Riemann-Liouville Fractional Derivative and Application to Model Chaotic Differential Equations

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Abstract: In this work, the stability analysis and numerical treatment of chaotic time-fractional differential equations are considered. The classical system of ordinary differential equations with initial conditions is generalized by replacing the first-order time derivative with the Riemann-Liouville fractional derivative of order α, for 0 < α ≤ 1. In the numerical experiments, we observed that analysis of pattern formation in time-fractional coupled differential equations at some parameter value is almost similar to a classical process. A range of chaotic systems with current and recurrent interests which have many applications in biology, physics and engineering are taken to address any points and queries that may naturally occur.

Keywords: Chaotic systems, fractional calculus, numerical simulations, Riemann-Liouville derivative.

1 Introduction

Fractional calculus has a long history [1–3]. It is seen as the generalization of ordinary differentiation and integration to non-integer orders. Over the years, fractional calculus was found to play an important role in the modeling of a considerable number of real-life or physical phenomena; for instance, the predator-prey dynamic [4, 5], the modeling of memory-dependent and complex media such as porous media [6–9] and references therein. It has emerged as an essential tool for the study of dynamical systems where standard methods are not effective with strong limitations. A lot of research findings have revealed that majority of models that are based only on the integer (classical) order derivatives do not provide enough information to describe the complexity of such phenomena, on the basis of their mathematical and physical considerations. Some system of equations present delays which may be finite, infinite, or state-dependent. Others are subject to an impulsive effect. This paper aims to capture a wide reader of specialists such as biologist, economist, engineers, mathematicians as well as physicists on the application of fractional calculus to address various nonlinear problems.

A general time-fractional differential equation of order α is given by

$$\varphi_n \mathcal{D}_t^\alpha u(t) + \cdots + \varphi_1 \mathcal{D}_t^{\alpha_1} u(t) + \varphi_0 \mathcal{D}_t^{\alpha_0} u(t) = 0$$

(1)

where \(\varphi_k \in \mathbb{R}, \delta = 0, 1, 2, \ldots, n\) are the coefficients of the differential equations. Without loss of generality, we let \(\alpha_n > \alpha_{n-1} > \alpha_{n-2} > \cdots > \alpha_0 \geq 0\). Following [3], we present the analytical solution of (1) in the general form

$$u(t) = \frac{1}{\varphi_n} \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \sum_{\delta_0 + \delta_1 + \cdots + \delta_{n-2} = k} (k; \delta_0, \delta_1, \delta_2, \ldots, \delta_{n-2})$$

$$\times \prod_{j=0}^{n-2} \left( \frac{\varphi_j}{\varphi_n} \right) \chi_k \left( t, -\frac{\varphi_{n-1}}{\varphi_n}; \alpha_0 - \alpha_{n-1}, \alpha_n + \sum_{j=0}^{n-2} (\alpha_{n-1} - \alpha_j) \delta_j + 1 \right),$$

(2)

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where \((k; \delta_0, \delta_1, \delta_2, \ldots, \delta_{n-2})\) are multinomial coefficients and the term \(\chi(t; \lambda; \xi, \tau)\) denotes the Mittag-Leffler function type, bear in mind that \(\delta_0 \geq 0; \delta_1 \geq 0; \ldots, \delta_{n-2} \geq 0\), as suggested by Podlubny [3, 10]. This function is given by

\[
\chi(t; \lambda; \xi, \tau) = t^\xi \Gamma(t) E_{\xi, \tau}^\delta \left( \lambda t^\delta \right), \quad \text{for } \delta = 0, 1, 2, \ldots
\]

where \(E_{\xi, \tau}^\delta(v)\) is the \(\delta\)–th derivative of two parameters Mittag-Leffler function [11]

\[
E_{\xi, \tau}^\delta(v) = \sum_{i=0}^{\infty} \frac{(i+\delta)! v^i}{\Gamma(i+\xi+\tau) \cdot i!}, \quad \text{for } \delta = 0, 1, 2, \ldots
\]

The Laplace transform of function \(\chi(t; \lambda; \alpha, \eta)\) is defined by

\[
\mathcal{L} \{\chi(t; \pm \lambda; \alpha, \eta)\} = \frac{\delta! y^{\alpha-\eta}}{(\alpha \pm \lambda)^{\delta+1}}.
\]

for \(y > |\lambda|^{1/\alpha}\). Readers are referred to [3, 12] for a list of some useful Laplace transforms and their inverse functions.

The non-local nature of fractional derivatives can be used to simulate accurately many natural phenomena containing long memory, for example, the groundwater and geo-hydrology models [6, 7, 13]. In recent years, there has been considerable interest in formulating different methods to numerically solve various types of differential and integral equations. One of the outstanding techniques is the use of spectral methods [14–21], due to their flexibility of implementation over finite and infinite intervals. A good survey of many appropriate numerical approximations can be found in [22, 23].

Precisely speaking, we consider the fractional ordinary differential system of the form

\[
D_0^\alpha u(t) = f(t, u(t)), \quad t \in (0, T], \quad T > 0,
\]

where \(f \in C([0, T] \times \mathbb{R}, \mathbb{R}), \alpha \in (0, 1),\) and \(n > 0\) is an integer not less than \(\alpha\). Here, we denote the term \(D_0^\alpha u(t)\) is the Riemann-Liouville \(\alpha\)th-order derivative of function \(u(t)\) given by

\[
D_0^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\xi)^{n-\alpha-1} u(\xi) d\xi, \quad n-1 < \alpha < n, \quad n \in \mathbb{Z}^+.
\]

The rest of this paper is structured as follows. There are several definitions of fractional derivatives, a quick tour based on the most useful and widely applied cases, such as the Riemann-Liouville, the Caputo, and the Grünwald-Letnikov types are briefly discussed in Section 2. Numerical method of approximation is given in Section 3. Modelling of three chaotic problems arising in biology, physics and financial economics is considered in Section 4. Finally, the main conclusion is outlined in Section 5.

## 2 Preliminary Background

In this section, we briefly introduced some of the useful preliminaries regarding fractional calculus theory [1, 3, 24–26].

The Riemann-Liouville fractional integral of order \(\alpha\) for a function \(u(t) \in C^1([0, b], \mathbb{R}^n); b > 0\) is given by

\[
D_0^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} u(\xi) d\xi, \quad 0 < \alpha < \infty,
\]

where \(\Gamma(\cdot)\) denotes the Euler’s gamma function.

The left Riemann-Liouville derivative and the right Riemann-Liouville derivative with order \(\alpha > 0\) of the given function \(u(t) \in C^1([0, b], \mathbb{R}^n)\) are respectively given as

\[
D_{a, l}^\alpha u(t) = \frac{d^n}{dt^n} \left[ D_{a, r}^{\alpha-n} u(t) \right] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\xi)^{n-\alpha-1} u(\xi) d\xi
\]

and

\[
D_{l, r}^\alpha u(t) = \frac{d^n}{dt^n} \left[ D_{l, a}^{\alpha-n} u(t) \right] = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (\xi-t)^{n-\alpha-1} u(\xi) d\xi,
\]

where \(n\) is an integer which satisfies \(n-1 < \alpha < n\).
The Riemann-Liouville fractional derivative of order $\alpha \in (0, 1]$ for a function $u(t) \in C^1([0, b], \mathbb{R}^n)$; $b > 0$ is given by

$$\frac{RL\varphi_0^{\alpha} u(t)}{dt} = \frac{d^n}{dt^n} \varphi_0^{1-\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\xi)^{n-\alpha-1} u(\xi) d\xi,$$  \hspace{1cm} (9)

for all $t \in [0, b]$ and $n - 1 < \alpha < n$, where $n > 0$ is an integer.

The left Caputo derivative and the right Caputo derivative with order $\alpha > 0$ of the given function $u(t)$, for $t \in (0, b)$ are respectively given as

$$C \varphi_{0,t}^{\alpha} u(t) = \varphi_{0,t}^{\alpha-n} \left[u^{(n)}(t)\right] = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} u^{(n)}(\xi) d\xi,$$  \hspace{1cm} (10)

and

$$C \varphi_{t,b}^{\alpha} u(t) = (-1)^n \varphi_{0,t}^{\alpha-n} \left[u^{(n)}(t)\right] = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\xi-t)^{n-\alpha-1} u^{(n)}(\xi) d\xi,$$  \hspace{1cm} (11)

where $n$ is an integer which satisfies $n - 1 < \alpha < n$.

The Caputo fractional derivative of order $\alpha \in (0, 1]$ for a function $u(t) \in C^1([0, b], \mathbb{R}^n)$; $b > 0$ is given by

$$C \varphi_{0,t}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} u^{(n)}(\xi) d\xi,$$  \hspace{1cm} (12)

for all $t \in [0, b]$.

The Grunwald-Letnikov fractional derivative of order $\alpha > 0$ of a function $u(t)$ is given by

$$GL\varphi_{0,t}^{\alpha} u(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{[\frac{t}{h}]} (-1)^k \binom{\alpha}{k} u(t-kh),$$  \hspace{1cm} (13)

where $h$ is the time-step.

The one-parameter Mittag-Leffler function $E_\alpha(z)$ is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$  \hspace{1cm} (14)

This function arises in the solution of fractional integral equations and interpolates between the power law and exponential law (when $\alpha = 1$, we obtain $E_1(z) = e^z$) scenarios for phenomena modelled by classical and fractional ordinary equations [27–31].

3 Numerical Approximation and Stability Analysis of Time-Fractional Differential System with Riemann-Liouville Derivative

In this section, we first present the numerical approximation of the Riemann-Liouville time-fractional derivative with the power law kernel [32], and later discuss the stability analysis of time-fractional differential equations.

3.1 Riemann-Liouville Approximation

We shall present directly the numerical approximation, by definition, we have

$$u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} f(\xi) d\xi,$$

$$RL\varphi_{0,t}^{\alpha} \{f(t)\} = \frac{d}{dt} u(t),$$

$$\frac{d}{dt} u(t) = \frac{u(t_{j+1}) - u(t_j)}{\Delta t},$$  \hspace{1cm} (15)
where

\[ u(t_{j+1}) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{j+1}} (t_{j+1} - \xi)^{-\alpha} f(\xi) d\xi \]

and

\[ u(t_j) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_j} (t_j - \xi)^{-\alpha} f(\xi) d\xi \]

Our numerical approximation is presented as follows:

\[
u(t_{j+1}) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{j+1}} (t_{j+1} - \xi)^{-\alpha} f(\xi) d\xi,
\]

\[= \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{j} \int_{t_s}^{t_{j+1}} (t_{j+1} - \xi)^{-\alpha} f(t_{s+1}) + f(t_s) \frac{1}{2} d\xi, \]

\[= \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{j} \frac{f(t_{s+1}) + f(t_s)}{2} \int_{t_s}^{t_{j+1}} (t_{j+1} - \xi)^{-\alpha} d\xi, \]

\[= \frac{1}{\Gamma(2-\alpha)} \sum_{s=0}^{j} \frac{f(t_{s+1}) + f(t_s)}{2} [(t_{j+1} - t_{s+1})^{1-\alpha} - (t_{j+1} - t_s)^{1-\alpha}], \]

similarly,

\[u(t_j) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_j} (t_j - \xi)^{-\alpha} f(\xi) d\xi,
\]

\[= \frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^{j} \frac{f(t_s) + f(t_{s-1})}{2} [(t_j - t_{s-1})^{1-\alpha} - (t_j - t_s)^{1-\alpha}] + O(\Delta t). \]

Thus

\[
R_{\alpha,j}^{RL} \{ f(t) \} = \frac{1}{\Delta t \Gamma(2-\alpha)} \left\{ \sum_{s=0}^{j} \frac{f(t_{s+1}) + f(t_s)}{2} [(t_{j+1} - t_{s+1})^{1-\alpha} - (t_{j+1} - t_s)^{1-\alpha}] \right. \\
- \sum_{s=1}^{j} \frac{f(t_s) + f(t_{s-1})}{2} [(t_j - t_{s-1})^{1-\alpha} - (t_j - t_s)^{1-\alpha}] \\
\left. + E_{\alpha,j} \right\}, \tag{18}
\]

where

\[
E_{\alpha,j} = \frac{1}{\Delta t \Gamma(1-\alpha)} \left\{ \sum_{s=0}^{j} \int_{t_s}^{t_{s+1}} \frac{f(u) - f(t_{s+1})}{(t_{j+1} - u)^{1-\alpha}} du - \sum_{s=0}^{j-1} \int_{t_s}^{t_{s+1}} \frac{f(u) - f(t_{s+1})}{(t_j - u)^{1-\alpha}} du \right\}. \]

**Theorem 31** (Atangana and Gómez-Aguilar [32]) Let \( f \) denotes a function not necessarily differentiable on \([a, T]\), then the fractional derivative of \( f \) of order \( \alpha \) in the Riemann-Liouville sense is defined by

\[
R_{\alpha,j}^{RL} \{ u(t) \} = \frac{1}{\Delta t \Gamma(2-\alpha)} \left\{ \sum_{s=0}^{j} \frac{f(t_{s+1}) + f(t_s)}{2} [(t_{j+1} - t_{s+1})^{1-\alpha} - (t_{j+1} - t_s)^{1-\alpha}] \right. \\
- \sum_{s=1}^{j} \frac{f(t_s) + f(t_{s-1})}{2} [(t_j - t_{s-1})^{1-\alpha} - (t_j - t_s)^{1-\alpha}] \\
\left. + E_{\alpha,j} \right\}, \]

where \( |E_{\alpha,j}| \leq C (t_{j+1}^{1-\alpha} - t_j^{1-\alpha}) \).

**Proof:** Readers are referred to Atangana and Gómez-Aguilar [32] for details.
3.2 Stability Analysis of Fractional Differential System in Riemann-Liouville Sense

Here, we shall consider the stability of fractional differential system in the sense of the Riemann-Liouville fractional derivative [24, 33]

\[ RL_{\alpha}u(t) = Fu(t), \quad 0 < \alpha < 1, \]

where \( u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^T \in \mathbb{R}^n, \ F = (f_{ij})_{n \times n} \in \mathbb{R}^{n \times n}, \) with initial condition taken the form

\[ RL_{\alpha}u(t)|_{t=0} = u_0 = [u_{10}, u_{20}, \ldots, u_{n0}]^T. \]

In what follows we analyze equation (19) for stability when \( F \) has zeros and non-zero eigenvalues.

**Theorem 32** If all the eigenvalues of \( F \) satisfies

\[ |\arg(\lambda(F))| > \frac{\alpha \pi}{2}, \]

then the zero solution of fractional differential system (19) is asymptotically stable.

**Proof.** By taking the Laplace transform on both sides of equation (19), subject to the initial condition (20), we get

\[ u(s),s^\alpha - u_0 = FU(s), \]

which means that the solution of fractional differential system (19) can be written as

\[ u(t) = u_0t^{\alpha-1}E_{\alpha,\alpha}(Ft^\alpha). \]

By following [10, 34], we assume that \( F \) is diagonalizable, such that

\[ \Lambda = A^{-1}FA = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \]

where \( A \) denotes an invertible matrix. Then,

\[ E_{\alpha,\alpha}(Ft^\alpha) = AE_{\alpha,\alpha}(A^t)A^{-1} = A \text{ diag}(E_{\alpha,\alpha}(\lambda_1t^\alpha), E_{\alpha,\alpha}(\lambda_2t^\alpha), E_{\alpha,\alpha}(\lambda_3t^\alpha), \ldots, E_{\alpha,\alpha}(\lambda_nt^\alpha))A^{-1}. \]

By using the two-parameter Mittag-Leffler function

\[ E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k)} + O(|z|^{-1-k}), \]

and (21), we have

\[ E_{\alpha,\alpha}(\lambda_j t^\alpha) = \sum_{k=2}^{\infty} \frac{(\lambda_j t^\alpha)^k}{\Gamma(\alpha k)} + O(|\lambda_j t^\alpha|^{-1-k}) \to 0, \ t \to +\infty, \ 1 \leq j \leq n. \]

So that,

\[ ||E_{\alpha,\alpha}(\Lambda t^\alpha)|| = ||\text{ diag}(E_{\alpha,\alpha}(\lambda_1t^\alpha), E_{\alpha,\alpha}(\lambda_2t^\alpha), E_{\alpha,\alpha}(\lambda_3t^\alpha), \ldots, E_{\alpha,\alpha}(\lambda_nt^\alpha))|| \to 0. \]

Again, we recall that any matrix \( F \in \mathbb{R}^{n \times n} \) can be written in its Jordan canonical form

\[ F = AJA^{-1} \]

where \( J \) denotes the diagonal block matrix \( J = \text{diag}(J_1, J_2, J_3, \ldots, J_M), \) and

\[ J_j = \begin{bmatrix} \lambda_j & 1 \\ & \ddots & \ddots \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}_{n_j \times n_j}, \quad j = 1, 2, \ldots, M. \]
Clearly, 

\[ E_{\alpha, \alpha}(F^t\alpha) = A \text{ diag}[E_{\alpha, \alpha}(J^{t_1\alpha}), E_{\alpha, \alpha}(J^{t_2\alpha}), \ldots, E_{\alpha, \alpha}(J^{t_M\alpha})]A^{-1}, \]

and

\[ E_{\alpha, \alpha}(J^{t\alpha}) = \sum_{s=0}^{\infty} \frac{(j^{t\alpha})^s}{\Gamma(\alpha + \alpha)} = \sum_{s=0}^{\infty} \frac{(t^{\alpha})^s}{\Gamma(\alpha + \alpha)} j^s \]

\[ = \sum_{s=0}^{\infty} \frac{(t^{\alpha})^s}{\Gamma(\alpha + \alpha)} \begin{bmatrix} \lambda_j^s K_j^1 \lambda_j^{s-1} \cdots K_s^{n_j-1} \lambda_j^{s-n_j+1} \\ \vdots \\ \vdots \\ \lambda_j^s \lambda_j^{s-1} \cdots \lambda_j^{s-n_j+1} \lambda_j \end{bmatrix} \]

\[ = \begin{bmatrix} E_{\alpha, \alpha}(\lambda_j^{t\alpha}) \frac{1}{i!} \frac{\partial}{\partial \lambda_j} E_{\alpha, \alpha}(\lambda_j^{t\alpha}) \cdots \frac{1}{(n_j-1)!} \left( \frac{\partial}{\partial \lambda_j} \right)^{n_j-1} E_{\alpha, \alpha}(\lambda_j^{t\alpha}) \\ E_{\alpha, \alpha}(\lambda_j^{t\alpha}) \cdots \\ \cdots \\ \frac{1}{i!} \frac{\partial}{\partial \lambda_j} E_{\alpha, \alpha}(\lambda_j^{t\alpha}) E_{\alpha, \alpha}(\lambda_j^{t\alpha}) \end{bmatrix} \]

(24)

The term \( K_i^1, 1 \leq i \leq n_j - 1 \) denotes the binomial coefficients. It is noticeable that if \( |\arg(\lambda_j(F))| > \frac{\alpha \pi}{2} \), \( 1 \leq j \leq M \) and \( t \to \infty \), with further calculations, we have

\[ |E_{\alpha, \alpha}(\lambda_j^{t\alpha})| \to 0, \quad \left| \frac{1}{i!} \left( \frac{\partial}{\partial \lambda_j} \right)^i E_{\alpha, \alpha}(\lambda_j^{t\alpha}) \right| \to 0, \text{ for } 0 \leq i \leq n_j, \quad 1 \leq j \leq M. \]

In fact, it is obvious that

\[ E_{\alpha, \alpha}(\lambda_j^{t\alpha}) = -\sum_{s=2}^{k} \frac{(\lambda_j^{t\alpha})^{-s}}{\Gamma(\alpha - \alpha s)} + \mathcal{O}(|\lambda_j^{t\alpha}|^{-1-k}), \]

which means that \( |E_{\alpha, \alpha}(\lambda_j^{t\alpha})| \to 0 \) as \( t \to \infty \), and

\[ \left| \frac{1}{i!} \left( \frac{\partial}{\partial \lambda_j} \right)^i E_{\alpha, \alpha}(\lambda_j^{t\alpha}) \right| \to 0, \quad 1 \leq i \leq n_j - 1 \text{ as } t \to \infty. \]

It follows that \( \|u(t)\| = \|u_0\alpha^{t-1} E_{\alpha, \alpha}(F^{t\alpha})\| \to 0 \) as \( t \to \infty \), for any non-zero initial value \( u_0 \). This completes the proof.

**Theorem 33** Let \( 0 < \alpha < 2 \). For internal stability, we consider the fractional autonomous system at \( a = 0 \), that is

\[ \mathcal{RL} \mathcal{D}_{0+}^\alpha u(t) = Fu(t), \]

(25)

with \( \alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T \) which has \( n \)-dimensional representation

\[ \mathcal{RL} \mathcal{D}_{0+}^{\alpha_1} u_1(t) = f_{11}u_1(t) + f_{12}u_2(t) + \cdots + f_{1n}u_n(t), \]

\[ \mathcal{RL} \mathcal{D}_{0+}^{\alpha_2} u_2(t) = f_{21}u_1(t) + f_{22}u_2(t) + \cdots + f_{2n}u_n(t), \]

\[ \vdots \]

\[ \mathcal{RL} \mathcal{D}_{0+}^{\alpha_n} u_n(t) = f_{n1}u_1(t) + f_{n2}u_2(t) + \cdots + f_{nn}u_n(t), \]

(26)
where \( \alpha_i \)'s are fractional numbers between 0 and 2. Suppose \( \varphi \) is the least common multiple of the denominators \( \chi_i \)'s of \( \alpha_i \)'s, where \( \alpha_i = y_i/x_i, \ x_i \in \mathbb{Z}^+ \) for \( i = 1, 2, \ldots, n \) and we set \( \beta = 1/\kappa \). We define

\[
\det \begin{pmatrix}
\lambda^{\kappa_0} - f_{11} & -f_{12} & \cdots & -f_{1n} \\
-f_{21} & \lambda^{\kappa_2} - f_{22} & \cdots & -f_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-f_{n1} & -f_{n2} & \cdots & \lambda^{\kappa_n} - f_{nn}
\end{pmatrix} = 0.
\tag{27}
\]

If all \( \alpha_i \)'s are fractional number, the characteristic polynomial equation \( \text{(27)} \) can be transformed to integer order polynomial. Then the zero solution of \( n \)-dimensional fractional differential system \( \text{(26)} \) is globally asymptotically stable provided all roots \( \lambda_i \) of the characteristic equation \( \text{(27)} \) satisfy

\[
|\arg(\lambda_i)| > \beta \frac{\pi}{2}, \forall i.
\]

By putting \( \lambda = h^\beta \) in equation \( \text{(27)} \), the characteristic equation can be written in the form \( \det(h^\beta I - F) = 0 \).

**Proof.** See Deng et al. [34] for a similar proof.

**Theorem 34** Assume all the eigenvalues of matrix \( F \) satisfy \( \text{(25)} \), that is,

\[
|\arg(\lambda(F))| \geq \frac{\alpha \pi}{2},
\]

and the critical eigenvalues found satisfying \( |\arg(\lambda(F))| = \frac{\alpha \pi}{2} \) have the same geometric and algebraic multiplicities, then we can say that the zero solution of fractional differential system \( \text{(19)} \) is not asymptotically stable but rather stable.

**Proof.** Let us assume there is a critical eigenvalue \( \lambda_0 \) which satisfies \( |\arg(\lambda_0)| = \frac{\alpha \pi}{2} \) with both geometric and algebraic multiplicity equal to unity. Then, from the above, we present the solution of \( \text{(19)} \) as

\[
u(t) = u_0 t^{\alpha-1} E_{\alpha,\alpha}(Ft^\alpha) = u_0 t^{\alpha-1} A \text{diag}[E_{\alpha,\alpha}(J_1^{t^\alpha}), \ldots, E_{\alpha,\alpha}(J_{q-1}^{t^\alpha}), E_{\alpha,\alpha}(\lambda_0^{t^\alpha}), E_{\alpha,\alpha}(J_{q+1}^{t^\alpha}), \ldots, E_{\alpha,\alpha}(J_n^{t^\alpha})] A^{-1}
\tag{28}
\]

where \( J_q \)'s are Jordan block matrices with order \( q_i \) \( |\arg(\lambda_q(F))| > \frac{\alpha \pi}{2} \), and \( \sum_{q=1}^{\rho-1} n_q + \sum_{q=\rho+1}^n n_q + 1 = n, q = 1, \ldots, \rho - 1, \rho + 1, \ldots, \omega \).

We obtain from above

\[
E_{\alpha,\alpha}(\lambda_0^{t^\alpha}) = \frac{1}{\alpha} (\lambda_0^{t^\alpha})^{(1-\alpha)/\alpha} \exp((\lambda_0^{t^\alpha})^{1/\alpha}) - \sum_{q=2}^i \frac{(\lambda_0^{t^\alpha})^{-q}}{\Gamma(\alpha - \alpha q)} + \mathcal{O}((|\lambda_0^{t^\alpha}|)^{-1-x}).
\tag{29}
\]

Assume

\[
\lambda_0 = s \left( \cos \frac{\alpha \pi}{2} + i \sin \frac{\alpha \pi}{2} \right),
\]

recall that \( t^2 = -1 \), and \( \omega \) denotes the modulus of \( \lambda_0 \). Then,

\[
E_{\alpha,\alpha}(\lambda_0^{t^\alpha}) = \frac{1}{\alpha} \left[ \omega^{t^\alpha} \left( \cos \frac{\alpha \pi}{2} + i \sin \frac{\alpha \pi}{2} \right) \right]^{(1-\alpha)/\alpha} \exp \left( \left[ \omega^{t^\alpha} \left( \cos \frac{\alpha \pi}{2} + i \sin \frac{\alpha \pi}{2} \right) \right]^{(1-\alpha)/\alpha} \right) - \sum_{q=2}^i \frac{(\omega^{t^\alpha})^{-q}}{\Gamma(\alpha - \alpha q)} + \mathcal{O}((|\omega^{t^\alpha}|)^{-1-x})
\]

\[
= \frac{1}{\alpha} \left( \omega^{(1-\alpha)/\alpha} \left( \sin \frac{\alpha \pi}{2} + i \cos \frac{\alpha \pi}{2} \right) \right) \exp \left( i \omega^{1/\alpha} \right) - \sum_{q=2}^i \frac{(\omega^{-q t^{-\alpha q}} \left( \sin \frac{\alpha \pi}{2} - i \cos \frac{\alpha \pi}{2} \right))^{-q}}{\Gamma(\alpha - \alpha q)} + \mathcal{O}((|\omega^{t^\alpha}|)^{-1-x}),
\tag{30}
\]

similarly, we have

\[
t^{\alpha-1} E_{\alpha,\alpha}(\lambda_0^{t^\alpha}) = \frac{1}{\alpha} \left( \omega^{(1-\alpha)/\alpha} \left( \sin \frac{\alpha \pi}{2} + i \cos \frac{\alpha \pi}{2} \right) \right) \exp \left( i \omega^{1/\alpha} \right)
\tag{31}
\]

\[
- t^{\alpha-1} \sum_{q=2}^i \frac{(\omega^{-q t^{-\alpha q}} \left( \sin \frac{\alpha \pi}{2} - i \cos \frac{\alpha \pi}{2} \right))^{-q}}{\Gamma(\alpha - \alpha q)} + \mathcal{O}((|\omega^{t^\alpha}|)^{-1-x}).
\]
From the above, it is obvious that the absolute value of the first term is \( \frac{1}{\alpha} \omega^{(1-\alpha)/\alpha} \), but the remainder terms tend to zero as \( t \to +\infty \). We can see from the result presented in Theorem 32 that \( E_{\alpha,q}(J_\alpha^{\alpha q}) \to 0 \) as \( t \to +\infty \), for \( q = 1, \ldots, \rho - 1, \rho + 1, \ldots, \omega \), which show that the solution of fractional differential system (19) is stable but not asymptotically stable. This completes the proof.

4 Numerical Experiments

In this section, we present some numerical experiments of the chaotic time-fractional differential equations arising from the fields of mathematical biology, physics and finance in which the time classical derivatives are replaced with the Riemann-Liouville fractional derivatives of order \( \alpha \in (0,1) \). Numerical approximation is done with the forward difference scheme which we execute with the Matlab R2012a package.

4.1 Lotka-Volterra System

Let us consider the time-fractional order Lotka-Volterra System [10]

\[
\begin{align*}
\mathcal{D}_0^\alpha u(t) &= au(t) + \varepsilon u^2(t) - \beta u(t)v(t) - \tau w(t)u^2(t) \\
\mathcal{D}_0^\alpha v(t) &= -cv(t) + du(t)v(t) \\
\mathcal{D}_0^\alpha w(t) &= -\rho w(t) + \tau w(t)u^2(t)
\end{align*}
\]

(32)

where \( u(t) > 0, v(t) > 0, w(t) > 0 \) are the two-predator and one-prey densities. The parameters \( a, \beta, c, d, \varepsilon, \rho, \tau \) are assumed positive. Whenever \( \rho = 0, \tau = 0 \), we recall the fractional order one-predator and one-prey model discussed in [35]. The Lotka-Volterra system (32) displays chaotic attractors in Figure 1 at different values of \( \alpha \) (as given in the caption) for the parameter values

\[
a = 1, \beta = 1, c = 1, d = 1, \varepsilon = 2, \rho = 3, \tau = 2.67, u(0) = 1, v(0) = 1.4, w(0) = 1.
\]

(33)

Fig. 1: Simulation result for fractional Lotka-Volterra system (32). The upper and lower rows correspond to species evolution at \( \alpha = 0.73 \) and \( \alpha = 0.95 \) respectively. Other parameters are given in (33). Simulation runs for \( t=100 \).
4.2 Hyperchaotic System

Here, we consider the time-fractional order problem which describes a novel hyperchaotic system given by the four-dimensional dynamics [36]

\[
\begin{align*}
RLD_0^\alpha u(t) &= \gamma(v(t) - u(t)) + w(t) + z(t) \\
RLD_0^\alpha v(t) &= \beta u(t) - u(t)w(t) + z(t) \\
RLD_0^\alpha w(t) &= u(t)v(t) - \tau w(t) \\
RLD_0^\alpha z(t) &= -\rho (u(t) + v(t))
\end{align*}
\]  

(34)

where \(u(t), v(t), w(t)\), and \(z(t)\) denotes the states, and \(\gamma, \beta, \tau, \rho\) are positive (constants) parameters. The 4D system (34) as

![Fig. 2](image1.png)

**Fig. 2:** Numerical results at \(\alpha = 0.91\) showing the 3D (upper-row) and 2D (lower-row) projections of the novel hyperchaotic system (34) with initial conditions \(u(0) = 2.6, v(0) = 2.1, w(0) = 2.5\) and \(z(0) = 2.3\). Other parameters are given in (35). Simulation runs for \(t=100\).

![Fig. 3](image2.png)

**Fig. 3:** Time series result of the novel hyperchaotic system (34) at \(\alpha = 0.87\) with different instances of simulation time \(t = 2, 10\). Other parameters are given in Figure 2.

presented in Figures 2 and 3, displays a strange hyperchaotic attractors for the parameter values

\[
\gamma = 18, \beta = 125, \tau = 4, \rho = 6.
\]  

(35)
4.3 Halvorsen Circulant Chaotic System

Lastly, we consider the Halvorsen circulant chaotic system, described by the 3D dynamics [37].

\[
\begin{align*}
\mathcal{RLD}_0^\alpha u(t) &= -\gamma u(t) - \beta (v(t) + w(t)) - v^2(t) \\
\mathcal{RLD}_0^\alpha v(t) &= -\gamma v(t) - \beta (u(t) + w(t)) - w^2(t) \\
\mathcal{RLD}_0^\alpha w(t) &= -\gamma w(t) - \beta (u(t) + v(t)) - u^2(t)
\end{align*}
\]  

(36)

which is symmetric with respect to cyclic interchanges of components \(u, v\) and \(w\). Halvorsen have shown that the circulant chaotic system described in (36) is chaotic when \(\gamma = 1.27\) and \(\beta = 4\). In the numerical experiment, we simulate with the initial conditions \(u(0) = 0.2, v(0) = 0.6\) and \(w(0) = 0.2\). Figure 4 illustrates the strange attractor of system (36) in \(\mathbb{R}^3\). Various 2D projections on \((u, v), (u, w)\) and \((v, w)\) coordinate are shown. It should be mentioned that, apart from the patterns obtained for the examples considered here, other chaotic and more-complex structures are obtainable, depending on the choices of initial conditions and other parameter values.
5 Conclusion

We have given a mathematical analysis with numerical treatment for general time-fractional differential equations. In the model, we the standard time derivative is replaced with the Riemann-Liouville fractional derivative of order $\alpha$, defined in the interval $[0,1]$. We observed that strange attractors can only occur if the right-hand side of the differential equation is coupled. Some numerical examples which have wide application in physics, biology and engineering are illustrated for different values of $\alpha$ to cover the pitfall and naturally arise question. In the future, the mathematical analysis and numerical treatment reported in this paper will be extended to time-space fractional reaction-diffusion equations.

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References


