1 L.$$

Also,

$$\begin{aligned} |v_k - \lambda|^2 &= (v_k - \lambda_1)^2 + \lambda_2^2 \\ &= v_k^2 + (\lambda_1^2 + \lambda_2^2) - 2\lambda_1 v_k \\ &= v_k^2 - 2\lambda_1(v_k - L) \\ &\geq v_k^2. \end{aligned}$$

Therefore

$$\left|\frac{v_k - \lambda}{v_k}\right| \geq 1, \text{ for all } k \in \mathbb{N}.$$

This completes the proof. □

Theorem 3.4. $\sigma_p(\Delta_v^*, l_p^*) = \{\lambda \in \mathbb{C} : |\lambda - L| < |L|\} \cup \{v_k : k \in \mathbb{N}\}$.

Proof. Suppose that $\Delta_v^* f = \lambda f$ for $f = (f_0, f_1, f_2, ...) \neq \theta$ in $l_p^* \cong l_q$, where $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. Then, by solving the system of equations

$$\begin{aligned} v_0 f_0 - v_0 f_1 &= \lambda f_0, \\ v_1 f_1 - v_1 f_2 &= \lambda f_1, \\ &\vdots \\ v_k f_k - v_k f_{k+1} &= \lambda f_k, \\ &\vdots \end{aligned}$$

we obtain

$$f_{k+1} = \frac{v_k - \lambda}{v_k} f_k, \ k \in \mathbb{N}.$$

Then $f_0 \neq 0$, since $f \neq \theta$.

It is clear that, for all $k \in \mathbb{N}$, the vector $f = (f_0, f_1, ..., f_k, 0, 0, ...)$ is an eigenvector of the operator Δ_v^* corresponding to the eigenvalue $\lambda = v_k$, where $f_0 \neq 0$ and $f_n = \frac{v_{n-1}-\lambda}{v_{n-1}} f_{n-1}$, for all $1 \leq n \leq k$. Thus $\{v_k : k \in \mathbb{N}\} \subseteq \sigma_p(\Delta_v^*, l_p^*)$. Also, if $\lambda \neq v_k$ for all $k \in \mathbb{N}$, then $f_k \neq 0$ for all $k \in \mathbb{N}$, and so, $\sum_k |f_k|^q < \infty$ if $\lim_{k\to\infty} \left|\frac{f_{k+1}}{f_k}\right|^q = \left|\frac{\lambda - L}{L}\right|^q < 1$. Thus $\{\lambda \in \mathbb{C} : |\lambda - L| < |L|\} \cup \{v_k : k \in \mathbb{N}\} \subseteq \sigma_p(\Delta_v^*, l_p^*)$.

Conversely, if $\lambda \in \sigma_p(\Delta_v^*, l_p^*)$, then there exists $f = (f_0, f_1, f_2, ...) \neq \theta$ in $l_p^* \cong l_q$, $\Delta_v^* f = \lambda f$. Then, $f_{k+1} = \frac{v_k - \lambda}{v_k} f_k, \ k \in \mathbb{N}$ and $\sum_k |f_k|^q < \infty$. Therefore $\lim_{k\to\infty} \left|\frac{f_{k+1}}{f_k}\right|^q = \left|\frac{\lambda - L}{L}\right|^q < 1$ or $\lambda \in \{v_k : k \in \mathbb{N}\}$ (note that $|L - \lambda| = |L|$ contradicts with $\sum_k |f_k|^q < \infty$, by using Lemma 3.2). This completes the proof. □

Theorem 3.5. $\sigma_r(\Delta_v, l_p) = \sigma_p(\Delta_v^*, l_p^*) \backslash \sigma_p(\Delta_v, l_p)$.

Proof. The proof follows immediately from the definition of the residual spectrum and Lemma 1.4. □

Theorem 3.6. $\sigma_r(\Delta_v, l_p) = \{\lambda \in \mathbb{C} : |\lambda - L| < |L|\}$.

Proof. The proof follows immediately from Theorems 3.3, 3.4 and 3.5. □

Theorem 3.7. $\sigma_c(\Delta_v, l_p) = \sigma(\Delta_v, l_p) \backslash \sigma_p(\Delta_v^*, l_p^*)$.

Proof. The proof follows immediately from Theorems 3.2, 3.3 and 3.5. □

Theorem 3.8. $\sigma_c(\Delta_v, l_p) = \{\lambda \in \mathbb{C} : |\lambda - L| = |L|\}$.

Proof. The proof follows immediately from Theorems 3.2, 3.3 and 3.6. □

Combining Theorems 3.1, 3.2, 3.3, 3.4, 3.6 and 3.8, we can have the following main theorem:

Theorem 3.9. *1. The operator $\Delta_v : l_p \to l_p$ is a bounded linear operator and*

$$2^{\frac{1}{p}} \sup_k |v_k| \le \|\Delta_v\|_{l_p} \le 2 \sup_k |v_k| .$$

2. $\sigma(\Delta_v, l_p) = D \cup E$.

3. $\sigma_p(\Delta_v, l_p) = E$.

4. $\sigma_p(\Delta_v^*, l_p^*) = \{\lambda \in \mathbb{C} : |\lambda - L| < |L|\} \cup E$.

5. $\sigma_r(\Delta_v, l_p) = \{\lambda \in \mathbb{C} : |\lambda - L| < |L|\}$.

6. $\sigma_c(\Delta_v, l_p) = \{\lambda \in \mathbb{C} : |\lambda - L| = |L|\}$.

We note that Condition (3.2) is important to be satisfied for the modified operator Δ_v. If Condition (3.2) is not satisfied, then we can see that some of the results in this section can not be applied in that context. Consider the following example.

Example 3.1. Let $p = 2$ and consider the sequence (v_k) such that $v_k = \frac{k+3}{2k+5}$, $k \in \mathbb{N}$. Clearly, (v_k) is a sequence of nonzero real numbers satisfying

$$\lim_{k\to\infty} v_k = L = \tfrac{1}{2} > 0, \text{ and}$$
$$\sup_k v_k = \tfrac{3}{5} > \tfrac{1}{2} = L.$$

Then, Condition (3.2) is not satisfied. We can easily prove that $1 \in \sigma_p(\Delta_v^*, l_2^*)$ and $1 \notin \{\lambda \in \mathbb{C} : |\lambda - L| < |L|\} \cup E$.

3.2 The spectrum of the modified operator Δ_v on c

The point spectrum of the adjoint operator Δ_v^* is given by the following theorem.

Theorem 3.10. $\sigma_p(\Delta_v^*, c^*) = \{\lambda \in \mathbb{C} : |\lambda - L| < |L|\} \cup E \cup \{0\}$.

Proof. Suppose that $\Delta_v^* f = \lambda f$ for $f = (f_0, f_1, f_2, ...) \neq \theta$ in $c^* \cong l_1$. Then, by solving the system of equations

$$\begin{aligned}
(0)\, f_0 &= \lambda f_0, \\
v_0 f_1 - v_0 f_2 &= \lambda f_1, \\
v_1 f_2 - v_1 f_3 &= \lambda f_2, \\
&\vdots \\
v_{k-2} f_{k-1} - v_{k-2} f_k &= \lambda f_{k-1}, \\
&\vdots
\end{aligned}$$

we obtain that

$$(0)\, f_0 = \lambda f_0 \text{ and } f_k = \frac{v_{k-2} - \lambda}{v_{k-2}} f_{k-1}, \;\; k \geq 2.$$

If $f_0 \neq 0,$ then $\lambda = 0.$ So, $\lambda = 0$ is an eigenvalue with the corresponding eigenvector $f = (f_0, 0, 0, ...)$, that is, $\lambda = 0 \in \sigma_p(\Delta_v^*, c^*)$. If $\lambda \neq 0$, then $f_0 = 0$ and so, using arguments similar to those in the proof of Theorem 3.4 one can see that $f \in l_1$. This completes the proof.

□

Since the spectrum of the operator Δ_v on the sequence space c can be obtained by arguments similar to those used in the case of the space l_p, where $1 < p < \infty$, we omit the details and give the results without proof.

Theorem 3.11.

1. *The operator* $\Delta_v : c \to c$ *is a bounded linear operator with the norm* $\|\Delta_v\|_c = \sup_k (|v_k| + |v_{k-1}|)$.
2. $\sigma(\Delta_v, c) = D \cup E.$
3. $\sigma_p(\Delta_v, c) = E.$
4. $\sigma_r(\Delta_v, c) = \{\lambda \in \mathbb{C} : |\lambda - L| < |L|\} \cup \{0\}.$
5. $\sigma_c(\Delta_v, c) = \{\lambda \in \mathbb{C} : |\lambda - L| = |L|\} \setminus \{0\}.$

4 Conclusion

In Section 2, we have considered the operator Δ_v which has been introduced by Srivastava and Kumar [19] and has been studied over the sequence space c by Akhmedov and El-Shabrawy [4] and over the sequence space l_p by El-Shabrawy [11]. We have summarized the main recent results concerning the fine spectrum of the operator Δ_v over the sequence spaces c and l_p**,** where $1 < p < \infty$. Also, we give some new results for the residual spectrum and the continuous spectrum of the operator Δ_v over the sequence spaces c and l_p**,** where $1 < p < \infty$. We note that, in Sections 2.1 and 2.2, the point spectrum of the adjoint operator Δ_v^*, and consequently the residual spectrum and the continuous spectrum of the operator Δ_v over the sequence spaces c and l_p, are not exactly determined as in the case of the operators $B(r, s)$ and Δ (*cf.*[1,5,6,9]). Also, we have shown that the condition that the sequence (v_k) is a strictly decreasing is not an effective condition. So, *In Section 3,* we have modified the definition of the operator Δ_v by dropping the condition that the sequence (v_k) is a strictly decreasing sequence of positive real numbers and replacing

Condition (2.3) by another condition. The modified operator Δ_v can be considered as a generalization of the difference operator Δ. We have determined the spectrum, the point spectrum, the residual spectrum and the continuous spectrum of the modified operator Δ_v over the sequence spaces c and l_p, where $1 < p < \infty$ in simple forms. Also, it should be noted that, part of the value of the modification of the operator Δ_v lies in the fact that the obtained results impove some of the corresponding results in [4,11].

Finally, we note that the spectrum of several special limitation matrices over the sequence spaces c and l_p is a region enclosed by a circle. It is interesting that the spectrum of the modified operator Δ_v over the sequence spaces c and l_p may include also a finite number of points outside the region enclosed by a circle. Also, we may have $\sigma_p(\Delta_v, c) \neq \emptyset$ and $\sigma_p(\Delta_v, l_p) \neq \emptyset$. Nevertheless, the point spectrum of several limitation matrices over the sequence spaces c and l_p is the empty set (*cf.*[[1], [4], [5], [6], [9], [11], [12], [13]]).

References

[1] A.M. Akhmedov, F. Başar, On the fine spectra of the difference operator Δ over the sequence space l_p, $(1 \leq p < \infty)$, Demonstratio Math. 39 (3) (2006) 585-595.

[2] A.M. Akhmedov, F. Başar, The fine spectra of the difference operator Δ over the sequence space bv_p, $(1 \leq p < \infty)$, Acta Math. Sin. (Engl. Ser.) 23 (2007) 1757-1768.

[3] A.M. Akhmedov, S.R. El-Shabrawy, On the fine spectrum of the operator $\Delta_{a,b}$ over the sequence space c, Comput. Math. Appl. 61 (2011) 2994-3002.

[4] A.M. Akhmedov, S.R. El-Shabrawy, The spectrum of the generalized lower triangle double-band matrix Δ_a over the sequence space c, Al-Azhar Univ. Eng. J., JAUES (special issue), Dec. 2010, 5 (9), 54-63.

[5] B. Altay, F. Başar, On the fine spectrum of the difference operator Δ on c_0 and c, Inform. Sci. 168 (2004) 217-224.

[6] B. Altay, F. Başar, On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces c_0 and c, Int. J. Math. Math. Sci. 18 (2005) 3005-3013.

[7] B. Altay, M. Karakuş, On the spectrum and the fine spectrum of the Zweier matrix as an operator on some sequence spaces, Thai J. Math. 3 (2) (2005) 153-162.

[8] F. Başar, B. Altay, On the space of sequences of *p*-bounded variation and related matrix mappings, Ukrainian Math. J. 55 (1) (2003) 136-147.

[9] H. Bilgiç, H. Furkan, On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces l_p and bv_p, $(1 < p < \infty)$, Nonlinear Anal. 68 (2008) 499-506.

[10] B. Choudhary, S. Nanda, Functional Analysis with Applications, John Wiley & Sons Inc., New York, Chichester, Brisbane, Toronto, Singapore, 1989.

[11] S.R. El-Shabrawy, On the spectrum of the operator Δ_v over the space l_p, $(1 < p < \infty)$, Baku Univ. News J., Phys. Math. Sci. Ser., to appear.

[12] H. Furkan, H. Bilgiç, B. Altay, On the fine spectrum of the operator $B(r, s, t)$ over c_0 and c, Comput. Math. Appl. 53 (2007) 989-998.

[13] H. Furkan, H. Bilgiç, F. Başar, On the fine spectrum of the operator $B(r, s, t)$ over the sequence spaces l_p and bv_p, $(1 < p < \infty)$, Comput. Math. Appl. 60 (2010) 2141-2152.

[14] S. Goldberg, Unbounded Linear Operators, Dover Publications, Inc., New York, 1985.

[15] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley & Sons Inc., New York, Chichester, Brisbane, Toronto, 1978.

[16] I. J. Maddox, Elements of Functional Analysis, Cambridge Univ. Press, London, 1970.

[17] B. De Malafosse, Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ, Hokkaido Math. J. 31 (2002) 283–299.

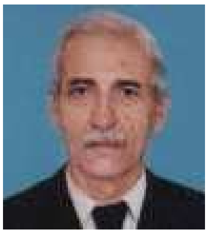
[18] B.L. Panigrahi, P.D. Srivastava, Spectrum and fine spectrum of generalized second order difference operator Δ^2_{uv} on sequence space c_0, Thai J. Math. 9 (1) (2011) 57–74.

[19] P.D. Srivastava, S. Kumar, On the fine spectrum of the generalized difference operator Δ_v over the sequence space c_0, Commun. Math. Anal. 6 (1) (2009) 8-21.

[20] P.D. Srivastava, S. Kumar, Fine spectrum of the generalized difference operator Δ_v on sequence space l_1, Thai J. Math. 8 (2) (2010) 221–233.

[21] A. Wilansky, Summability Through Functional Analysis, in: North-Holland Mathematics Studies, vol. 85, North-Holland, Amsterdam, 1984.

.



Ali M. Akhmedov is Professor of Mathematics and Head of the department of ”Theory of functions and functional analysis”, Mechanics and Mathematics Faculty at Baku State University since 1995. He has obtained his B.Sc. with honors and M.Sc. from Mathematics Faculty of Azerbaijan State University (now Baku State University) in 1961 and 1966, respectively. He has received his Ph.D. from the same university in 1970. He has obtained the Doctor of Science in 1994 from Baku State University. His research interests include Functional Analysis, Spectral Theory of Operator and Differential Equations on Complex Domain.

Saad R. El-Shabrawy has obtained his B.Sc. and M.Sc. from Mansoura University (Damietta Branch), Faculty of Science, New Damietta, Egypt in 2002 and 2007, respectively. He is working toward the Ph.D. at Baku state University, Mechanics and Mathematics Faculty, Baku, Azerbaijan. His research interests include Functional Analysis and Topology.